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CHARACTERIZATIONS OF CONTACT PSEUDO-SLANT SUBMANIFOLDS OF A PARA-KENMOTSU MANIFOLD

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Abstract

In this paper, the geometry of contact pseudo-slant submanifolds of a para Kenmotsu manifold have been studied. The necessary and sufficient conditions for a submanifold to be a contact pseudo- slant submanifolds of a para Kenmotsu manifold are given.

Key Words: Para-Kenmotsu manifold. Contact pseudo-slant submanifold, totally umbilic.

Özet

Bu makalede, bir para Kenmotsu manifoldunun kontak pseudo slant alt manifoldlarının geometrisi çalışılmıştır. Bir para-Kenmotsu manifoldunun kontak pseudo slant altmanifoldu için gerek ve yeterli koşullar verilmiştir

Anahtar Kelimeler: Para-Kenmotsu manifold, kontak pseudo slant altmanifold, total umbilik.

1. Introduction

Slant submanifolds are known to generalize invariant and anti-invariant submanifolds, many geometrs have expressed an interest in this research. Chen [8], [9] started this research on complex manifolds. Lotta[17] pioneered slant immersions in an almost contact metric manifold. Carriazo defined a new class of submanifolds known as hemi-slant submanifolds (Also known as anti-slant or pseudo-slant submanifolds) [6] . The contact version of a pseudo-slant submanifold in a Sasakian manifold was then defined and studied by V. A. Khan and M. A. Khan. [13]. Later many geometers like ([10], [11], [12], [14], [16]) studied pseudo-slant submanifolds on various manifolds. Recently, M. Atçeken and S. Dirik studied contact pseudo-slant submanifold on various manifolds ([1],[11],[12]).

In the light of the above studies, our article, the following is how this paper is structured: Section 2 includes some fundamental formulas and definitons of the para-Kenmotsu manifold and it is submanifolds. Section 3 we review some definitions and proves some basic results on the contact pseudo-slant submanifolds of the para-Kenmotsu manifold. The final section looks at the totally umbilical contact pseudo-slant in para Kenmotsu manifolds.

Let $\widetilde{M}$ be a (2n+1)-dimensional smoot manifold φ a tensor field of type (1,1), ξ a vector field and η a 1-form. We say that (φ,ξ,η) is an almost paracontact structure on $\widetilde{M}$ if

$$\varphi^2 X = -X + \eta(X)\xi, \quad (1)$$

$$\eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta(\varphi X) = 0 \quad (2)$$

for all vector field X on $\widetilde{M}$.

Para Kenmotsu manifold, contact pseudo-slant submanifolds, totally umbilical, mixed geodesic.

If an almost paracontact Manifold admits a pseudo-Riemannian metric g of signature $(n+1, n)$ satisfying

$$g(\varphi X, \varphi Y) = -g(X,Y) + \eta(X)\eta(Y) \quad (3)$$

$$g(X,\xi) = \eta(X) \quad (4)$$

for all vector field X, Y on $\widetilde{M}$. Then $\widetilde{M}$ equipped with an almost paracontact metric structure (φ, ξ, η, g) is referred to as an almost paracontact metric manifold.

An almost paracontact metric manifold $\widetilde{M}(\varphi, \xi, \eta, g)$ is a para-Kenmotsu manifold if the Levi Civita connection $\tilde{\nabla}$ of g satisfies,

$$(\tilde{\nabla}_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X, \quad (5)$$

for all vector field X , Y on $\widetilde{M}$on $\widetilde{M}$. From (5), taking instead of ξ

$$\tilde{\nabla}_X \xi = X - \eta(X)\xi. \tag{6}$$

Asume $\widetilde{M}$ is a submanifold of a para-Kenmotsu manifold $\widetilde{M}$.

