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Testing Complete Spatial Randomness on Linear Networks: Leon County Traffic Accident Example

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ABSTRACT

A relatively new sub-area within this is the statistical analysis of point processes on linear networks, that is, processes of events occurring randomly in space but with locations constrained to lie on a linear network. For example, traffic accidents occur at random locations constrained to lie on a network of streets. In this case, the network is idealized as a network of line segments in the plane or three-dimensional space. The development of statistical techniques for the analysis of point processes on linear networks is still in its infancy. Many standard statistical techniques for analyzing point processes cannot be directly applied to data arising from linear networks and require suitable modification. Test of Complete Spatial Randomness (CSR) for point processes on the plane based on quadrat counts or nearest neighbors cannot be applied to point processes on linear networks. This paper defines a Voronoi tessellation of the linear network which uses the shortest path distance along the network instead of Euclidean distance, and then develops a chi-square test of CSR for linear networks based on the event counts in the tiles of this tessellation. This test is applied to data on traffic accidents in Leon County, Florida, USA.

Keywords: Spatial statistics, Point pattern, Complete spatial randomness, Linear network.

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Introduction

Point processes are used as models for "events" or "points" occurring randomly in some space. As defined in [1] a point process X on $\mathbb{R}^d$ is a random countable subset of a region S $\subseteq \mathbb{R}^d$, and a realization of such a process is a point pattern $x = \{x_1, \dots, x_n\}$ of $n \geq 0$ points contained in S. For any set x, let $n(x)$ denote the cardinality of x. Point processes X used in applications are usually assumed to be simple and locally finite. In a simple point process, no two points coincide, that is, no two "events" occur in the same position in $\mathbb{R}^d$. A locally finite point process X has only finitely many points in any bounded set. Stated more precisely, all realizations x are locally finite subsets of S, meaning that $n(x_A) < \infty$ for all bounded sets $A \subseteq S$ where $x_A = x \cap A$ is the restriction of the realization x to A.

The terms "point pattern" and "point process" are used interchangeably in most sources [2]. The purpose of a point process is to serve as a model for the pattern of "things" and their distributions. The things have locations in one, two, or three-dimensional spaces. Examples of things are tree locations in a forest, cancer cells in a tissue, bird migration routes, and tornado paths. The things are constructed using points and marks. Points are the locations of things, and marks are additional information associated with the points [2]. The general theory of point processes has been developed for arbitrary dimensions.

The methods suggested for determining randomness can be roughly separated into two types, which are referred to as quadrant methods and distance approaches, respectively [3]. The article [4] has investigated the efficacy of randomness tests, in particular tests based on nearest-neighbor distance, inter-point distances, and estimators of moment measures. In addition to using distances and quadrants, alternative approaches have also been developed in several studies. The author of article [5] proposed assessing spatial randomness using angles between vectors connecting each sample location to its nearest neighbors. Additionally, [6] also mentioned a way of defining spatial patterns in which sample points travel in a regular arrangement that resembles a hexagonal lattice. However, none of the tests were developed for the test of Complete Spatial Randomness (CSR) on linear networks. The only method developed for the linear network was proposed by [7], which is a distance-based approach for testing CSR on a linear network, however, she advises conditioning on the positions of two arbitrary points to get the cumulative distribution function (CDF) of inter-event distance for complete spatial random (CSR) point pattern on the mxn grid network. Finally, to test two CSR test methods she suggested are based on inter-event distance and nearest-neighbor distance, respectively are based on Monte Carlo simulations. The method is based on simulation and depends on writing the CDF of the function which is not always straightforward to write CDF of a function. In this paper, we first define linear networks and spatial point process on liner networks, then we review complete spatial randomness on both planar space and linear networks then we define a Voronoi tessellation of the linear network which uses the shortest path distance along the network instead of Euclidean distance, and finally, we developed a chi-square test of CSR for linear networks based on the event counts in the tiles of this tessellation. This test is applied to data on traffic accidents in Leon County, Florida, USA.

