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On Directed Length Ratios in the Lorentz-Minkowski Plane

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ABSTRACT

The linear structure of the Lorentz-Minkowski plane is almost the same as Euclidean plane. But, there is one different aspect. These planes have different distance functions. So, it can be interesting to study the Lorentz analogues of topics that include the distance concept in the Euclidean plane. Thus, in this study, we show that the relationship between Euclidean and Lorentz distances is given depending on the slope of the line segment. Following, we investigate Lorentz analogues of Thales' theorem, Angle Bisector theorems, Menelaus' theorem and Ceva's theorem.

Keywords: Lorentz distance, Directed lengths, Angle bisector theorems, Menelaus' theorem, Ceva's theorem.

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Introduction

Lorentz-Minkowski geometry is created by taking the Lorentz distance function instead of the Euclidean distance function. The basic notions, inner product, metric and vector classification in Lorentz space are given in [1, 3, 5, 6].

Lorentz-Minkowski plane (L^2) is the vector space $\mathbb{R}^2$ provided with Lorentz inner product

$$\langle x, y\rangle_L = x_1y_1 - x_2y_2$$

where $x = (x_1, x_2) \in \mathbb{R}^2$, $y = (y_1, y_2) \in \mathbb{R}^2$. The arbitrary vector $x = (x_1, x_2) \in L^2$ is classified according to the sign of $\langle x, x\rangle_L$ as follows:

- **(i)** x is timelike vector if $\langle x, x\rangle_L < 0$,
- **(ii)** x is spacelike vector if $\langle x, x\rangle_L > 0$ and $x = 0$,
- **(iii)** x is lightlike vector if $\langle x, x\rangle_L = 0$ ve $x \neq 0$.

Let $e = (0,1)$. A timelike vector $x = (x_1, x_2)$ is future-pointing (past-pointing) if $\langle x, e\rangle_L < 0$ ($\langle x, e\rangle_L > 0$). The norm $\|.\|$ of any $x = (x_1, x_2) \in L^2$ is defined by $\|x\|_L = \sqrt{|\langle x, x\rangle|}$ [1]. Then the distance function between two points is defined by

$$d_L(x, y) = \|x - y\|_L = \sqrt{|(x_1 - y_1)^2 - (x_2 - y_2)^2|}$$

where $x = (x_1, x_2) \in L^2$, $y = (y_1, y_2) \in L^2$.

The Lorentz-Minkowski plane is almost the same as the Euclidean plane since the points and the lines are the same. The angles are measured in the same way. But, the distance function is different. Since the distance function is different, the properties in the Euclidean plane can be reproduced faithfully in L^2. A few such topics have been studied by some authors [1, 2, 4, 7, 8] in this plane. So, in this study we show that the relationship between Euclidean and Lorentz distances is given depending on the slope of the line segment. Following, we investigate Lorentz analogues of Thales' theorem, Angle Bisector theorems, Menelaus' theorem and Ceva's theorem

Materials and Methods

In this section, we mention the basic concepts that would be the basis of our study.

Proposition 2.1 Let d_E denote the Euclidean distance function and $P = (x_1, x_2)$ and $Q = (y_1, y_2)$ be two points in the analytical plane and the slope of the line $\overleftrightarrow{PQ}$ be m. Then

- **i)** $d_E(P, Q) = \sqrt{\frac{1+m^2}{|1-m^2|}}.d_L(P, Q)$, if $|m| \neq 1$ and $m \in \mathbb{R}$,
- **ii)** $d_E(P, Q) = d_L(P, Q)$, if $m = 0$ or $m \to \infty$.

Proof:

i) Let $P = (x_1, x_2)$ and $Q = (y_1, y_2)$ be two points in the analytical plane and the slope of the line $\overleftrightarrow{PQ}$ be m, ($|m| \neq 1$). We will show that

$$d_E(P, Q) = p(m).d_L(P, Q)$$

where $p(m) = \sqrt{\frac{1+m^2}{|1-m^2|}}$. We can write

$d_E(P,Q) = |x_1 - y_1|\sqrt{1+m^2}$
and

$d_L(P,Q) = |x_1 - y_1|\sqrt{|1-m^2|}$.

