Cumhuriyet Sci. J., 44(3) (2023) 522-530
DOI: https://doi.org/10.17776/csj.1145020

Cumhuriyet Science Journal

| csj.cumhuriyet.edu.tr | Founded: 2002 ISSN: 2587-2680 e-ISSN: 2587-246X Publisher: Sivas Cumhuriyet University

Construction of New Ostrowski's Type Inequalities By Using Multistep Linear Kernel

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Research Article

History
Received: 19/07/2022
Accepted: 31/03/2023

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ABSTRACT

In this paper, we construct a generalisation of Ostrowski's type inequalities with the help of new identity. By using this identity, we construct further results for $\acute{g}' \in L^1[\dot{c},\breve{d}], \acute{g}' \in L^2[\dot{c},\breve{d}], \acute{g}'' \in L^2[\dot{c},\breve{d}]$. To prove our main and related results, we utilized some famous inequalities such as Gruss-inequality, Diaz-Mıtcaf's inequality and Cauchy's inequality. To prove our main results, we used a new multistep kernel (9-step linear kernel). Some related results are also discussed. In the end, we apply our results to numerical integration also.

Keywords: Ostrowski inequalities, Numerical integration, Linear Kernel.

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Introduction

In 1970, Mitrinovi'c [1-3] stressed the significance of inequalities. Ostrowski type integral inequalities for 2-times differentiable mappings. Barnett *et al.* [4] released research about Ostrowski type integral inequalities for $L_p(c,d)$ and $L_1(c,d)$. Qayyum and Husain[5] generalized Ostrowski type integral inequalities to present new estimates. Qayyum *et al.* [6-11] provided a generalized form of Ostrowski type Gruss-inequality for twice derivable mappings. Barnett *et al.* [4] stressed another new concept i.e. proved Ostrowski type integral inequalities by utilizing $\beta-$ function for 1st and 2nd differential mappings and they applied their all findings for numerical quadrature rules. Few people (for example [9, 10, 12]) worked on different type of inequalities.

Main Findings

Lemma 1 Let $\acute{g}: [\dot{c},\breve{d}] \to \mathbb{R}$ be such that $\acute{g}'$ is absolutely continuous on $[\dot{c},\breve{d}]$. Define the kernel $P(u,\hat{U})$ as:

$$P(u,\hat{U}) = \begin{cases} \hat{U}-\dot{c}, & \hat{U} \in \left(\dot{c}, \frac{7\dot{c}+u}{8}\right] \\ \hat{U}-\frac{15\dot{c}+\breve{d}}{16}, & \hat{U} \in \left(\frac{7\dot{c}+u}{8}, \frac{3\dot{c}+u}{4}\right] \\ \hat{U}-\frac{7\dot{c}+\breve{d}}{8}, & \hat{U} \in \left(\frac{3\dot{c}+u}{4}, \frac{\dot{c}+u}{2}\right] \\ \hat{U}-\frac{3\dot{c}+\breve{d}}{4}, & \hat{U} \in \left(\frac{\dot{c}+u}{2}, u\right] \\ \hat{U}-\frac{\dot{c}+\breve{d}}{2}, & \hat{U} \in \left(u, \dot{c}+\breve{d}-u\right] \\ \hat{U}-\frac{\dot{c}+3\breve{d}}{4}, & \hat{U} \in \left(\dot{c}+\breve{d}-u, \frac{\dot{c}+2\breve{d}-u}{2}\right] \\ \hat{U}-\frac{\dot{c}+7\breve{d}}{8}, & \hat{U} \in \left(\frac{\dot{c}+2\breve{d}-u}{2}, \frac{\dot{c}+4\breve{d}-u}{4}\right] \\ \hat{U}-\frac{\dot{c}+15\breve{d}}{16}, & \hat{U} \in \left(\frac{\dot{c}+4\breve{d}-u}{4}, \frac{\dot{c}+8\breve{d}-u}{8}\right] \\ \hat{U}-\breve{d}, & \hat{U} \in \left(\frac{\dot{c}+8\breve{d}-u}{8}, \breve{d}\right] \end{cases} \tag{1}$$

for all $ų \in \left[\dot{c}, \frac{\dot{c}+\breve{d}}{2}\right]$, the following identity holds:

$$\frac{1}{\breve{d}-\dot{c}}\int_{\dot{c}}^{\breve{d}} P(\mathrm{u},\widehat{U})\mathrm{g}'(\widehat{U})d\widehat{U}$$
$$=\frac{1}{16}\left[\mathrm{g}\left(\frac{7\dot{c}+\mathrm{u}}{8}\right)+\mathrm{g}\left(\frac{3\dot{c}+\mathrm{u}}{4}\right)+2\mathrm{g}\left(\frac{\dot{c}+\mathrm{u}}{2}\right)+4\mathrm{g}(\mathrm{u})+4\mathrm{g}(\dot{c}+\breve{d}-\mathrm{u})+2\mathrm{g}\left(\frac{\dot{c}+2\breve{d}-2\mathrm{u}}{2}\right)+\mathrm{g}\left(\frac{\dot{c}+4\breve{d}-\mathrm{u}}{4}\right)\right.$$
$$\left.+\mathrm{g}\left(\frac{\dot{c}+8\breve{d}-\mathrm{u}}{8}\right)\right]-\frac{1}{\breve{d}-\dot{c}}\int_{\dot{c}}^{\breve{d}}\mathrm{g}(\widehat{U})d\widehat{U}. \tag{2}$$

Proof. We obtain the desired identity (2) by applying integration by parts on (1);

$$\int_{\dot{c}}^{\breve{d}} P(\mathrm{u},\widehat{U})\mathrm{g}'(\widehat{U})d\widehat{U}=\int_{\dot{c}}^{\frac{7\dot{c}+\mathrm{u}}{8}}(\widehat{U}-\dot{c})\mathrm{g}'(\widehat{U})d\widehat{U}+\int_{\frac{7\dot{c}+\mathrm{u}}{8}}^{\frac{3\dot{c}+\mathrm{u}}{4}}\left(\widehat{U}-\frac{15\dot{c}+\breve{d}}{16}\right)\mathrm{g}'(\widehat{U})d\widehat{U}+\int_{\frac{3\dot{c}+\mathrm{u}}{4}}^{\frac{\dot{c}+\mathrm{u}}{2}}\left(\widehat{U}-\frac{7\dot{c}+\breve{d}}{8}\right)\mathrm{g}'(\widehat{U})d\widehat{U}$$
$$+\int_{\frac{\dot{c}+\mathrm{u}}{2}}^{\mathrm{u}}\left(\widehat{U}-\frac{3\dot{c}+\breve{d}}{4}\right)\mathrm{g}'(\widehat{U})d\widehat{U}+\int_{\mathrm{u}}^{\dot{c}+\breve{d}-\mathrm{u}}\left(\widehat{U}-\frac{\dot{c}+\breve{d}}{2}\right)\mathrm{g}'(\widehat{U})d\widehat{U}+\int_{\dot{c}+\breve{d}-\mathrm{u}}^{\frac{\dot{c}+2\breve{d}-\mathrm{u}}{2}}\left(\widehat{U}-\frac{\dot{c}+3\breve{d}}{4}\right)\mathrm{g}'(\widehat{U})d\widehat{U}$$
$$+\int_{\frac{\dot{c}+2\breve{d}-\mathrm{u}}{2}}^{\frac{\dot{c}+4\breve{d}-\mathrm{u}}{4}}\left(\widehat{U}-\frac{\dot{c}+7\breve{d}}{8}\right)\mathrm{g}'(\widehat{U})d\widehat{U}+\int_{\frac{\dot{c}+4\breve{d}-\mathrm{u}}{4}}^{\frac{\dot{c}+8\breve{d}-\mathrm{u}}{8}}\left(\widehat{U}-\frac{\dot{c}+15\breve{d}}{16}\right)\mathrm{g}'(\widehat{U})d\widehat{U}+\int_{\frac{\dot{c}+8\breve{d}-\mathrm{u}}{8}}^{\breve{d}}(\widehat{U}-\breve{d})\mathrm{g}'(\widehat{U})d\widehat{U}$$

After simplification, we get(2).