Let M be a submanifold of a paracontact metric manifold $\widetilde{M}$ whit the same symbol g for the induced metric. Then the Gauss and Weingarteen formulae are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y) \tag{7}$$

$$\tilde{\nabla}_X V = -A_V X + \nabla^{\perp}{}_X V \tag{8}$$

where ∇ and $\nabla^{\perp}$ are induced connections on the tangent bundle TM and $T^{\perp}M$ of M respectively, With respect to σ and A_V are the second fundamental form and shape operator, respectively. Then the shape operator A_V and the second fundamental form σ are interconnected by

$$g(A_V X, Y) = g(\sigma(X,Y), V) \tag{9}$$

for all $X, Y \in \Gamma(TM)$, $V \in \Gamma(T^{\perp}M)$.

The mean curvature vector H of M is given by

$$H = \frac{tr(\sigma)}{r} = \frac{1}{r}\sum_{i=1}^{r} \sigma(e_i, e_i) = 0 \tag{10}$$

where r is the dimension of M and $\{e_1, e_2, , , e_r\}$ is the local orthonormal frame of M.

- If a submanifold M of a Riemannian manifold $\widetilde{M}$ is a said to be totally umbilical,

$$\sigma(X,Y) = g(X,Y)H. \tag{11}$$

- If $\sigma(X,Y)=0$ a submanifold is said to be totally geodesic, where for all $X, Y \in \Gamma(TM)$.
- If $H = 0$, a submanifold is said to be minimal.

Contact Pseudo-Slant Submanifolds of a Para-Kenmotsu Manifold

Let M be a submanifold of a para Kenmotsu manifold $\widetilde{M}$. Then for any $X \in \Gamma(TM)$ we can write

$$\varphi X = EX + FX \tag{12}$$

where EX is the tangent component and FX is the normal component of φX .
Also, for any $V \in (T^{\perp}M)$, φV can be written in the following way;

$$\varphi V = BV + CV, \tag{13}$$

where BV and CV are also the tangent and normal component of φV, respectively.
We can deduce from (12) and (13) that the tensor field, E, F, B and C are also anti-symmetric because φ is anti-symmetric. By using (1), (12) and (13), we obtein

$$E^2 + BF = I - \eta o \xi, \quad FE + CF = 0 \tag{14}$$

$$C^2 = I - FB, \quad EB + BC = 0 \ . \tag{15}$$

Furthermore, (4), (12) and (13) show that E and C are skew symmetric tensor fields.

$$g(EX,Y) = -g(X,EY) \tag{16}$$

$$g(CX,Y) = -g(X,CY) \ . \tag{17}$$

Also, we can optein relation between F and B as

$$g(FX,V) = -g(X,BV) \tag{18}$$

for all $X, Y \in \Gamma(TM)$, $V \in \Gamma(T^{\perp}M)$.
The covariant derivatives of the tensor fields E, F, B, and C are defined.

$$(\nabla_X E)Y = \nabla_X EY - E\nabla_X Y \tag{19}$$

$$(\nabla_X F)Y = \nabla^{\perp}{}_X FY - F\nabla_X Y \tag{20}$$

$$(\nabla_X B)V = \nabla_X BV - B\nabla^{\perp}{}_X V \tag{21}$$

and

$$(\nabla_X C)V = \nabla^{\perp}{}_X CV - C\nabla^{\perp}{}_X V \ . \tag{22}$$

The covariant derivative of φ, $\tilde{\nabla}\varphi$ can be defined by

$$(\tilde{\nabla}_X \varphi)Y = \tilde{\nabla}_X \varphi Y - \varphi\tilde{\nabla}_X Y \tag{23}$$

for all $X, Y \in \Gamma(TM)$, $V \in \Gamma(T^{\perp}M)$. White $\tilde{\nabla}$ is the Riemannian connection on $\tilde{M}$.