From above equations, we obtain that

$$\frac{d_E(P,Q)}{d_L(P,Q)} = \frac{|x_1 - y_1|\sqrt{1+m^2}}{|x_1 - y_1|\sqrt{|1-m^2|}}$$

$$d_E(P,Q) = \sqrt{\frac{1+m^2}{|1-m^2|}}.d_L(P,Q)$$

$$d_E(P,Q) = p(m).d_L(P,Q).$$

ii) If $m = 0$ or $m \to \infty$, it is clear that $d_E(P,Q) = d_L(P,Q)$.

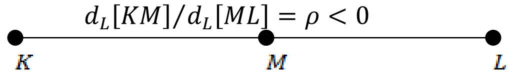
Corollary 2.1 Let P, Q and X be three collinear points in analytical plane. Then, $d_E(P,X) = d_E(Q,X)$ if and only if $d_L(P,X) = d_L(Q,X)$.

Definition 2.1 Let $d_L[AB]$ denote the Lorentz directed distance from A to B along the line l in L^2. We define Lorentz directed distance of the segment $[AB]$ as follows:

$$d_L[AB] = \begin{cases} d_L(A,B), & if \text{ AB and } l \text{ have same direction,} \\ -d_L(A,B), & if \text{ AB and } l \text{ have opposite direction.} \end{cases}$$

If K, L, M are points on the same directed line and M is between points K and L, they are denoted KML. If KML then the point M divides the line segment $[KL]$ internally and becomes $d_L[KM]/d_L[ML] = \rho > 0$. If KLM and MKL then the point M divides the line segment $[KL]$ externally and becomes $d_L[KM]/d_L[ML] = \rho < 0$ [9].

$d_L[KM]/d_L[ML] = \rho < 0$



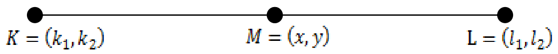
$d_L[KM]/d_L[ML] = \rho > 0$

Theorem 2.1 Let $K = (k_1, k_2)$ and $L = (l_1, l_2)$ be any two different points in the analytical plane. If $M = (x, y)$ is a point on the line passing through K and L, then we can write

$$\frac{d_E(K,M)}{d_E(M,L)} = \frac{d_L(K,M)}{d_L(M,L)}.$$

That is, the ratios of the Euclidean and Lorentz directed lengths are the same [9].

Proof:



If $K = M$ then both ratios are equal to 0. Therefore without loss of generality, let $K \neq M \neq L$. It is enough to show that

$$\frac{|(k_1-x)^2 - (k_2-y)^2|}{|(x-l_1)^2 - (y-l_2)^2|} = \frac{(k_1-x)^2 + (k_2-y)^2}{(x-l_1)^2 + (y-l_2)^2}.$$

Let examine the cases of the line $\overleftrightarrow{KL}$ be spacelike, timelike and lightlike separately.

Case 1: If the line $\overleftrightarrow{KL}$ is spacelike, since $|k_1 - x| > |k_2 - y|$ and $|x - l_1| > |y - l_2|$, we obtain

$$\frac{(k_1-x)^2 - (k_2-y)^2}{(x-l_1)^2 - (y-l_2)^2} = \frac{(k_1-x)^2 + (k_2-y)^2}{(x-l_1)^2 + (y-l_2)^2}$$

$$\Rightarrow (k_1-x)^2(x-l_1)^2 + (k_1-x)^2(y-l_2)^2 - (k_2-y)^2(x-l_1)^2 - (k_2-y)^2(y-l_2)^2$$

$$= (k_1-x)^2(x-l_1)^2 - (k_1-x)^2(y-l_2)^2 + (k_2-y)^2(x-l_1)^2 - (k_2-y)^2(y-l_2)^2$$