Spatial Point Processes on Linear Networks

Although spatial point processes on the line and plane have been studied since at least the 1950s, work on spatial point processes on linear networks is a recent development. There are important differences between the analysis of point processes on Euclidean spaces (e.g., the line or plane) and point processes on a linear network. For example, the Euclidean distance metric figures prominently in the development of point processes on the plane but may be an insufficient or misleading distance metric for spatial point patterns on linear networks, such as those arising in the study of the locations of traffic accidents, crime on sidewalks, road-kill, the population distribution of dendrites, or the heterogeneity of tree species along a river. For point processes on a linear network, it is usually more appropriate to define the distance between two points on the linear network to be the length of the shortest path between these two points traveling along the linear network. Another important difference between point processes on the plane and on a linear network is illustrated in Figure 1. When one looks at Figure 1(a), one may not think the points are randomly distributed, but Figure 1(b) clearly shows that the points are randomly distributed [8] on a linear network. The notions of “randomness” or “uniformity” are very different on the plane and on a linear network.

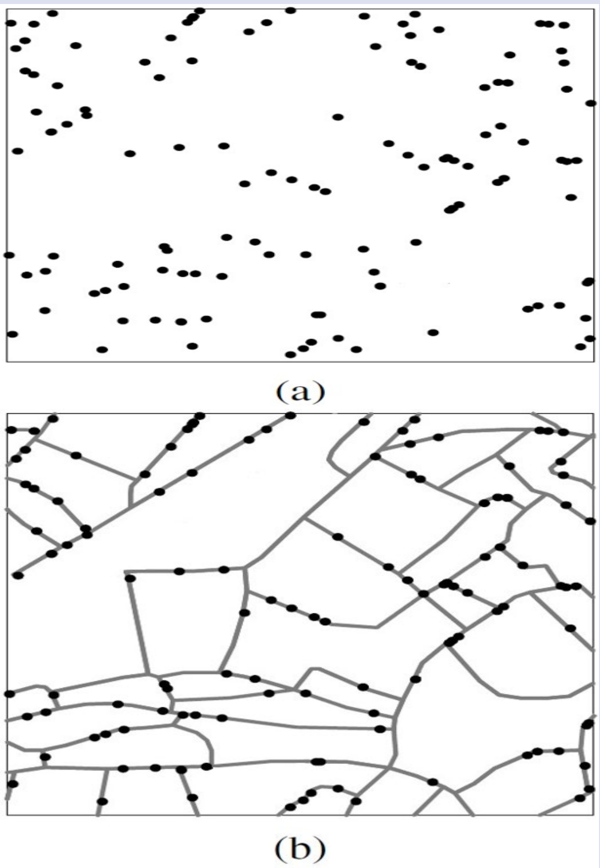

Figure 1: Points on a planar and linear network (Note: (a) and (b) are the same data points

Definitions Relating to Linear Networks

The line segment *l* on the plane with endpoints $u, v \in R^2$, $u \neq v$, can be written in any of the following ways:
$$l = l_{u,v} = [u,v] = \{tu + (1-t)v : 0 \leq t \leq 1\}$$

The length of this segment can be written as

$$|l| = |l_{u,v}| = \|u - v\|,$$

where $\|.\|$ is the usual Euclidean norm in R^2 which for

$z = (z_1, z_2)$ is defined by $\|z\| = \sqrt{z_1^2 + z_2^2}$,[9,10].

A linear network *L* is a combination of line segments (edges) l_i:

$$L = \bigcup_{i=1}^{n} l_i$$

The total length of the linear network *L* is defined by

$$|L| = \sum_{i=1}^{n} |l_i|$$

According to [8], the endpoints of segments are called nodes or vertices, and the degree of a node *u*, written as *d*(*u*), is the number of segments that are connected to the node. When *d*(*u*) = 1, then *u* is called a *terminal* node [11].