Now by using (2) , we construct five different cases:

Case. 1: $When\ \acute{g}' \in L^1[\dot{c},\breve{d}]$

Theorem 1 Let $\acute{g}:[\dot{c},\breve{d}]\to\mathbb{R}$ be differentiable on $(\dot{c},\breve{d})$. If $\acute{g}'\in L^1[\dot{c},\breve{d}]$ and $\gamma\le\acute{g}'(\widehat{U})\le\Gamma$, for all $\widehat{U}\in[\dot{c},\breve{d}]$, then,

$$\left|\frac{1}{16}\left[\mathrm{g}\left(\frac{7\dot{c}+\mathrm{u}}{8}\right)+\mathrm{g}\left(\frac{3\dot{c}+\mathrm{u}}{4}\right)+2\mathrm{g}\left(\frac{\dot{c}+\mathrm{u}}{2}\right)+4\mathrm{g}(\mathrm{u})+4\mathrm{g}(\dot{c}+\breve{d}-\mathrm{u})+2\mathrm{g}\left(\frac{\dot{c}+2\breve{d}-2\mathrm{u}}{2}\right)+\mathrm{g}\left(\frac{\dot{c}+4\breve{d}-\mathrm{u}}{4}\right)\right.\right.$$
$$\left.\left.+\mathrm{g}\left(\frac{\dot{c}+8\breve{d}-\mathrm{u}}{8}\right)\right]-\frac{1}{\breve{d}-\dot{c}}\int_{\dot{c}}^{\breve{d}}g(\widehat{U})d\widehat{U}\right|\le\frac{1}{64}(\breve{d}-\dot{c})(\Gamma-\gamma) \tag{3}$$

holds for all $ų \in \left[\dot{c}, \frac{\dot{c}+\breve{d}}{2}\right]$.

Proof. As we know that for all $\widehat{U}\in[\dot{c},\breve{d}]$ and $ų \in \left[\dot{c}, \frac{\dot{c}+\breve{d}}{2}\right]$, we have

$$\mathrm{u}-\frac{15\dot{c}+\breve{d}}{16}\le P(\mathrm{u},\widehat{U})\le\mathrm{u}-\dot{c}.$$

Using Gruss-inequality [5] on the mappings $P(\mathrm{u},\widehat{U})$ and $\acute{g}'(\widehat{U})$,

$$\left|\frac{1}{\breve{d}-\dot{c}}\int_{\dot{c}}^{\breve{d}}P(\mathrm{u},\widehat{U})\mathrm{g}'(\widehat{U})d\widehat{U}-\frac{1}{\breve{d}-\dot{c}}\int_{\dot{c}}^{\breve{d}}P(\mathrm{u},\widehat{U})d\widehat{U}\frac{1}{\breve{d}-\dot{c}}\int_{\dot{c}}^{\breve{d}}\mathrm{g}'(\widehat{U})d\widehat{U}\right|\le\frac{1}{64}(\breve{d}-\dot{c})(\Gamma-\gamma) \tag{4}$$

for all $ų \in \left[\dot{c}, \frac{\dot{c}+\breve{d}}{2}\right]$.

It is straight forward exercise to show that

$$\frac{1}{\breve{d}-\dot{c}}\int_{\dot{c}}^{\breve{d}}P(\mathrm{u},\widehat{U})d\widehat{U}=\int_{\dot{c}}^{\frac{7\dot{c}+\mathrm{u}}{8}}(\widehat{U}-\dot{c})d\widehat{U}+\int_{\frac{7\dot{c}+\mathrm{u}}{8}}^{\frac{3\dot{c}+\mathrm{u}}{4}}\left(\widehat{U}-\frac{15\dot{c}+\breve{d}}{16}\right)d\widehat{U}+\int_{\frac{3\dot{c}+\mathrm{u}}{4}}^{\frac{\dot{c}+\mathrm{u}}{2}}\left(\widehat{U}-\frac{7\dot{c}+\breve{d}}{8}\right)d\widehat{U}$$
$$+\int_{\frac{\dot{c}+\mathrm{u}}{2}}^{\mathrm{u}}\left(\widehat{U}-\frac{3\dot{c}+\breve{d}}{4}\right)d\widehat{U}+\int_{\mathrm{u}}^{\dot{c}+\breve{d}-\mathrm{u}}\left(\widehat{U}-\frac{\dot{c}+\breve{d}}{2}\right)d\widehat{U}+\int_{\dot{c}+\breve{d}-\mathrm{u}}^{\frac{\dot{c}+2\breve{d}-\mathrm{u}}{2}}\left(\widehat{U}-\frac{\dot{c}+3\breve{d}}{4}\right)d\widehat{U}$$
$$+\int_{\frac{\dot{c}+2\breve{d}-\mathrm{u}}{2}}^{\frac{\dot{c}+4\breve{d}-\mathrm{u}}{4}}\left(\widehat{U}-\frac{\dot{c}+7\breve{d}}{8}\right)d\widehat{U}+\int_{\frac{\dot{c}+4\breve{d}-\mathrm{u}}{4}}^{\frac{\dot{c}+8\breve{d}-\mathrm{u}}{8}}\left(\widehat{U}-\frac{\dot{c}+15\breve{d}}{16}\right)d\widehat{U}+\int_{\frac{\dot{c}+8\breve{d}-\mathrm{u}}{8}}^{\breve{d}}(\widehat{U}-\breve{d})d\widehat{U}.$$

Again after simplification, we have

$$\frac{1}{\breve{d}-\dot{c}}\int_{\dot{c}}^{\breve{d}}P(\mathrm{u},\widehat{U})d\widehat{U}=0 \tag{5}$$

And

$$\frac{1}{\breve{d}-\dot{c}}\int_{\dot{c}}^{\breve{d}} g'(\widehat{U})d\widehat{U} = \frac{g(\breve{d})-g(\dot{c})}{\breve{d}-\dot{c}}. \quad (6)$$

Hence using $(4)-(6)$, we get our required result (3).

Now we will discuss some corollaries.