Lemma: Let M be a submanifold of a para-Kenmotsu manifold $\tilde{M}$. Then

$$(\nabla_X E)Y = B\sigma(X,Y) + A_{FY}X + g(EX,Y)\xi - \eta(Y)EX \ , \tag{24}$$

$$(\nabla_X F)Y = C\sigma(X,Y) - \sigma(X,EY) - \eta(Y)FX, \tag{25}$$

$$(\nabla_X B)V = A_{CV}X - EA_V X \tag{26}$$

$$(\nabla_X C)V = -\sigma(X, BV) - FA_V X \tag{27}$$

for all $X, Y \in \Gamma(TM)$, $V \in \Gamma(T^{\perp}M)$.

Using (5), (6), and (7), we have that ξ is tangent to M.

$$\nabla_X \xi = X - \eta(X)\xi, \tag{28}$$

$$\sigma(X, \xi) = 0 \tag{29}$$

for all $X \in \Gamma(TM)$.

Let us now same definetions of classes submanifolds.

- If F is identically zero in (12), then the submanifold is invariant.
- If E is identically zero in (12), then the submanifold is anti-invariant,
- If there is a constant angle $\theta(x) \in \left[0, \frac{\pi}{2}\right]$ between φX and TM for all nonzero vector X tangent to M at x, the manifold is called slant.
- If there are distribution $D_\theta, D^{\perp}$ is there a contact pseudo-slant submanifold. Such that
  1) orthogonal direct composition is allowed in

     $TM = D^{\perp} \oplus D_\theta, \xi \in D_\theta$
  2) D_θ is slant with slant angle , $\theta = \frac{\pi}{2}$
  3) $D^{\perp}$ is an anti-invariant [13].

  From the definitions, we can see that a slant submanifold is a generalization of invariant (if $\theta = 0$)and anti-invariant $\left(\text{if } \theta = \frac{\pi}{2}\right)$ submanifolds.

A proper slant submanifold is one that is not invariant or anti-invariant. i. e. As a result, the following theorem characterized slant submanifolds of almost contact metric manifolds;

Theorem 1: *[5].* Let M be a slant submanifolds of an almost contact metric manifold $\widetilde{M}$ such that $\xi \in \Gamma(TM)$, then, M is a slant if and only if a constant $\lambda \in [0, 1]$ exists such that

$$E^2 = -\lambda(\mathrm{I} - \eta \otimes \xi) \tag{30}$$

furthermore, in this situation, if θ is the slant angle of M. Then it satisfies $\lambda = \cos^2\theta$.

Corollary*: [5].* Let M be a slant submanifolds of an almost contact metric manifold $\widetilde{M}$. Then for all $X, Y \in \Gamma(TM)$ we have

$$g(EX,EY)=-\cos^2\theta\{g(X,Y)-\eta(X)\eta(Y)\} \quad (31)$$

$$g(FX,FY)=-\sin^2\theta\{g(X,Y)-\eta(X)\eta(Y)\}. \quad (32)$$

If the orthogonal complementary of φTM in $T^{\perp}M$ is denoted by V, then the normal bundle $T^{\perp}M$ can be decomposed as follows.

$$T^{\perp}M = FD_{\theta}\oplus FD^{\perp}\oplus \nu, \qquad FD_{\theta}\perp FD^{\perp}$$

In the following sections, contact pseudo-slant submanifold was called (CPSS) for short.

Definition 1 A (CPSS) M of a para-Kenmotsu manifold $\widetilde{M}$ is said to be mixed-geodesic submanifold if $\sigma(X,Y)=0$ for all $X\in\Gamma(D_{\theta}),\ \ Y\in\Gamma(D^{\perp})$.

Teoreme 2. Let M be proper (CPSS) of a para-Kenmotsu manifold $\widetilde{M}$. M is either an anti-invariant or a mixed geodesic if B is parallel.

Proof: For all $X\in\Gamma(D_{\theta}),\ \ Y\in\Gamma(D^{\perp})$, from (25) and (26)

B parallel if and only if F parallel, so $\nabla F=0$.