$$\Rightarrow (k_1-x)(y-l_2) = (k_2-y)(x-l_1)$$

$$\Rightarrow (k_1-x)(y-l_2) - (k_2-y)(x-l_1) = 0 \quad (1)$$

$$\Rightarrow k_1y - k_1l_2 - xy + xl_2 - k_2x + k_2l_1 + yx - yl_1 = 0$$

$$\Rightarrow (k_1 - l_1)y = (k_2 - l_2)x + k_1l_2 - k_2l_1$$

$$y = \frac{k_2 - l_2}{k_1 - l_1}x + \frac{k_1l_2 - k_2l_1}{k_1 - l_1}. \quad (2)$$

Thus, from equation (2), the point M is on the line $\overleftrightarrow{KL}$. Using this value of y in the equation (1) we get

$$(k_1-x)\left(\frac{k_2x-l_2x+k_1l_2-k_2l_1}{k_1-l_1}-l_2\right)-\left(k_2-\frac{k_2x-l_2x+k_1l_2-k_2l_1}{k_1-l_1}\right)(x-l_1)=0$$
$$(k_1-x)\left[\frac{k_2x-l_2x+k_1l_2-k_2l_1-l_2k_1+l_2l_1}{k_1-l_1}\right]-\left[\frac{k_2k_1-k_2l_1-k_2x+l_2x-k_1l_2+k_2l_1}{k_1-l_1}\right](x-l_1)=0$$
$$(k_1-x)\left[\frac{x(k_2-l_2)-l_1(k_2-l_2)}{k_1-l_1}\right]-(x-l_1)\left[\frac{k_1(k_2-l_2)-x(k_2-l_2)}{k_1-l_1}\right]=0$$
$$\frac{1}{k_1-l_1}[(k_1-x)(x-l_1)(k_2-l_2)-(x-l_1)(k_1-x)(k_2-l_2)]=0.$$

Thus, the equation (1) is satisfied.

Case 2: If the line $\overleftrightarrow{KL}$ is timelike, since $|k_1-x|<|k_2-y|$ and $|x-l_1|<|y-l_2|$, we obtain

$$\frac{(k_2-y)^2-(k_1-x)^2}{(y-l_2)^2-(x-l_1)^2}=\frac{(k_1-x)^2+(k_2-y)^2}{(x-l_1)^2+(y-l_2)^2}$$
$$\Rightarrow (k_2-y)^2(x-l_1)^2+(k_2-y)^2(y-l_2)^2-(k_1-x)^2(x-l_1)^2-(k_1-x)^2(y-l_2)^2$$
$$=(k_1-x)^2(y-l_2)^2-(k_1-x)^2(x-l_1)^2+(k_2-y)^2(y-l_2)^2-(k_2-y)^2(x-l_1)^2$$
$$\Rightarrow (k_2-y)(x-l_1)=(k_1-x)(y-l_2)$$
$$\Rightarrow (k_2-y)(x-l_1)-(k_1-x)(y-l_2)=0 \qquad (3)$$
$$\Rightarrow k_2x-k_2l_1-yx+yl_1-k_1y+k_1l_2+xy-xl_2=0$$
$$\Rightarrow y(l_1-k_1)=x(l_2-k_2)+k_2l_1-k_1l_2$$
$$\Rightarrow y=\frac{l_2-k_2}{l_1-k_1}x+\frac{k_2l_1-k_1l_2}{l_1-k_1} \qquad (4)$$

Thus, from equation (4), the point M is on the line $\overleftrightarrow{KL}$. Using this value of y in the equation (3) we get