Corollary 1 By substituting ų$=\frac{\dot{c}+\breve{d}}{2}$ in (3), then

$$\left|\frac{1}{16}\left[g\left(\frac{15\dot{c}+\breve{d}}{16}\right)+g\left(\frac{7\dot{c}+\breve{d}}{8}\right)+2g\left(\frac{3\dot{c}+\breve{d}}{4}\right)+4g\left(\frac{\dot{c}+\breve{d}}{2}\right)+4g\left(\frac{\dot{c}+\breve{d}}{2}\right)+2g\left(\frac{\breve{d}}{2}\right)+g\left(\frac{\dot{c}+7\breve{d}}{8}\right)\right.\right.$$
$$\left.\left.+g\left(\frac{\dot{c}+15\breve{d}}{16}\right)\right]-\frac{1}{\breve{d}-\dot{c}}\int_{\dot{c}}^{\breve{d}} g(\widehat{U})d\widehat{U}\right|\leq\frac{1}{64}(\breve{d}-\dot{c})(\Gamma-\gamma).$$

Corollary 2 By substituting ų$=\frac{3\dot{c}+\breve{d}}{4}$ in (3), we get

$$\left|\frac{1}{16}\left[g\left(\frac{31\dot{c}+\breve{d}}{32}\right)+g\left(\frac{15\dot{c}+\breve{d}}{16}\right)+2g\left(\frac{7\dot{c}+\breve{d}}{8}\right)+4g\left(\frac{3\dot{c}+\breve{d}}{4}\right)+4g\left(\frac{\dot{c}+3\breve{d}}{4}\right)+2g\left(\frac{3\breve{d}-\dot{c}}{4}\right)\right.\right.$$
$$\left.\left.+g\left(\frac{\dot{c}+15\breve{d}}{16}\right)+g\left(\frac{\dot{c}+31\breve{d}}{32}\right)\right]-\frac{1}{\breve{d}-\dot{c}}\int_{\dot{c}}^{\breve{d}} g(\widehat{U})d\widehat{U}\right|\leq\frac{1}{64}(\breve{d}-\dot{c})(\Gamma-\gamma). \quad (7)$$

Corollary 3 By substituting ų$=\frac{\dot{c}+3\breve{d}}{4}$ in (3), we get

$$\left|\frac{1}{16}\left[g\left(\frac{29\dot{c}+3\breve{d}}{32}\right)+g\left(\frac{13\dot{c}+3\breve{d}}{16}\right)+2g\left(\frac{5\dot{c}+3\breve{d}}{8}\right)+4g\left(\frac{\dot{c}+3\breve{d}}{4}\right)+4g\left(\frac{3\dot{c}+\breve{d}}{4}\right)+2g\left(\frac{\dot{c}+\breve{d}}{4}\right)\right.\right.$$
$$\left.\left.+g\left(\frac{3\dot{c}+13\breve{d}}{16}\right)+g\left(\frac{3\dot{c}+29\breve{d}}{32}\right)\right]-\frac{1}{\breve{d}-\dot{c}}\int_{\dot{c}}^{\breve{d}} g(\widehat{U})d\widehat{U}\right|.\leq\frac{1}{64}(\breve{d}-\dot{c})(\Gamma-\gamma). \quad (8)$$

Case: 2 For $\acute{g}'\in L^1[\dot{c},\breve{d}]$

Theorem 2 Let $I:\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I^0, the interior of the interval I, and let $\dot{c},\breve{d}\in I$ with $\dot{c}<\breve{d}$. If $\acute{g}'\in L^1[\dot{c},\breve{d}]$,and $\gamma\leq\acute{g}'(\widehat{U})\leq\Gamma\forall$ ų$\in[\dot{c},\breve{d}]$, then the following inequality holds for all ų$\in\left[\dot{c},\frac{\dot{c}+\breve{d}}{2}\right]$, we have

$$\left|\frac{1}{16}\left[g\left(\frac{7\dot{c}+u}{8}\right)+g\left(\frac{3\dot{c}+u}{4}\right)+2g\left(\frac{\dot{c}+u}{2}\right)+4g(u)+4g(\dot{c}+\breve{d}-u)+2g\left(\frac{\dot{c}+2\breve{d}-2u}{2}\right)\right.\right.$$
$$\left.\left.+g\left(\frac{\dot{c}+4\breve{d}-u}{4}\right)+g\left(\frac{\dot{c}+8\breve{d}-u}{8}\right)\right]-\frac{1}{\breve{d}-\dot{c}}\int_{\dot{c}}^{\breve{d}} g(\widehat{U})d\widehat{U}\right|\leq\frac{1}{128(\breve{d}-\dot{c})}\left[(43\dot{c}+21\breve{d}-64u)(\dot{c}-u)\right](\Gamma-\gamma). \quad (9)$$

Proof. Let

$$c=\frac{\Gamma+\gamma}{2}$$

then

$$\frac{1}{\breve{d}-\dot{c}}\int_{\dot{c}}^{\breve{d}} P(u,\widehat{U})g'(\widehat{U})d\widehat{U}-\frac{c}{\breve{d}-\dot{c}}\int_{\dot{c}}^{\breve{d}} P(u,\widehat{U})d\widehat{U}=\frac{1}{\breve{d}-\dot{c}}\int_{\dot{c}}^{\breve{d}} P(u,\widehat{U})[g'(\widehat{U})-c]d\widehat{U}$$
$$=\frac{1}{16}\left[g\left(\frac{7\dot{c}+u}{8}\right)+g\left(\frac{3\dot{c}+u}{4}\right)+2g\left(\frac{\dot{c}+u}{2}\right)+4g(u)+4g(\dot{c}+\breve{d}-u)+2g\left(\frac{\dot{c}+2\breve{d}-2u}{2}\right)+g\left(\frac{\dot{c}+4\breve{d}-u}{4}\right)\right.$$
$$\left.+g\left(\frac{\dot{c}+8\breve{d}-u}{8}\right)\right]-\frac{1}{\breve{d}-\dot{c}}\int_{\dot{c}}^{\breve{d}} g(\widehat{U})d\widehat{U},$$

where

$$\int_{\dot{c}}^{\breve{d}} P(u,\widehat{U})d\widehat{U}=0.$$

On the other hand,

$$\left|\frac{1}{\breve{d}-\dot{c}}\int_{\dot{c}}^{\breve{d}} P(\mathrm{u},\widehat{U})[\mathrm{g}'(\widehat{U})-c]d\widehat{U}\right| \leq \frac{1}{\breve{d}-\dot{c}}\max_{\widehat{U}\in[\dot{c},\breve{d}]}\left|\mathrm{g}'(\widehat{U})-c\right|\int_{\dot{c}}^{\breve{d}} \left|P(\mathrm{u},\widehat{U})\right|d\widehat{U}. \tag{10}$$

Since

$$\max_{\widehat{U}\in[\dot{c},\breve{d}]}\left|\mathrm{g}'(\widehat{U})-c\right| \leq \frac{\Gamma+\gamma}{2} \tag{11}$$

and

$$\frac{1}{\breve{d}-\dot{c}}\int_{\dot{c}}^{\breve{d}} \left|P(\mathrm{u},\widehat{U})\right|d\widehat{U} = \frac{21}{64(\breve{d}-\dot{c})}\left[(2\dot{c}+\breve{d}-3\mathrm{u})(\dot{c}-\mathrm{u})+\frac{1}{64}(\dot{c}-\mathrm{u})^2\right]. \tag{12}$$

From $(10)-(12)$, we get (9).