This implies

$$C\sigma(X,Y)+\sigma(X,EY)-\eta(Y)FX=0.$$

Replacing X by EX in the above equation, we get

$$C\sigma(EX,Y)+\sigma(EX,EY)=0$$

for $Y\in\Gamma(D^{\perp})$, $EY=0$. Hence

$$C\sigma(EX,Y)=0.$$

Replacing X by EX in the above equation, we have

$$C\sigma(E^2X,Y)=C\cos^2\theta\sigma(X,Y)=0.$$

Hence we have either $\sigma(X,Y)=0$ (M is mixed geodesic) or $\theta=\frac{\pi}{2}$ (M is anti-invariant).

Theorem 3. Let M be a (CPSS) of a para-Kenmotsu manifold $\widetilde{M}$. Then $D^{\perp}$ is integrable at all times.

Proof: For all $Z,U\in\Gamma(D^{\perp})$, from (5), we have

$$(\widetilde{\nabla}_Z\varphi)U = g(\varphi Z,U)\xi-\eta(U)\varphi Z=0.$$

By using (7), (8) , (12) and (13) we have

$$-A_{FU}Z + \nabla_Z^{\perp}U - E\nabla_Z U - F\nabla_Z U - B\sigma(Z,U) - C\sigma(Z,U) = 0.$$

Comparing the tangent companents, we have

$$-A_{FU}Z - E\nabla_Z U - B\sigma(Z,U) = 0 \tag{33}$$

interchangin Z and U , we get

$$-A_{FZ}U - E\nabla_U Z - B\sigma(U,Z) = 0\ . \tag{34}$$

Subtracting equation (33) from (34) and using the fact that σ is symmetric , we get

$$A_{FU}Z - A_{FZ}U + E[\nabla_Z U - \nabla_U Z] = 0,$$

$$A_{FU}Z - A_{FZ}U + E[Z,U] = 0,$$

$$E[U,Z] = A_{FU}Z - A_{FZ}U. \tag{35}$$

On the other hand, for all $W \in \Gamma(TM)$. By using (5), (7) (8) and (9), we have

$$\begin{aligned} g(A_{FU}Z - A_{FZ}U, W) &= g(\sigma(Z,W), FU) - g(\sigma(U,W), FZ) \\ &= g(\sigma(Z,W), FU) - g(\widetilde{\nabla}_W U, FZ) \\ &= g(\sigma(Z,W), FU) + g(\varphi\widetilde{\nabla}_W U, Z) \\ &= g(\sigma(Z,W), FU) + g(\widetilde{\nabla}_W \varphi U - (\widetilde{\nabla}_W \varphi)^U, Z) \\ &= g(\sigma(Z,W), FU) + g(-A_{FU}W + \nabla_W^{\perp}FU, Z) \\ &= g(\sigma(Z,W), FU) - g(A_{FU}W, Z) \\ &= g(\sigma(Z,W), FU) + g(\sigma(Z,W), FU) = 0 \end{aligned}$$

here

$$A_{FU}z = A_{FZ}U.$$

So, from (35), [U,Z]$\in \Gamma(D^{\perp})$, for all $Z, U \in \Gamma(D^{\perp})$. That is, $D^{\perp}$ is every time integrable.

Theorem 4. Let M be a (CPSS) of a para-Kenmotsu manifold $\widetilde{M}$. Then the D_θ is integrable if and only if

$$\varpi_1\{\nabla_X EY - A_{FY}X - E\nabla_Y X - B\sigma(X,Y) - g(EX,Y)\xi - \eta(Y)EX\} = 0$$

for all X, Y $\in \Gamma(D_\theta)$.