$$(x-l_1)\left(k_2-\frac{l_2x-k_2x+k_2l_1-k_1l_2}{l_1-k_1}\right)-(k_1-x)\left(\frac{l_2x-k_2x+k_2l_1-k_1l_2}{l_1-k_1}-l_2\right)=0$$
$$(x-l_1)\left[\frac{k_2l_1-k_2k_1-l_2x+k_2x-k_2l_1+k_1l_2}{l_1-k_1}\right]-(k_1-x)\left[\frac{l_2x-k_2x+k_2l_1-k_1l_2-l_1l_2+k_1l_2}{l_1-k_1}\right]=0$$
$$(x-l_1)\left[\frac{k_1(l_2-k_2)-x(l_2-k_2)}{l_1-k_1}\right]-(k_1-x)\left[\frac{x(l_2-k_2)-l_1(l_2-k_2)}{l_1-k_1}\right]=0$$
$$\frac{1}{l_1-k_1}[(x-l_1)(k_1-x)(l_2-k_2)-(k_1-x)(x-l_1)(l_2-k_2)]=0$$

Thus, the equation (3) is satisfied.
Case 3: If the line $\overleftrightarrow{KL}$ is lightlike, then $|k_1-x|=|k_2-y|$ and $|x-l_1|=|y-l_2|$. Thus, it is obvious.

If the point M divides the line segment $[KL]$ externally, the proof is similar.

Conclusion and Discussion

In this section, we give Lorentz versions of some Euclidean theorems.
Theorem 3.1 (Thales' Theorem) In the Lorentz-Minkowski plane, if we have three or more parallel lines, and they cut the other two lines, then they produce proportional segments.
Proof:

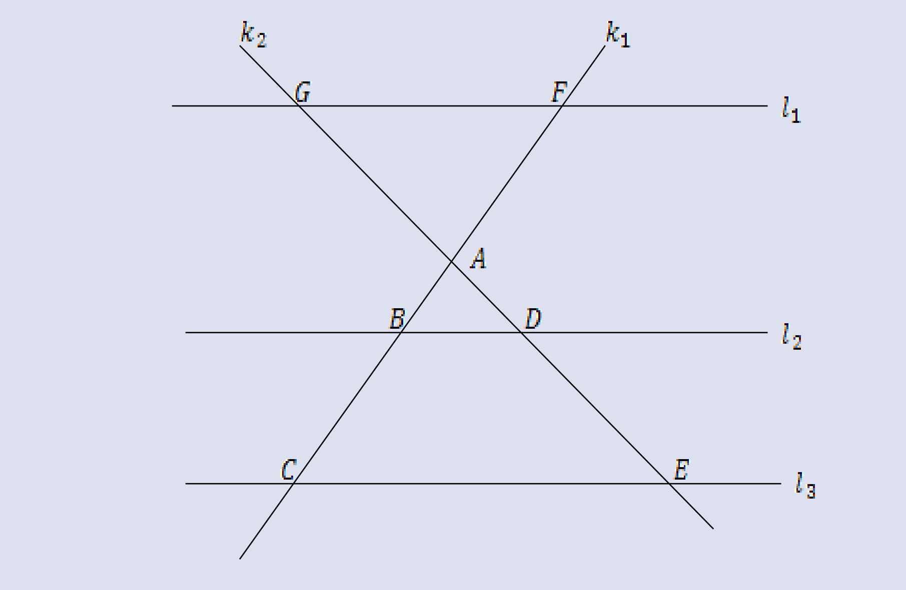


Figure 1. The parallel lines and two other lines intersecting these lines.

Let the lines l_1, l_2, l_3 are parallel and the lines k_1, k_2 cut of them like in the above figure. Since the lines l_1, l_2, l_3 are parallel, the slopes of this lines are same. Thus, let the slopes of the lines l_1, l_2, l_3 be m_1 and the slope of the lines k_1, k_2 be m_2, m_3, respectively. Then, from Proposition 2.1, we can write that

$$\frac{d_E(A,B)}{d_E(A,C)} = \frac{\sqrt{\frac{1+{m_2}^2}{|1-{m_2}^2|}}.d_L(A,B)}{\sqrt{\frac{1+{m_2}^2}{|1-{m_2}^2|}}.d_L(A,C)} = \frac{d_L(A,B)}{d_L(A,C)},$$