Case. 3:

Theorem 3 Let ģ: $[\dot{c},\breve{d}] \to \mathbb{R}$ be differentiable mapping on $(\dot{c},\breve{d})$. If $ģ' \in L^1[\dot{c},\breve{d}]$ and $\gamma \leq ģ'(\widehat{U}) \leq \Gamma$

$$\left|\frac{1}{16}\left[\mathrm{g}\left(\frac{7\dot{c}+\mathrm{u}}{8}\right)+\mathrm{g}\left(\frac{3\dot{c}+\mathrm{u}}{4}\right)+2\mathrm{g}\left(\frac{\dot{c}+\mathrm{u}}{2}\right)+4\mathrm{g}(\mathrm{u})+4\mathrm{g}(\dot{c}+\breve{d}-\mathrm{u})+2\mathrm{g}\left(\frac{\dot{c}+2\breve{d}-2\mathrm{u}}{2}\right)\right.\right.$$
$$\left.\left.+\mathrm{g}\left(\frac{\dot{c}+4\breve{d}-\mathrm{u}}{4}\right)+\mathrm{g}\left(\frac{\dot{c}+8\breve{d}-\mathrm{u}}{8}\right)\right]-\frac{1}{\breve{d}-\dot{c}}\int_{\dot{c}}^{\breve{d}} \mathrm{g}(\widehat{U})d\widehat{U}\right| \leq \Omega(S-\gamma) \tag{13}$$

And

$$\left|\frac{1}{16}\left[\mathrm{g}\left(\frac{7\dot{c}+\mathrm{u}}{8}\right)+\mathrm{g}\left(\frac{3\dot{c}+\mathrm{u}}{4}\right)+2\mathrm{g}\left(\frac{\dot{c}+\mathrm{u}}{2}\right)+4\mathrm{g}(\mathrm{u})+4\mathrm{g}(\dot{c}+\breve{d}-\mathrm{u})+2\mathrm{g}\left(\frac{\dot{c}+2\breve{d}-2\mathrm{u}}{2}\right)\right.\right.$$
$$\left.\left.+\mathrm{g}\left(\frac{\dot{c}+4\breve{d}-\mathrm{u}}{4}\right)+\mathrm{g}\left(\frac{\dot{c}+8\breve{d}-\mathrm{u}}{8}\right)\right]-\frac{1}{\breve{d}-\dot{c}}\int_{\dot{c}}^{\breve{d}} \mathrm{g}(\widehat{U})d\widehat{U}\right| \leq \Omega(S-\Gamma) \tag{14}$$

for all ųϵ $\left[\dot{c};\frac{\dot{c}+\breve{d}}{2}\right]$, where

$$\Omega = \max_{\widehat{U}\in[\dot{c},\breve{d}]}\left|P(\mathrm{u},\widehat{U})\right|,$$
$$S = \frac{\mathrm{g}(\breve{d})-\mathrm{g}(\dot{c})}{\breve{d}-\dot{c}},$$
$$\gamma = \inf_{\widehat{U}\in[\dot{c},\breve{d}]}\mathrm{g}'(\widehat{U}),$$
$$\Gamma = \sup_{\widehat{U}\in[\dot{c},\breve{d}]}\mathrm{g}'(\widehat{U}).$$

Proof. As we know

$$\frac{1}{\breve{d}-\dot{c}}\int_{\dot{c}}^{\breve{d}} P(\mathrm{u},\widehat{U})\mathrm{g}'(\widehat{U})d\widehat{U}-\frac{1}{(\breve{d}-\dot{c})^2}\int_{\dot{c}}^{\breve{d}} P(\mathrm{u},\widehat{U})d\widehat{U}.\int_{\dot{c}}^{\breve{d}} \mathrm{g}'(\widehat{U})d\widehat{U} = \frac{1}{16}\left[\mathrm{g}\left(\frac{7\dot{c}+\mathrm{u}}{8}\right)+\mathrm{g}\left(\frac{3\dot{c}+\mathrm{u}}{4}\right)+2\mathrm{g}\left(\frac{\dot{c}+\mathrm{u}}{2}\right)\right.$$
$$\left.+4\mathrm{g}(\mathrm{u})+4\mathrm{g}(\dot{c}+\breve{d}-\mathrm{u})+2\mathrm{g}\left(\frac{\dot{c}+2\breve{d}-2\mathrm{u}}{2}\right)+\mathrm{g}\left(\frac{\dot{c}+4\breve{d}-\mathrm{u}}{4}\right)+\mathrm{g}\left(\frac{\dot{c}+8\breve{d}-\mathrm{u}}{8}\right)\right] \tag{15}$$

We denote

$$R_n(\mathrm{u}) = \frac{1}{\breve{d}-\dot{c}}\int_{\dot{c}}^{\breve{d}} P(\mathrm{u},\widehat{U})\mathrm{g}'(\widehat{U})d\widehat{U}-\frac{1}{(\breve{d}-\dot{c})^2}\int_{\dot{c}}^{\breve{d}} P(\mathrm{u},\widehat{U})d\widehat{U}.\int_{\dot{c}}^{\breve{d}} \mathrm{g}'(\widehat{U})d\widehat{U}. \tag{16}$$

If $c \in R$ is an arbitrary constant

$$R_n(\mathrm{u}) = \frac{1}{\breve{d}-\dot{c}}\int_{\dot{c}}^{\breve{d}} \left(\mathrm{g}'(\widehat{U})-c\right)\left[P(\mathrm{u},\widehat{U})-\frac{1}{\breve{d}-\dot{c}}\int_{\dot{c}}^{\breve{d}} P(\mathrm{u},s)ds\right]d\widehat{U}. \tag{17}$$

Since

$$\int_{\dot{c}}^{\breve{d}} \left[P(\mathrm{u},\widehat{U})-\frac{1}{\breve{d}-\dot{c}}\int_{\dot{c}}^{\breve{d}} P(\mathrm{u},s)ds\right]d\widehat{U} = 0.$$

Further more, we have

$$|R_n(\mathrm{u})| \le \frac{1}{\breve{d}-\dot{c}} \max_{\widehat{U}\in[\dot{c},\breve{d}]} \left|P(\mathrm{u},\widehat{U})-0\right| \int_{\dot{c}}^{\breve{d}} \left|\mathrm{g}'(\widehat{U})-c\right| d\widehat{U}$$

and

$$\max_{\widehat{U}\in[\dot{c},\breve{d}]} \left|P(\mathrm{u},\widehat{U})\right| = \Omega. \quad (18)$$

From [1]-[3], we get

$$\int_{\dot{c}}^{\breve{d}} \left|\mathrm{g}'(\widehat{U})-\gamma\right| d\widehat{U} = (S-\gamma)(\breve{d}-\dot{c}), \tag{19}$$

$$\int_{\dot{c}}^{\breve{d}} \left|\mathrm{g}'(\widehat{U})-\Gamma\right| d\widehat{U} = (\Gamma-S)(\breve{d}-\dot{c}). \tag{20}$$

By using $(5), (6), (15), (18)-(20)$, we get (13) and (14).