A *path* between *u* and *v* in a linear network *L* is a sequence $x_0, x_1, \ldots, x_m$ of points in *L* such that $x_0 = u, x_m = v$,and $[x_i, x_{i+1}] \subset L$ for each $i = 0, \ldots, m-1$. This path is denoted by $P(u, x_1, \ldots, x_{m-1}, v)$. The *length of a path* $P(u, x_1, \ldots, x_{m-1}, v)$ on *L* is defined to be

$$\|u - x_1\| + \|x_1 - x_2\| + \ldots + \|x_{m-1} - v\|.$$

The *shortest-path distance* between two points *u* and *v* in a linear network *L* is the length of the shortest path in *L* between *u* and *v*; this distance is denoted by $d_L(u, v)$.

[9] notes that a point process *X* on a linear network *L* is a special case of a point process on a planar space. We assume that *X* is simple, meaning that it does not have any coincident points. Each realization of *X* is a finite set $x = \{x_1, \ldots, x_n\}$ of distinct points $x_i \in L$, where $n \geq 0$ is (typically) random and not fixed in advance.

When analyzing point patterns on the plane (e.g., the locations of crimes, stores, tree species, etc.), the ordinary Euclidean distance is usually the most appropriate measure of the distance between events. However, when it is known that the events of interest occur on a street network, this is often not a proper choice. In many cities' streets are arranged (at least roughly) in a rectangular grid and the Euclidean distance can sometimes differ substantially from the true street distance (the shortest-path distance along the network) as is illustrated in Figure 2(a). For this reason (as noted by [7]) some researchers began using the “grid distance” (also known as the taxi-cab distance or the L_1 distance) in their analyses, such as the crime pattern along network analysis in [8]. For two points with *x-y* coordinates (x_1,y_1) and (x_2,y_2), the grid distance between them is $|x_1 - x_2| + |y_1 - y_2|$. The grid distance between two points P_1 and P_2 is illustrated in Figure 2(b); it is computed as the summation of the horizontal length *x* and the vertical length *h*. However, [12] observe that grid distance is not ideal for two reasons: first, not every city uses grid roads, and many cities are using a circle-radial system. Second, even when a city uses a grid-style network, using grid distance to compute true street distance can be inaccurate. For example, in Figure 2(c), the grid distance between P_1 and P_2 is $x + h$ but the shortest-path distance between P_1 and P_2 is substantially greater than this. Therefore, neither Euclidean distance nor grid distance is an appropriate distance metric to use while analyzing events on a linear network. The shortest-path distance or true street distance is the proper distance [12].