$$\frac{d_E(A,D)}{d_E(A,E)} = \frac{\sqrt{\frac{1+{m_3}^2}{|1-{m_3}^2|}}.d_L(A,D)}{\sqrt{\frac{1+{m_3}^2}{|1-{m_3}^2|}}.d_L(A,E)} = \frac{d_L(A,D)}{d_L(A,E)},$$

$$\frac{d_E(B,D)}{d_E(C,E)} = \frac{\sqrt{\frac{1+{m_1}^2}{|1-{m_1}^2|}}.d_L(B,D)}{\sqrt{\frac{1+{m_1}^2}{|1-{m_1}^2|}}.d_L(C,E)} = \frac{d_L(B,D)}{d_L(C,E)}.$$

Since

$$\frac{d_E(A,B)}{d_E(A,C)} = \frac{d_E(A,D)}{d_E(A,E)} = \frac{d_E(B,D)}{d_E(C,E)},$$

we can obtain as follows

$$\frac{d_L(A,B)}{d_L(A,C)} = \frac{d_L(A,D)}{d_L(A,E)} = \frac{d_L(B,D)}{d_L(C,E)}.$$

Theorem 3.2 (Interior Angle Bisector Theorem) Let interior angle bisector of vertex A of the triangle ΔABC intersects side $[BC]$ at point D, $a_L = d_L(B,C)$, $b_L = d_L(C,A)$, $c_L = d_L(A,B)$, $p_L = d_L(B,D)$, $q_L = d_L(D,C)$ and slopes of sides $[AB]$, $[BC]$, $[AC]$ be m_c, m_a, m_b, respectively, in the Lorentz-Minkowski plane. Then we can write as follows:

i) $\frac{p_L}{q_L} = \frac{c_L}{b_L}.\left[\frac{1+m_c{}^2}{|1-m_c{}^2|}.\frac{|1-m_b{}^2|}{1+m_b{}^2}\right]$,

ii) $\frac{p_L}{q_L} = \frac{c_L}{b_L}.\left[\frac{1+m_c{}^2}{|1-m_c{}^2|}\right]$, if $m_b = 0$ or $m_b \to \infty$,

iii) $\frac{p_L}{q_L} = \frac{c_L}{b_L}.\left[\frac{|1-m_b{}^2|}{1+m_b{}^2}\right]$, if $m_c = 0$ or $m_c \to \infty$,

where m_c, m_a, $m_b \neq \pm 1$. Here, sides of the triangle ΔABC must be same kinds. That is, three sides of the triangle ΔABC are either spacelike lines or timelike lines.

Proof.

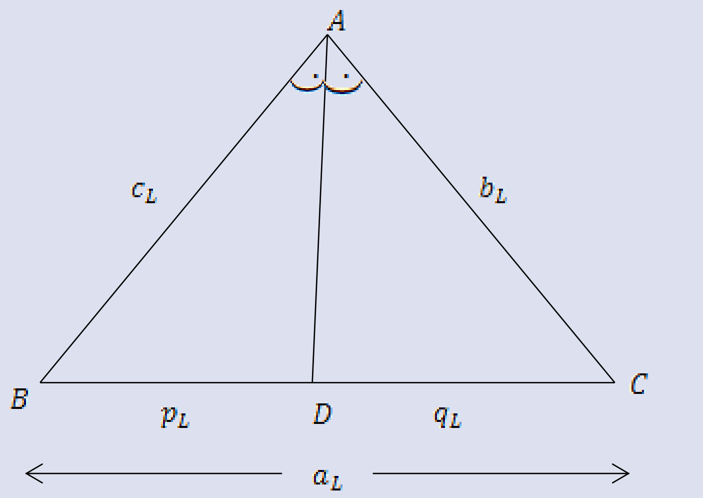


Figure 2. AD is the internal bisector of the angle ∠BAC which meets BC at point D, for the triangle ΔABC

i) Let slopes of sides $[AB]$, $[BC]$, $[AC]$ be m_c, m_a, m_b, respectively. It is clear that,