Case. 4: When $\acute{\mathrm{g}}' \in L^2[\dot{c},\breve{d}]$

Theorem 4 Let $\acute{\mathrm{g}}: [\dot{\mathrm{c}},\breve{\mathrm{d}}] \to \mathbb{R}$ be an absolutely continuous mapping in $(\dot{\mathrm{c}},\breve{\mathrm{d}})$. If $\acute{\mathrm{g}}' \in \mathrm{L}^2[\dot{\mathrm{c}},\breve{\mathrm{d}}]$, then we have

$$\begin{aligned}&\Big|\tfrac{1}{16}\Big[\mathrm{g}\left(\tfrac{7\dot{c}+\mathrm{u}}{8}\right)+\mathrm{g}\left(\tfrac{3\dot{c}+\mathrm{u}}{4}\right)+2\mathrm{g}\left(\tfrac{\dot{c}+\mathrm{u}}{2}\right)+4\mathrm{g}(\mathrm{u})+4\mathrm{g}(\dot{c}+\breve{d}-\mathrm{u})+2\mathrm{g}\left(\tfrac{\dot{c}+2\breve{d}-2\mathrm{u}}{2}\right)\\&+\mathrm{g}\left(\tfrac{\dot{c}+4\breve{d}-\mathrm{u}}{4}\right)+\mathrm{g}\left(\tfrac{\dot{c}+8\breve{d}-\mathrm{u}}{8}\right)\Big]-\tfrac{1}{\breve{d}-\dot{c}}\int_{\dot{c}}^{\breve{d}}\mathrm{g}(\widehat{U})d\widehat{U}\Big|\\&\le\sqrt{\tfrac{\sigma(\mathrm{g}')}{\breve{d}-\dot{c}}}\times\left[\tfrac{1}{3072}\left(697\dot{c}^2+805\dot{c}\breve{d}+256\breve{d}^2-2199\dot{c}\mathrm{u}-1317\breve{d}\mathrm{u}+1758\mathrm{u}^2\right)\right]^{\frac{1}{2}}\end{aligned} \tag{21}$$

for all ų∈ $\left[\dot{c},\frac{\dot{c}+\breve{d}}{2}\right]$, where

$$\sigma(\mathrm{g}') = \|\mathrm{g}''\|_2^2 - \frac{\left(\mathrm{g}(\breve{d})-\mathrm{g}(\dot{c})\right)^2}{\breve{d}-\dot{c}} = \|\mathrm{g}''\|_2^2 - S^2(\breve{d}-\dot{c}).$$

Proof. Let $R_n(\mathrm{u})$ is defined as in (16) then from (15), we get

$$\begin{aligned}R_n(\mathrm{u}) = &\left|\frac{1}{16}\left[\mathrm{g}\left(\frac{7\dot{c}+\mathrm{u}}{8}\right)+\mathrm{g}\left(\frac{3\dot{c}+\mathrm{u}}{4}\right)+2\mathrm{g}\left(\frac{\dot{c}+\mathrm{u}}{2}\right)+4\mathrm{g}(\mathrm{u})+4\mathrm{g}(\dot{c}+\breve{d}-\mathrm{u})+2\mathrm{g}\left(\frac{\dot{c}+2\breve{d}-2\mathrm{u}}{2}\right)\right.\right.\\&\left.\left.+\mathrm{g}\left(\tfrac{\dot{c}+4\breve{d}-\mathrm{u}}{4}\right)+\mathrm{g}\left(\tfrac{\dot{c}+8\breve{d}-\mathrm{u}}{8}\right)\right]-\tfrac{1}{\breve{d}-\dot{c}}\int_{\dot{c}}^{\breve{d}}\mathrm{g}(\widehat{U})d\widehat{U}\right|.\end{aligned}$$

If we choose

$$c = \frac{1}{\breve{d}-\dot{c}}\int_{\dot{c}}^{\breve{d}} \mathrm{g}'(s)ds$$

in (17) and using the Cauchy's inequality;

$$\begin{aligned}|R_n(\mathrm{u})| &\le \frac{1}{\breve{d}-\dot{c}}\int_{\dot{c}}^{\breve{d}}\left|\mathrm{g}'(\widehat{U})-\frac{1}{\breve{d}-\dot{c}}\int_{\dot{c}}^{\breve{d}}\mathrm{g}'(s)ds\right|\left|P(\mathrm{u},\widehat{U})-\frac{1}{\breve{d}-\dot{c}}\int_{\dot{c}}^{\breve{d}}P(\mathrm{u},s)ds\right|d\widehat{U}\\&\le\tfrac{1}{\breve{d}-\dot{c}}\left[\int_{\dot{c}}^{\breve{d}}\left(\mathrm{g}'(\widehat{U})-\tfrac{1}{\breve{d}-\dot{c}}\int_{\dot{c}}^{\breve{d}}\mathrm{g}'(s)ds\right)^2d\widehat{U}\right]^{\frac{1}{2}}\times\left[\int_{\dot{c}}^{\breve{d}}\left(P(\mathrm{u},\widehat{U})-\tfrac{1}{\breve{d}-\dot{c}}P(\mathrm{u},s)ds\right)^2d\widehat{U}\right]^{\frac{1}{2}}\\&\le\sqrt{\sigma(\mathrm{g}')}(\breve{d}-\dot{c})^{\frac{-1}{2}}\times\left[\tfrac{1}{3072}\left(697\dot{c}^2+805\dot{c}\breve{d}+256\breve{d}^2-2199\dot{c}\mathrm{u}-1317\breve{d}\mathrm{u}+1758\mathrm{u}^2\right)\right]^{\frac{1}{2}}.\end{aligned}$$

Corollary 4 If we substitute ų= $\frac{\dot{c}+\breve{d}}{2}$, in (21), we get

$$\left|\tfrac{1}{16}\left[\mathrm{g}\left(\tfrac{15\dot{c}+\breve{d}}{16}\right)+\mathrm{g}\left(\tfrac{7\dot{c}+\breve{d}}{8}\right)+2\mathrm{g}\left(\tfrac{3\dot{c}+\breve{d}}{4}\right)+8\mathrm{g}\left(\tfrac{\dot{c}+\breve{d}}{2}\right)+2\mathrm{g}\left(\tfrac{\breve{d}}{2}\right)+\mathrm{g}\left(\tfrac{\dot{c}+7\breve{d}}{8}\right)+\mathrm{g}\left(\tfrac{\dot{c}+15\breve{d}}{16}\right)\right]-\tfrac{1}{\breve{d}-\dot{c}}\int_{\dot{c}}^{\breve{d}}\mathrm{g}(\widehat{U})d\widehat{U}\right|$$

$$\leq \sqrt{\frac{\sigma(g')}{\breve{d}-\dot{c}}}\left[\frac{37}{3072}\left(\dot{c}-\breve{d}\right)^2\right]^{\frac{1}{2}}. \tag{22}$$

Now we state another case.