Proof: Let ϖ_1 and ϖ_2 the projections on $D^{\perp}$ and D_θ , respectively. For all X, Y $\in \Gamma(D_\theta)$ from (5), we have

$$(\widetilde{\nabla}_X\varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X = 0.$$

On applying (7), (8), (12) and (13), we get

$$\nabla_X EY + \sigma(X, EY) - A_{FY}X + \nabla_X^{\perp}FY - E\nabla_X Y - F\nabla_X Y - B\sigma(X,Y)$$

$$-C\sigma(X,Y) - g(\varphi X, Y)\xi - \eta(Y)\varphi X = 0.$$

Comparig the tangential components

$$\nabla_X EY - A_{FY}X - E\nabla_X Y - B\sigma(X,Y) - g(\varphi X, Y)\xi - \eta(Y)EX = 0,$$

$$\nabla_X EY - A_{FY}X - E\nabla_Y X + E\nabla_Y X - E\nabla_X Y - B\sigma(X,Y) - g(\varphi X, Y)\xi - \eta(Y)EX = 0,$$

$$E[X,Y] = \nabla_X EY - A_{FY}X - E\nabla_Y X - B\sigma(X,Y) - g(\varphi X, Y)\xi - \eta(Y)EX \quad (36)$$

$X, Y \in \Gamma(D_\theta)$, $[X, Y] \in \Gamma(D_\theta)$, so $\varpi_1[X, Y]=0$.

As a result, we conclude our theorem by applying ϖ_1 to both sides of (36) equation.

Theorem 5. Let M be totally umbilical proper (CPSS) of a para-Kenmotsu manifold $\widetilde{M}$. If B is parallel, then M is either minimal or anti-invariant submanifold.

Proof: For all $X \in \Gamma(D_\theta)$, $Y \in \Gamma(D^\perp)$, from (25) and (26), we have

B parallel if and only if F parallel, so $\nabla F = 0$.

This implies

$$C\sigma(X,Y) + \sigma(X,EY) - \eta(Y)FX = 0.$$

Replacing X by EX in the above equation, we get

$$C\sigma(EX,Y) + \sigma(EX,EY) = 0$$

for $Y \in \Gamma(D^\perp)$, $EY = 0$. Hence

$$C\sigma(EX,Y) = 0.$$

Since M is totally umbilical, from (11)

$$Cg(EX,Y)H = 0$$

replacing X by EX in the above equation, we have

$$Cg(E^2X,Y)H = -Cg(EX,EY)H = C\cos^2\theta g(X,Y)H = 0.$$

Hence we have either $\theta = \frac{\pi}{2}$ (M is anti-invariant) or $H = 0$ (M is minimal).

Theorem 6. Let M be a totally umbilical (CPSS) of a para-Kenmotsu manifold $\widetilde{M}$. Then at least one of the following satements is true.

1- M is proper (CPSS).
2- $H \in \Gamma(\nu)$.
3- Dim $(D^\perp) = 1$.

Proof: Let $X \in \Gamma(D^{\perp})$ and using (5), we obtain

$$(\widetilde{\nabla}_X \varphi)X = g(\varphi X, X)\xi - \eta(X)\varphi X = 0.$$

On applying (7), (8), (12) and (13), we get

$$-A_{FX}X + \nabla_X^{\perp}FX - F\nabla_X X - B\sigma(X,X) - C\sigma(X,X) = 0.$$

Comparig the tangential components

$$-A_{FX}X + B\sigma(X,X) = 0.$$

Taking the product by $U \in \Gamma(D^{\perp})$,we obtain

$$g(A_{FX}X, U) + g(B\sigma(X,X), U) = 0.$$

Because M is a totally umbilical , we get

$$g(A_{FX}U, X) + g(B\sigma(X,X), U) = 0$$
$$g(\sigma(U,X), FX) - g(\sigma(X,X), FU) = 0$$
$$g(U,X)g(H,FX) - g(X,X)g(H,FU) = 0$$
$$g(X,X)g(BH,U) - g(U,X)g(BH,X) = 0$$

that is

$$g(BH,U)X - g(BH,X)U = 0.$$

Here BH is either zero or X and U are linearly dependent vector fields. If $BH \neq 0$, than dim $(D^{\perp}) = 1$.

Othervise $H \in \Gamma(\mu)$. Since $D_\theta \neq 0$ M is (CPSS). Since $\theta \neq 0$ and $d_1.d_2 \neq 0$ proper (CPSS).

Conflicts of interest

The authors declare that there are no potential conflicts of interest relevant to this article.

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