$$\frac{d_E(A,B)}{d_E(A,C)} = \frac{d_E(B,D)}{d_E(D,C)} \tag{5}$$

is satisfied in the Euclidean plane. According to Proposition 2.1, we obtain that,

$$d_E(A,B) = \sqrt{\frac{1+m_c{}^2}{|1-m_c{}^2|}}.c_L, \qquad d_E(A,C) = \sqrt{\frac{1+m_b{}^2}{|1-m_b{}^2|}}.b_L,$$

$$d_E(B,D) = \sqrt{\frac{1+m_a{}^2}{|1-m_a{}^2|}}.p_L, \qquad d_E(D,C) = \sqrt{\frac{1+m_a{}^2}{|1-m_a{}^2|}}.q_L.$$

If the above values are substituted in the equation (5), we get

$$\frac{\sqrt{\frac{1+m_c{}^2}{|1-m_c{}^2|}}.c_L}{\sqrt{\frac{1+m_b{}^2}{|1-m_b{}^2|}}.b_L,} = \frac{\sqrt{\frac{1+m_a{}^2}{|1-m_a{}^2|}}.p_L}{\sqrt{\frac{1+m_a{}^2}{|1-m_a{}^2|}}.q_L},$$

$$\frac{p_L}{q_L} = \frac{c_L}{b_L}.\left[\frac{1+m_c{}^2}{|1-m_c{}^2|}.\frac{|1-m_b{}^2|}{1+m_b{}^2}\right]. \tag{6}$$

ii) It is clear that, for $m_b = 0$ or $m_b \to \infty$, the equation (6) is obtained that

$\frac{p_L}{q_L} = \frac{c_L}{b_L} \cdot \left[\frac{1+{m_c}^2}{|1-{m_c}^2|}\right].$

iii) It is clear that, for $m_c = 0$ or $m_c \to \infty$, the equation (6) is obtained that

$\frac{p_L}{q_L} = \frac{c_L}{b_L} \cdot \left[\frac{|1-{m_b}^2|}{1+{m_b}^2}\right].$

Theorem 3.3 (Exterior Angle Bisector Theorem) Let exterior angle bisector of vertex A of the triangle ΔABC intersects side $\overleftrightarrow{BC}$ at point D, $a_L = d_L(B,C)$, $b_L = d_L(C,A)$, $c_L = d_L(A,B)$, $p_L = d_L(B,D)$, $q_L = d_L(D,C)$ and slopes of sides $[AB]$, $[BC]$, $[AC]$ be m_c, m_a, m_b, respectively, in the Lorentz-Minkowski plane. Then we can write as follows:

i) $\frac{p_L}{q_L} = \frac{c_L}{b_L} \cdot \left[\frac{1+{m_c}^2}{|1-{m_c}^2|} \cdot \frac{|1-{m_b}^2|}{1+{m_b}^2}\right],$

ii) $\frac{p_L}{q_L} = \frac{c_L}{b_L} \cdot \left[\frac{1+{m_c}^2}{|1-{m_c}^2|}\right]$, if $m_b = 0$ or $m_b \to \infty$,

iii) $\frac{p_L}{q_L} = \frac{c_L}{b_L} \cdot \left[\frac{|1-{m_b}^2|}{1+{m_b}^2}\right]$, if $m_c = 0$ or $m_c \to \infty$,

where $m_c, m_a, m_b \neq \pm 1$. Here, sides of the triangle ΔABC must be same kinds. That is, three sides of the triangle ΔABC are either spacelike lines or timelike lines.