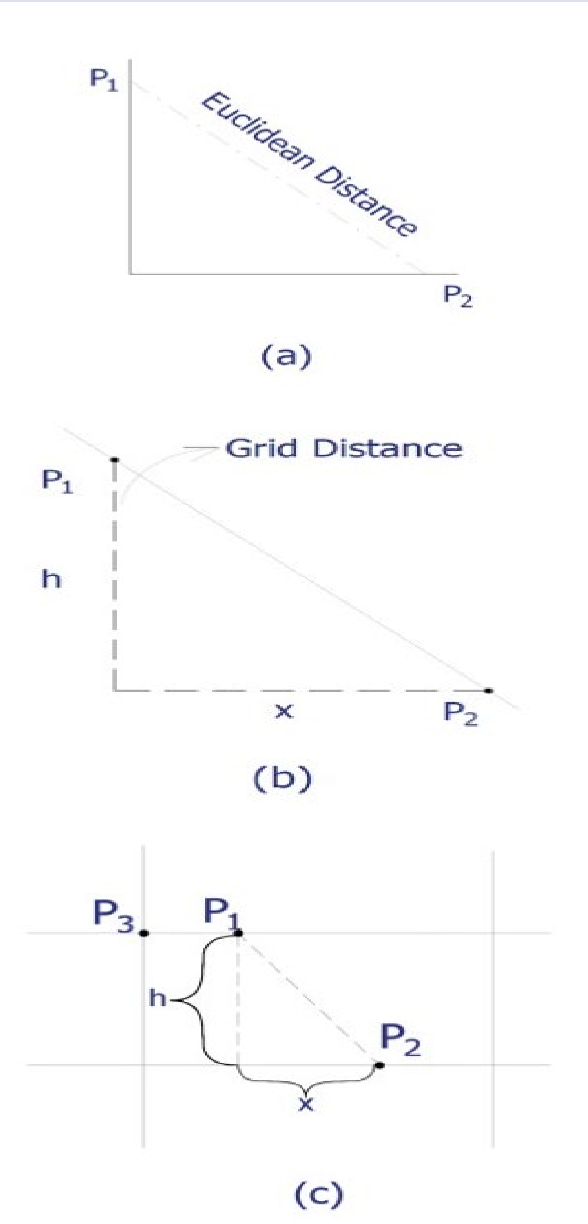

Figure 2: (a) Situation in which Euclidean distance differs greatly from shortest-path distance. (b) Illustration of grid distance between P_1 and P_2. (c) The situation in which grid distance differs greatly from shortest-path distance. (Note: Solid lines are streets.)

Complete Spatial Randomness

Point pattern analysis differs from other spatial processes in that the number of events and their locations occur randomly. Point pattern analysis has been studied since the early 1920s in the fields of ecology and forestry. The simplest point patterns are those which exhibit *Complete Spatial Randomness* (CSR), which is (roughly speaking) the absence of structure [14]. Mathematically, CSR is equivalent to a point process being a *homogeneous Poisson process*. Checking for CSR is of paramount importance for point pattern analysis. The lack of sufficient evidence for rejecting CSR implies that events can be modeled with a uniform distribution and hence there is no spatial dependence. Therefore, there is no further reason to carry out a spatial analysis because there will be no or limited gain. Secondly, CSR analysis carries a fundamental role in exploring and learning about the data [15].

Many methods have been developed for testing CSR for point patterns in the plane, for example, the quadrat method is commonly used (although [14] notes that this method is not powerful enough to catch characteristics of the pattern on multiple scales). However, CSR on a linear network has not been widely studied. In this paper, we will explore the use of the Voronoi diagram to test CSR on a linear network.

Complete Spatial Randomness on R^2

CSR is synonymous with a homogeneous Poisson process. For such a process, the points (events) which occur in any bounded region $B \subset R^2$ are uniformly distributed over this region and are independent and do not interact with each other. Let $N(B)$ denote the number of events of the process in B. Given that $N(B) = n$, the ordered n-tuple of events $(u_1, u_2, \ldots, u_n) \in B_n$ satisfies

$$P(u_1 \in A_1, \ldots, u_n \in A_n) = \prod_{i=1}^{n} \left(\frac{|A_i|}{|B|} \right), \quad A_1, \ldots, A_n \subset B,$$

where $|A| \equiv \int_A du$. This implies the events have the same probability to occur anywhere within B with no interaction between them, either repulsively or attractively.

It is difficult to identify whether points are distributed randomly through visual methods. Figure 3 shows three-point patterns, each containing 39 points in the same study area but generated by different methods. For example, Figure 3(c) shows 39 points created by a CSR process. Intuition can be misleading. People frequently do not expect a homogeneous Poisson process to display the apparent clustering and gaps observed in Figure 3(c); they expect the Poisson process to look more like Figure 3(c) which in fact is a process exhibiting fairly strong repulsion between the points leading to the more regular spacing between them. Formal statistical methods are needed to test for complete spatial randomness on R^2. Some commonly used tests are based on quadrat analysis, the nearest neighbor distance, and the K function.