2.5 Case. 5: When $\acute{g}'' \in L^2[\dot{c}, \breve{d}]$

Theorem 5 Let $\acute{g}: [\dot{c}, \breve{d}] \to \mathbb{R}$ be a twice absolutely continuous differentiable mapping in $(\dot{c}, \breve{d})$ with $\acute{g}'' \in L^2[\dot{c}, \breve{d}]$.

$$\left|\frac{1}{16}\left[g\left(\frac{7\dot{c}+u}{8}\right)+g\left(\frac{3\dot{c}+u}{4}\right)+2g\left(\frac{\dot{c}+u}{2}\right)+4g(u)+4g\left(\dot{c}+\breve{d}-u\right)+2g\left(\frac{\dot{c}+2\breve{d}-2u}{2}\right)\right.\right.$$
$$\left.\left.+g\left(\frac{\dot{c}+4\breve{d}-u}{4}\right)+g\left(\frac{\dot{c}+8\breve{d}-u}{8}\right)\right]-\frac{1}{\breve{d}-\dot{c}}\int_{\dot{c}}^{\breve{d}} g(\widehat{U})d\widehat{U}\right|$$
$$\leq \left[\frac{1}{3072\pi}\left(697\dot{c}^2+805\dot{c}\breve{d}+256\breve{d}^2-2199\dot{c}u-1317\breve{d}u+1758u^2\right)\right]^{\frac{1}{2}} \times \left(\breve{d}-\dot{c}\right)^{\frac{3}{2}}\|g''\|_2 \tag{23}$$

for all ų∈ $\left[\dot{c}, \frac{\dot{c}+\breve{d}}{2}\right]$.

Proof. Let $R_n(u)$ be defined by (16) from (15)

$$R_n(u) = \left|\frac{1}{16}\left[g\left(\frac{7\dot{c}+u}{8}\right)+g\left(\frac{3\dot{c}+u}{4}\right)+2g\left(\frac{\dot{c}+u}{2}\right)\right.\right.$$
$$\left.\left.+4g(u)+4g\left(\dot{c}+\breve{d}-u\right)+2g\left(\frac{\dot{c}+2\breve{d}-2u}{2}\right)+g\left(\frac{\dot{c}+4\breve{d}-u}{4}\right)+g\left(\frac{\dot{c}+8\breve{d}-u}{8}\right)\right]-\frac{1}{\breve{d}-\dot{c}}\int_{\dot{c}}^{\breve{d}} g(\widehat{U})d\widehat{U}\right|.$$

If we choose $C = \acute{g}'\left(\frac{\dot{c}+\breve{d}}{2}\right)$ in (17) and use the Cauchy's Inequality, we get

$$|R_n(u)| \leq \frac{1}{\breve{d}-\dot{c}}\int_{\dot{c}}^{\breve{d}} \left|g'(\widehat{U})-g'\left(\frac{\dot{c}+\breve{d}}{2}\right)\right|\left|P(u,\widehat{U})-\frac{1}{\breve{d}-\dot{c}}\int_{\dot{c}}^{\breve{d}} P(u,s)ds\right|d\widehat{U}$$
$$\leq \frac{1}{\breve{d}-\dot{c}}\left[\int_{\dot{c}}^{\breve{d}}\left(g'(\widehat{U})-g'\left(\frac{\dot{c}+\breve{d}}{2}\right)\right)^2 d\widehat{U}\right]^{\frac{1}{2}} \times \left[\int_{\dot{c}}^{\breve{d}}\left(P(u,\widehat{U})-\frac{1}{\breve{d}-\dot{c}}P(u,s)ds\right)^2 d\widehat{U}\right]^{\frac{1}{2}}.$$

We may apply Diaz-Metcalf inequality[1] or [13], to obtain

$$\int_{\dot{c}}^{\breve{d}}\left(g'(\widehat{U})-g'\left(\frac{\dot{c}+\breve{d}}{2}\right)\right)^2 d\widehat{U} \leq \frac{\left(\breve{d}-\dot{c}\right)^2}{\pi^2}\|g''\|_2^2.$$

We also have

$$\int_{\dot{c}}^{\breve{d}}\left(P(u,\widehat{U})-\frac{1}{\breve{d}-\dot{c}}P(u,s)ds\right)^2 d\widehat{U} = \int_{\dot{c}}^{\breve{d}}\left(P(u,\widehat{U})\right)^2 d\widehat{U}$$
$$= \frac{1}{3072}\left(\breve{d}-\dot{c}\right)\left(697\dot{c}^2+805\dot{c}\breve{d}+256\breve{d}^2-2199\dot{c}u-1317\breve{d}u+1758u^2\right). \tag{24}$$

Corollary 5 If we substitute ų= $\frac{\dot{c}+3\breve{d}}{4}$, in (23) we get

$$\left|\frac{1}{16}\left[g\left(\frac{29\dot{c}+3\breve{d}}{32}\right)+g\left(\frac{13\dot{c}+3\breve{d}}{16}\right)+2g\left(\frac{5\dot{c}+3\breve{d}}{8}\right)+4g\left(\frac{\dot{c}+3\breve{d}}{4}\right)+4g\left(\frac{3\dot{c}+\breve{d}}{4}\right)+2g\left(\frac{\dot{c}+\breve{d}}{4}\right)\right.\right.$$
$$\left.\left.+g\left(\frac{3\dot{c}+13\breve{d}}{16}\right)+g\left(\frac{3\dot{c}+29\breve{d}}{32}\right)\right]-\frac{1}{\breve{d}-\dot{c}}\int_{\dot{c}}^{\breve{d}} g(\widehat{U})d\widehat{U}\right| \leq \left[\frac{1}{3072\pi}\left(\frac{2057}{8}\left(\dot{c}-\breve{d}\right)^2\right)\right]^{\frac{1}{2}}\left(\breve{d}-\dot{c}\right)^{\frac{3}{2}}\|g''\|_2.$$

An application to Composite Quadrature Rules

Let $I_n: \dot{c} = ų_0 < ų_1 <\ldots< ų_{n-1} < ų_n = \breve{d}$ be a division of the interval $[\dot{c}, \breve{d}]$, $\xi_i \in [u_i, u_{i+1}] (i = 0,1,\ldots n-1)$; a sequence of intermediate points $h_i = ų_{i+1} - ų_i (i = 0,1,\ldots n-1)$. We have the following quadrature formula:

When $ǵ' \in L^1[\dot{c}, \breve{d}]$

Theorem 6 Let $ǵ: I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I^0, the interior of the interval I, and let $\dot{c}, \breve{d} \in I$ with $\dot{c} < \breve{d}$. If $ǵ' \in L^1[\dot{c}, \breve{d}]$ and $\gamma \leq ǵ'(\widehat{U}) \leq \Gamma \forall ų \in \left[\dot{c}, \frac{\dot{c}+\breve{d}}{2}\right]$,

$$\int_{\dot{c}}^{\breve{d}} g(\widehat{U}) d\widehat{U} = \dot{c}(g, I_n) + R(g, I_n), \tag{25}$$

where

$$\dot{c}(g, I_n) = \frac{1}{16} \sum_{i=0}^{n-1} h_i \left[g\left(\frac{29u_i + 3u_{i+1}}{32}\right) + g\left(\frac{13u_i + 3u_{i+1}}{16}\right) + 2g\left(\frac{5u_i + 3u_{i+1}}{8}\right) + 4g\left(\frac{u_i + 3u_{i+1}}{4}\right) + 4g\left(\frac{3u_i + u_{i+1}}{4}\right) + 2g\left(\frac{u_i + u_{i+1}}{4}\right) + g\left(\frac{3u_i + 13u_{i+1}}{16}\right) + g\left(\frac{3u_i + 29u_{i+1}}{32}\right)\right] \tag{26}$$

and

$$|R(g, I_n)| \leq \frac{1}{64}(\Gamma - \gamma) \sum_{i=0}^{n-1} h_i \tag{27}$$

for all $\xi_i \in [u_i, u_{i+1}]$, where $h_i = ų_{i+1} - ų_i, (i = 0,1,\ldots n-1)$.