Proof:

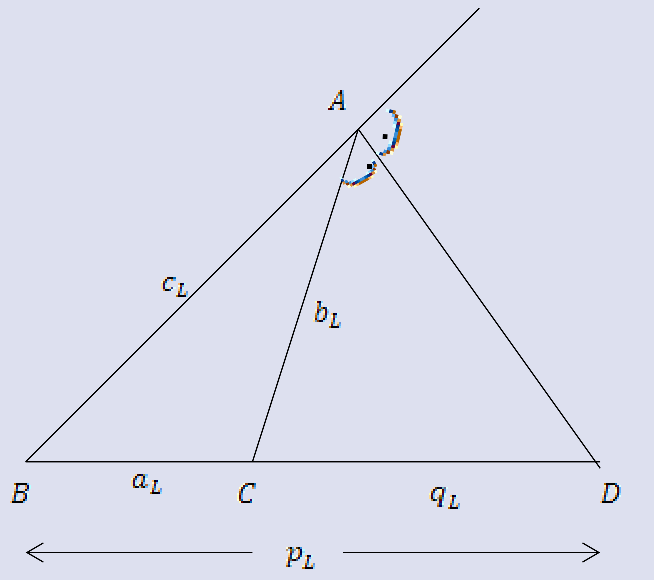


Figure 3. AD is the external bisector of the angle ∠BAC which meets BC at point D, for the triangle ΔABC

i) Let slopes of the sides $[AB]$, $[BC]$, $[AC]$ be m_c, m_a, m_b, respectively. It is clear that,

$$\frac{d_E(A,B)}{d_E(A,C)} = \frac{d_E(B,D)}{d_E(C,D)} \tag{7}$$

is satisfied in the Euclidean plane. According to Proposition 2.1, we obtain that,

$$d_E(A,B) = \sqrt{\frac{1+{m_c}^2}{|1-{m_c}^2|}} \cdot c_L, \qquad d_E(A,C) = \sqrt{\frac{1+{m_b}^2}{|1-{m_b}^2|}} \cdot b_L,$$

$$d_E(B,D) = \sqrt{\frac{1+{m_a}^2}{|1-{m_a}^2|}}.p_L, \qquad d_E(C,D) = \sqrt{\frac{1+{m_a}^2}{|1-{m_a}^2|}}.q_L.$$

If the above values are substituted in the equation (7), we get

$$\frac{\sqrt{\frac{1+{m_c}^2}{|1-{m_c}^2|}}.c_L}{\sqrt{\frac{1+{m_b}^2}{|1-{m_b}^2|}}.b_L,} = \frac{\sqrt{\frac{1+{m_a}^2}{|1-{m_a}^2|}}.p_L}{\sqrt{\frac{1+{m_a}^2}{|1-{m_a}^2|}}.q_L},$$

$$\frac{p_L}{q_L} = \frac{c_L}{b_L}.\left[\frac{1+{m_c}^2}{|1-{m_c}^2|}.\frac{|1-{m_b}^2|}{1+{m_b}^2}\right]. \tag{8}$$

ii) It is clear that, for $m_b = 0$ or $m_b \to \infty$, the equation (8) is obtained that

$$\frac{p_L}{q_L} = \frac{c_L}{b_L}.\left[\frac{1+{m_c}^2}{|1-{m_c}^2|}\right].$$

iii) It is clear that, for $m_c = 0$ or $m_c \to \infty$, the equation (8) is obtained that

$$\frac{p_L}{q_L} = \frac{c_L}{b_L}.\left[\frac{|1-{m_b}^2|}{1+{m_b}^2}\right].$$

Theorem 3.4 (Menelaus' Theorem) Let ΔABC be a triangle and P_1, P_2, P_3 be on the lines that contain the sides BC, CA, AB, respectively, in the Lorentz-Minkowski plane. If P_1, P_2, P_3 are collinear, then

$$\frac{d_L[BP_1]}{d_L[P_1C]}.\frac{d_L[CP_2]}{d_L[P_2A]}.\frac{d_L[AP_3]}{d_L[P_3B]} = -1$$

where none of P_1, P_2, P_3 coincide with any of A, B, C. Here, the points P_1, P_2, P_3, A, B, C must be same kinds. That is, all of them are either spacelike or timelike.