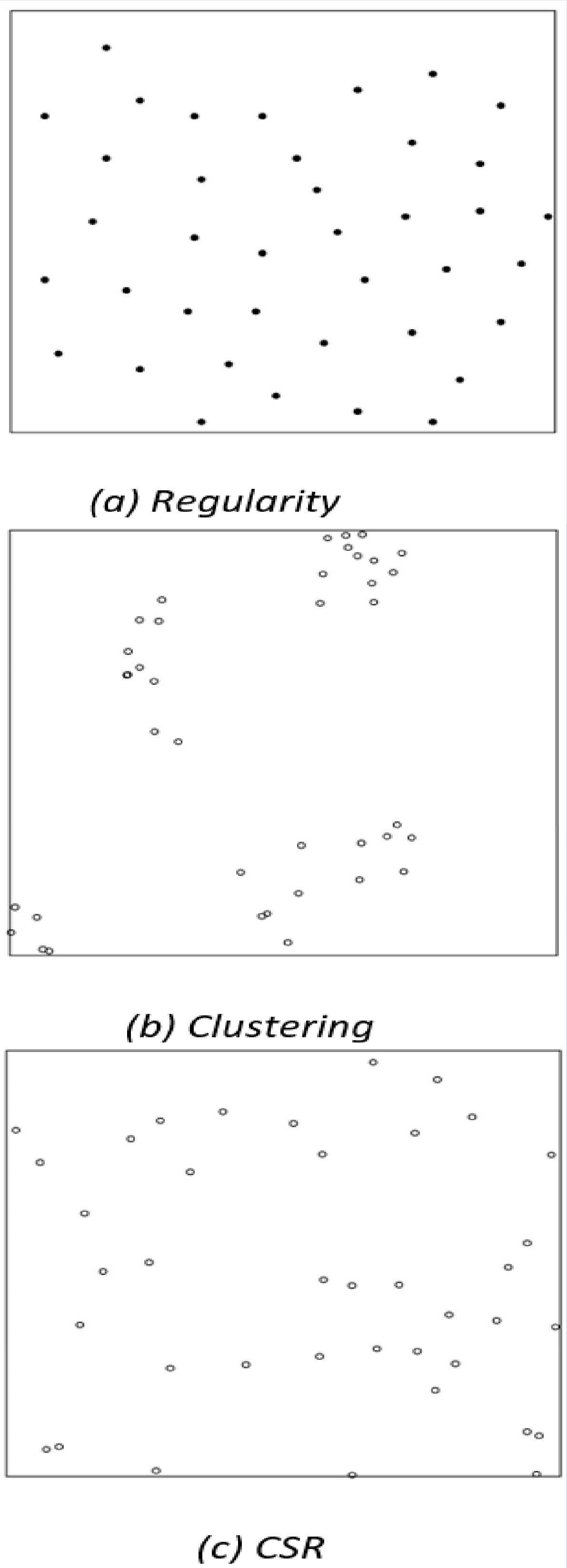

Figure 3: Three-point patterns produced by different processes, each with 39 points.

The current focus is on quadrat analysis because quadrat analysis and a test for CSR based on Voronoi diagrams will be compared in the next subsection. Details about the nearest neighbor distance and *K* function methods can be found in many books, including [16], [14], and [15]).

Quadrat Analysis

Quadrats are defined as unions of *regular* sub-regions [17]. The name quadrat comes from dividing the study area in R^2 into small sub-regions of equal areas, preferably square or rectangular, and counting the number of events in each sub-region. The intensity of events in each quadrat is the ratio of the number of events it contains to the area.

Two main assumptions about quadrat methods are:

1. The study area is represented by the Euclidean space.
2. The events in the study area are homogeneous, which implies that the probability of a random event happening in any part of the study area is constant despite the location of the event.

Testing the randomness assumes the event counts in each quadrat follow a Poisson distribution. If there are *n* quadrats with equal areas having events counts $x_!, x_2, \ldots, x_n$, the test statistics will be

$$X^2 = \frac{(n-1)S_x^2}{\bar{x}} = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{\bar{x}}, \qquad (1)$$

where, $\bar{x}$ is the mean of observed counts and S_x^2 is the variance of observed counts. Under the null hypothesis of CSR, the statistic X^2 has approximately a chi-square distribution with $n-1$ degrees of freedom so long as $\bar{x}$ is not too small.