Proof. Apply (7) on the interval $[u_i, u_{i+1}], \xi_i \in [u_i, u_{i+1}]$ where $h_i = ų_{i+1} - ų_i, (i = 0,1,\ldots n-1)$,

$$R(g, I_n) = \int_{u_i}^{u_{i+1}} g(\widehat{U}) d\widehat{U} - \frac{1}{16} \sum_{i=0}^{n-1} h_i \left[g\left(\frac{29u_i + 3u_{i+1}}{32}\right) + g\left(\frac{13u_i + 3u_{i+1}}{16}\right) + 2g\left(\frac{5u_i + 3u_{i+1}}{8}\right) + 4g\left(\frac{u_i + 3u_{i+1}}{4}\right) + 4g\left(\frac{3u_i + u_{i+1}}{4}\right) + 2g\left(\frac{u_i + u_{i+1}}{4}\right) + g\left(\frac{3u_i + 13u_{i+1}}{16}\right) + g\left(\frac{3u_i + 29u_{i+1}}{32}\right)\right].$$

Adding over i from 0 to $n-1$,

$$R(g, I_n) = \sum_{i=0}^{n-1} \int_{u_i}^{u_{i+1}} g(\widehat{U}) d\widehat{U} - \frac{1}{16} \sum_{i=0}^{n-1} h_i \left[g\left(\frac{29u_i + 3u_{i+1}}{32}\right) + g\left(\frac{13u_i + 3u_{i+1}}{16}\right) + 2g\left(\frac{5u_i + 3u_{i+1}}{8}\right) + 4g\left(\frac{u_i + 3u_{i+1}}{4}\right) + 4g\left(\frac{3u_i + u_{i+1}}{4}\right) + 2g\left(\frac{u_i + u_{i+1}}{4}\right) + g\left(\frac{3u_i + 13u_{i+1}}{16}\right) + g\left(\frac{3u_i + 29u_{i+1}}{32}\right)\right]$$

$$R(g, I_n) = \int_{\dot{c}}^{\breve{d}} g(\widehat{U}) d\widehat{U} - \frac{1}{16} \sum_{i=0}^{n-1} h_i \left[g\left(\frac{29u_i + 3u_{i+1}}{32}\right) + g\left(\frac{13u_i + 3u_{i+1}}{16}\right) + 2g\left(\frac{5u_i + 3u_{i+1}}{8}\right) + 4g\left(\frac{u_i + 3u_{i+1}}{4}\right) + 4g\left(\frac{3u_i + u_{i+1}}{4}\right) + 2g\left(\frac{u_i + u_{i+1}}{4}\right) + g\left(\frac{3u_i + 13u_{i+1}}{16}\right) + g\left(\frac{3u_i + 29u_{i+1}}{32}\right)\right].$$

From (7),

$$|R(g, I_n)| = \left| \int_{\dot{c}}^{\breve{d}} g(\widehat{U}) d\widehat{U} - \frac{1}{16} \sum_{i=0}^{n-1} h_i \left[g\left(\frac{29u_i + 3u_{i+1}}{32}\right) + g\left(\frac{13u_i + 3u_{i+1}}{16}\right) + 2g\left(\frac{5u_i + 3u_{i+1}}{8}\right) + 4g\left(\frac{u_i + 3u_{i+1}}{4}\right) + 4g\left(\frac{3u_i + u_{i+1}}{4}\right) + 2g\left(\frac{u_i + u_{i+1}}{4}\right) + g\left(\frac{3u_i + 13u_{i+1}}{16}\right) + g\left(\frac{3u_i + 29u_{i+1}}{32}\right)\right]\right| \leq \frac{1}{64} h_i (\Gamma - \gamma).$$

Hence proved.

When $ǵ' \in \boldsymbol{L^2[\dot{c}, \breve{d}]}$

Theorem 7 Let $h_i = ų_{i+1} - ų_i = h = \frac{\breve{d}-\dot{c}}{n} (i = 0,1,\ldots, n-1)$ and let $ǵ: [\dot{c}, \breve{d}] \to \mathbb{R}$ be an absolutely continuous mapping in $(\dot{c}, \breve{d})$ with $ǵ' \in L^2[\dot{c}, \breve{d}]$.

$$\int_{\dot{c}}^{\breve{d}} \mathrm{g}(\mathrm{u})d\mathrm{u} = \dot{c}(\mathrm{g}, I_n) + R(\mathrm{g}, I_n),$$

and

$$|R(\mathrm{g}, I_n)| \leq \sqrt{\frac{37(\breve{d}-\dot{c})}{3072}}\sigma(\mathrm{g}').$$

Proof. Applying (22) to the interval $[\mathrm{u}_i, \mathrm{u}_{i+1}]$, then

$$\left|\frac{h}{16}\left[\mathrm{g}\left(\frac{15\mathrm{u}_i+\mathrm{u}_{i+1}}{16}\right)+\mathrm{g}\left(\frac{7\mathrm{u}_i+\mathrm{u}_{i+1}}{8}\right)+2\mathrm{g}\left(\frac{3\mathrm{u}_i+\mathrm{u}_{i+1}}{4}\right)+8\mathrm{g}\left(\frac{\mathrm{u}_i+\mathrm{u}_{i+1}}{2}\right)+2\mathrm{g}\left(\frac{\mathrm{u}_{i+1}}{2}\right)+\mathrm{g}\left(\frac{\mathrm{u}_i+7\mathrm{u}_{i+1}}{8}\right)\right.\right.$$
$$\left.\left.+\mathrm{g}\left(\frac{\mathrm{u}_i+15\mathrm{u}_{i+1}}{16}\right)\right]-\int_{\mathrm{u}_i}^{\mathrm{u}_{i+1}}\mathrm{g}(\widehat{U})d\widehat{U}\right| \leq \sqrt{\frac{37h}{3072}}(\mathrm{u}_i-\mathrm{u}_{i+1})\left[\int_{\mathrm{u}_i}^{\mathrm{u}_{i+1}}\left(\mathrm{g}(\widehat{U})\right)^2 d\widehat{U}-\frac{(\mathrm{g}(\mathrm{u}_{i+1})-\mathrm{g}(\mathrm{u}_i))^2}{h}\right]^{\frac{1}{2}}$$

for $i=0,1,\ldots,n-1$.