Proof:

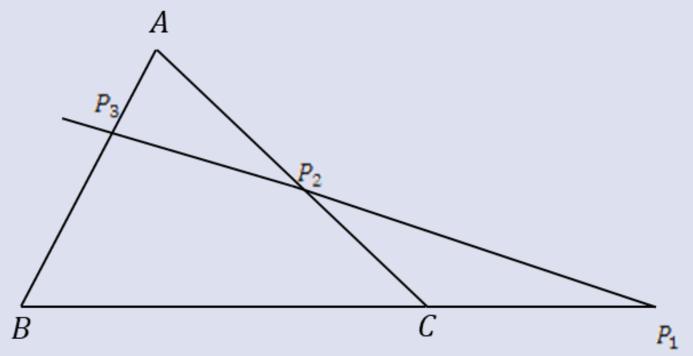


Figure 4. Collinear points P_1, P_2, P_3 on the sides BC, CA, AB, respectively.

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Let slopes of the lines $\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$, $\overleftrightarrow{CA}$ be m_1, m_2, m_3, respectively and $m_1, m_2, m_3 \neq \pm 1$. It is clear that

$$\frac{d_E[BP_1]}{d_E[P_1C]}.\frac{d_E[CP_2]}{d_E[P_2A]}.\frac{d_E[AP_3]}{d_E[P_3B]} = -1 \tag{9}$$

is satisfied in the Euclidean plane. According to Proposition 2.1, we obtain that

$$d_E(B,P_1) = \sqrt{\frac{1+m_2{}^2}{|1-m_2{}^2|}}.d_L(B,P_1), \qquad d_E(P_1,C) = \sqrt{\frac{1+m_2{}^2}{|1-m_2{}^2|}}.d_L(P_1,C)$$

$$d_E(C,P_2) = \sqrt{\frac{1+m_3{}^2}{|1-m_3{}^2|}}.d_L(C,P_2), \qquad d_E(P_2,A) = \sqrt{\frac{1+m_3{}^2}{|1-m_3{}^2|}}.d_L(P_2,A)$$

$$d_E(A,P_3) = \sqrt{\frac{1+m_1{}^2}{|1-m_1{}^2|}}.d_L(A,P_3), \qquad d_E(P_3,B) = \sqrt{\frac{1+m_1{}^2}{|1-m_1{}^2|}}.d_L(P_3,B).$$

From Definition 2.1, $|BP_1|$ and $|P_1C|$ have opposite direction. If the above values are substituted in the equation (9), we get

$$\frac{d_E[BP_1]}{d_E[P_1C]}.\frac{d_E[CP_2]}{d_E[P_2A]}.\frac{d_E[AP_3]}{d_E[P_3B]} = -\frac{\sqrt{\frac{1+m_2{}^2}{|1-m_2{}^2|}}.d_L[BP_1]}{\sqrt{\frac{1+m_2{}^2}{|1-m_2{}^2|}}.d_L[P_1C]}.\frac{\sqrt{\frac{1+m_3{}^2}{|1-m_3{}^2|}}.d_L[CP_2]}{\sqrt{\frac{1+m_3{}^2}{|1-m_3{}^2|}}.d_L[P_2A]}.\frac{\sqrt{\frac{1+m_1{}^2}{|1-m_1{}^2|}}.d_L[AP_3]}{\sqrt{\frac{1+m_1{}^2}{|1-m_1{}^2|}}.d_L[P_3B]} = -1.$$

Theorem 3.5 (Ceva's Theorem) Let ΔABC be a triangle and lines l_1, l_2, l_3 pass through the vertices A, B, C, respectively and intersect lines containing the opposite sides at points P_1, P_2, P_3, in the Lorentz-Minkowski plane. Then the lines l_1, l_2, l_3 are concurrent if and only if

$$\frac{d_L[BP_1]}{d_L[P_1C]}.\frac{d_L[CP_2]}{d_L[P_2A]}.\frac{d_L[AP_3]}{d_L[P_3B]} = 1\,.$$

Here, none of P_1, P_2, P_3 are of A, B, C. The points P_1, P_2, P_3, A, B, C must be same kinds. That is, all of them are either spacelike or timelike.

Conflict of interests

The authors state that did not have conflict of interests.

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