When χ^2 is too big, it is a sign of clustering. When χ^2 is too small, it may indicate regularity. In [16], the *index of dispersion* is computed by $\frac{S_x^2}{\bar{x}}$, and $\frac{S_x^2}{\bar{x}} - 1$ is defined as the *index of cluster size*(ICS). Based on the expected value of ICS one can conclude whether or not the events follow CSR;

$$\begin{cases} E(ICS) > 0, & Clustered \\ E(ICS) = 0, & CSR \\ E(ICS) < 0, & Regularity \end{cases}$$

One drawback of this method is choosing the optimal number of quadrats. There is no agreed-upon number or a mechanism to check. Therefore, it is possible that some quadrat areas are too small or too big, and some quadrats might even be empty which will affect the number of observed events. That would make the interpretation difficult or misleading. Another weakness is that quadrat analysis is not actually a *measure of pattern* analysis because the test statistics in Equation (1), do not include the location or distance between points. Even though they are in the same quadrat, how many of the points are in particular quadrats is the only interest.

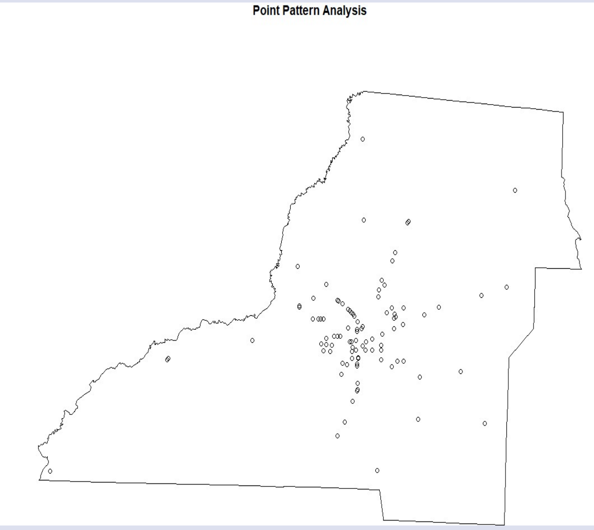

Figure 4: The location of 100 Leon County accidents occurring

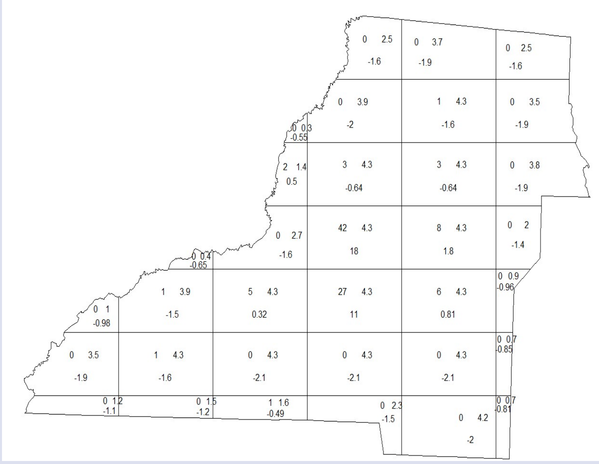

Figure 6: Quadrat analysis: equal-distance 6x7 square quadrats (In each box, the top left number is the observed value, the right top is the number of expected value and the bottom number is residual)

For illustration, Figure 4 shows 100 traffic accidents sampled in Leon County, Florida, between January 1, 2008, and December 31, 2013. A quadrat analysis with 3x3 and 6x7 quadrats was executed. Figure 5 shows an example of a quadrat analysis using a 3x3 quadrat setup. In each quadrat, the values represent *observed, expected*, and *residuals*. Where $\mathrm{residual_i} = (O_i - E_i)/\sqrt{E_i}$. As seen in the 3x3 square quadrat analysis, the quadrat size affects the observed and expected counts and the residuals, which affects the test statistics.

$$\chi^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i} \tag{2}$$

Figure 6 shows a 6x7 quadrat analysis for the same study area with the same number of events. Increasing the number of quadrats can create smaller, possibly empty quadrats (right bottom of Figure 6) that will affect the test statistics.