Now adding over i from 0 to $n-1$, using the triangle Inequality and Cauchy's inequality twice, we get

$$\left|\frac{h}{16}\sum_{i=0}^{n-1}\left[\mathrm{g}\left(\frac{15\mathrm{u}_i+\mathrm{u}_{i+1}}{16}\right)+\mathrm{g}\left(\frac{7\mathrm{u}_i+\mathrm{u}_{i+1}}{8}\right)+2\mathrm{g}\left(\frac{3\mathrm{u}_i+\mathrm{u}_{i+1}}{4}\right)+8\mathrm{g}\left(\frac{\mathrm{u}_i+\mathrm{u}_{i+1}}{2}\right)+2\mathrm{g}\left(\frac{\mathrm{u}_{i+1}}{2}\right)+\mathrm{g}\left(\frac{\mathrm{u}_i+7\mathrm{u}_{i+1}}{8}\right)\right.\right.$$
$$\left.\left.+\mathrm{g}\left(\frac{\mathrm{u}_i+15\mathrm{u}_{i+1}}{16}\right)\right]-\int_{\mathrm{u}_i}^{\mathrm{u}_{i+1}}\mathrm{g}(\widehat{U})d\widehat{U}\right| \leq \sqrt{\frac{37h}{3072}}\sum_{i=0}^{n-1}\left((\mathrm{u}_i-\mathrm{u}_{i+1})\left[\int_{\mathrm{u}_i}^{\mathrm{u}_{i+1}}\left(\mathrm{g}(\widehat{U})\right)^2 d\widehat{U}-\frac{\left(\mathrm{g}(\mathrm{u}_{i+1})-\mathrm{g}(\mathrm{u}_i)\right)^2}{h}\right]^{\frac{1}{2}}\right)$$
$$\leq \sqrt{\frac{37h}{3072}}\sqrt{n}\left[\|\mathrm{g}'\|_2^2-\frac{n}{\breve{d}-\dot{c}}\sum_{i=0}^{n-1}\left(\mathrm{g}(\mathrm{u}_{i+1})-\mathrm{g}(\mathrm{u}_i)\right)^2\right]^{\frac{1}{2}}$$
$$\leq \sqrt{\frac{37h}{3072}}\sqrt{n}\left[(\mathrm{u}_i-\mathrm{u}_{i+1})\left(\|\mathrm{g}'\|_2^2-\frac{\left(\mathrm{g}(\breve{d})-\mathrm{g}(\dot{c})\right)^2}{\breve{d}-\dot{c}}\right)\right]^{\frac{1}{2}}$$
$$=\sqrt{\frac{37(\breve{d}-\dot{c})}{3072}}(\mathrm{u}_i-\mathrm{u}_{i+1})\sigma(\mathrm{g}').$$

When $\acute{g}'' \in \boldsymbol{L^2[\dot{c},\breve{d}]}$

Theorem 8 Let $\mathrm{h_i}=\mathrm{u_{i+1}}-\mathrm{u_i}=\mathrm{h}=\frac{\breve{\mathrm{d}}-\dot{\mathrm{c}}}{\mathrm{n}}(\mathrm{i}=0,1,\ldots,\mathrm{n}-1)$ and let $\acute{g}:[\dot{\mathrm{c}},\breve{\mathrm{d}}]\to\mathbb{R}$ be a twice continuously differentiable mapping in $(\dot{\mathrm{c}},\breve{\mathrm{d}})$ with $\acute{g}'' \in \mathrm{L}^2[\dot{\mathrm{c}},\breve{\mathrm{d}}]$. Then,

$$\int_{\dot{c}}^{\breve{d}} \mathrm{g}(\mathrm{u})d\mathrm{u} = \dot{c}(\mathrm{g}, I_n) + R(\mathrm{g}, I_n),$$

where

$$|R(\mathrm{g}, I_n)|$$

$$\leq \left[\frac{2057}{24576\pi}\right]^{\frac{1}{2}}\frac{(\breve{d}-\dot{c})^{\frac{5}{2}}}{n^{\frac{5}{2}}}(\dot{c}-\breve{d})\|\mathrm{g}''\|_2.$$

Proof. Applying (25) to the interval $[\mathrm{u}_i, \mathrm{u}_{i+1}]$, we get

$$\left|\frac{h}{16}\left[\mathrm{g}\left(\frac{29\mathrm{u}_i+3\mathrm{u}_{i+1}}{32}\right)+\mathrm{g}\left(\frac{13\mathrm{u}_i+3\mathrm{u}_{i+1}}{16}\right)+2\mathrm{g}\left(\frac{5\mathrm{u}_i+3\mathrm{u}_{i+1}}{8}\right)+4\mathrm{g}\left(\frac{\mathrm{u}_i+3\mathrm{u}_{i+1}}{4}\right)+4\mathrm{g}\left(\frac{3\mathrm{u}_i+\mathrm{u}_{i+1}}{4}\right)+2\mathrm{g}\left(\frac{\mathrm{u}_i+\mathrm{u}_{i+1}}{4}\right)\right.\right.$$
$$\left.\left.+\mathrm{g}\left(\frac{3\mathrm{u}_i+13\mathrm{u}_{i+1}}{16}\right)+\mathrm{g}\left(\frac{3\mathrm{u}_i+29\mathrm{u}_{i+1}}{32}\right)\right]-\int_{\mathrm{u}_i}^{\mathrm{u}_{i+1}}\mathrm{g}(\widehat{U})d\widehat{U}\right| \leq \left[\frac{2057}{24576\pi}\right]^{\frac{1}{2}}h^{\frac{5}{2}}(\mathrm{u}_i-\mathrm{u}_{i+1})\left[\int_{\mathrm{u}_i}^{\mathrm{u}_{i+1}}\mathrm{g}''(\widehat{U})d\widehat{U}\right]^{\frac{1}{2}}.$$

By adding over i from 0 to $n-1$, applying the triangle inequality and Cauchy's inequality, we have

$$\left|\frac{h}{16}\sum_{i=0}^{n-1}\left[g\left(\frac{29u_i+3u_{i+1}}{32}\right)+g\left(\frac{13u_i+3u_{i+1}}{16}\right)+2g\left(\frac{5u_i+3u_{i+1}}{8}\right)\right.\right.$$

$$\left.\left.+4g\left(\frac{u_i+3u_{i+1}}{4}\right)+4g\left(\frac{3u_i+u_{i+1}}{4}\right)+2g\left(\frac{u_i+u_{i+1}}{4}\right)+g\left(\frac{3u_i+13u_{i+1}}{16}\right)+g\left(\frac{3u_i+29u_{i+1}}{32}\right)\right]-\int_{u_i}^{u_{i+1}} g(\widehat{U})d\widehat{U}\right|$$

$$\leq\left[\frac{2057}{24576\pi}\right]^{\frac{1}{2}} h^{\frac{5}{2}}\sum_{i=0}^{n-1}\left((u_i-u_{i+1})\left[\int_{u_i}^{u_{i+1}} g''(\widehat{U})d\widehat{U}\right]^{\frac{1}{2}}\right)$$

$$\leq\left[\frac{2057n}{24576\pi}\right]^{\frac{1}{2}} h^{\frac{5}{2}}\sum_{i=0}^{n-1}(u_i-u_{i+1})\left[\sum_{i=0}^{n-1}\int_{u_i}^{u_{i+1}} g''(\widehat{U})d\widehat{U}\right]^{\frac{1}{2}}\left[\frac{2057}{24576\pi}\right]^{\frac{1}{2}}\frac{(\breve{d}-\dot{c})^{\frac{5}{2}}}{n^{\frac{5}{2}}}(\dot{c}-\breve{d})\|g''\|_2.$$

Conclusion

In this paper, we constructed a generalization of Ostrowski's type inequalities for different norms by using some famous inequalities. Some perturbed results are also discussed. In addition, we gave a new idea of peano kernel i.e. 9-step linear kernel. In the last section, we applied our obtained results to numerical integration.

Conflict of interests

There are no conflicts of interest in this work.

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