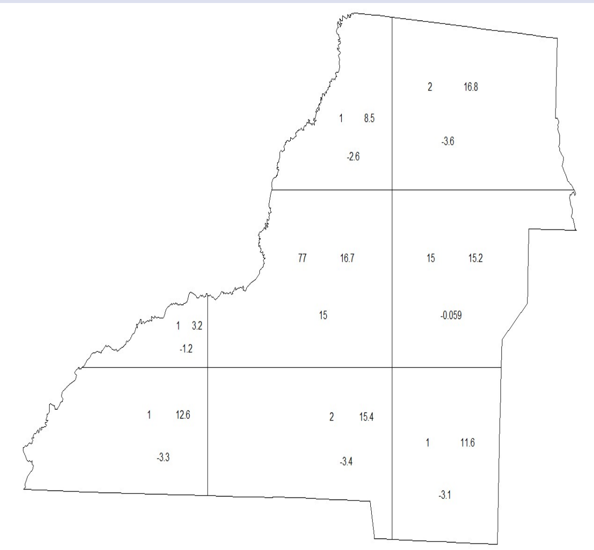

Figure 5: Quadrat analysis: equal-distance 3x3 square quadrats (Note: In each box, the top left is the number of observed points, the top right is the number of expected points and the bottom is the residual)

Voronoi tessellation

An alternative method for testing CSR on planar spaces is Voronoi tessellation, also known as Voronoi diagrams or Dirichlet tessellation.

Suppose n distinct points $z_1, z_2, \dots, z_n$ (with n ≥ 2) are chosen in planar space. The Voronoi tessellation determined by these points (which are called *generator* or *seed* points) is a subdivision of the planar space into n regions (known as *tiles*, cells, or *polygons*) denoted $\Phi(z_1), \Phi(z_2), \dots, \Phi(z_n)$. Each of these tiles surrounds one of the generator points. The tile $\Phi(z_i)$ consists of all points in the space that are closer to z_i then to any of the other generators z_j , $j \neq i$. That is,

$$\Phi(z_i) = \{z | d_E(z, z_i) \leq d_E(z, z_j), j = 1, \dots, n\} \quad (3)$$

where $d_E(z, z_i) = \parallel z - z_i \parallel$ is the Euclidean distance in R^2 between z and z_i. (For information on Dirichlet tessellations see [18] and [19].)

Complete spatial randomness on linear networks

One of the assumptions for quadrat analysis is the study area is represented by Euclidean space. Some cases show this assumption holds, especially on planar space examples, but some cases show this assumption does not hold such as street crimes, store locations, or traffic accidents. These events are observed along a linear network.

In Figure 8, a linear network is divided into 4x3 quadrats, and the two events z_1 and z_2 are in the same quadrat. The shortest-path distance between these two events could be far because there may or may not be a direct connection between them. Therefore, even if these events are in the same quadrats, they may not be close. [12] noted that one needs to update these assumptions based on the problem;

1. The study area is represented by a linear network.
2. The events on the network are homogeneous, which implies the probability of a random event happening on any segment is constant despite the location of the segment.

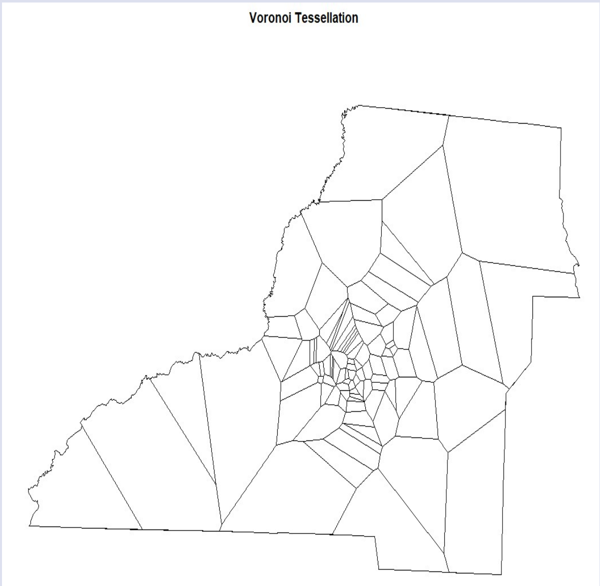

Figure 7: Illustration of Voronoi tessellation in Leon County

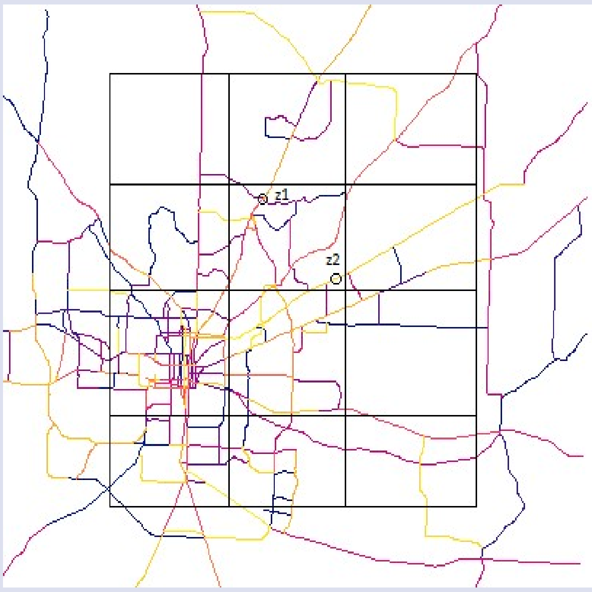

Figure 8: Illustration of quadrat analysis on a linear network in Leon County

In Figure 8, a linear network is divided into 4x3 quadrats, and the two events z_1 and z_2 are in the same quadrat. The shortest-path distance between these two events could be far because there may or may not be a direct connection between them. Therefore, even if these events are in the same quadrats, they may not be close. [12] noted that one needs to update these assumptions based on the problem;

1. The study area is represented by a linear network.
2. The events on the network are homogeneous, which implies the probability of a random event happening on any segment is constant despite the location of the segment.

[17] notes that there have been some attempts to use quadrat analysis on a linear network such as a river network, or on a road network to identify zones with high concentrations of traffic accidents, but [17] uses a heuristic approach. Creating quadrats on a linear network is not as straightforward as in planar space. Therefore, this paper proposes to use Voronoi tessellation on a linear network to test complete spatial randomness.

In Equation (3), [8] showed how to create Voronoi tessellation on planar networks using the Euclidean distance between two events. Therefore,

Equation (3) is updated with the shortest path between z and z_i in Equation (4) for Network Voronoi diagram;

$$\Phi(z_i) = \{z \mid d_L(z, z_i) \leq d_L(z, z_j)\}, j = 1, \ldots, n \quad (4)$$

Voronoi tessellation sets are shown as $\Phi(\boldsymbol{Z}) = \{\Phi(z_1), \ldots \Phi(z_n)\}$. The configuration of the Voronoi tessellation is determined by the number and location of the generator points and the particular distance metric which is used [9]. Various numbers of tiles have been tried and finally, 15 tiles have been used.

There is no a straightforward method to decide the number of tiles. In Figure 9, using Voronoi tessellation, Leon County's Road network is divided into 15 tiles.

Data Analysis

There are two parts to this section. The first one is about how to obtain a linear network and the second part is about where to get the accident data. Road information of Leon County is downloaded from the Florida Department of Transportation (FDOT)'s website. More detail on how Leon County linear network was created is explained in [20]. The data set, which totals 59,773 accident records, consists of accidents occurring from 2013 through 2019 and was provided by the GeoPlan Center affiliated with the Department of Urban & Regional Planning at the University of Florida.

The programming languages used in this paper are R 4.02 and ArcMap [21, 22]. ArcMap has been used to read and manipulate shapefiles that contains road information from Leon County. It is the first step which is creating the linear network. R programming has been used for creating Voronoi tessellation on the linear network and analyzing it, mainly the package *spatstat* has been used [23].

The locations of reported traffic accidents between January 2013 and December 2019 are placed on this tessellation, and the number of accidents in each tile is counted. Figure 10 shows these events. A histogram in Figure 11 shows the number of events in each tile, and tile 13 has the most events. Afterward, the observed number of events in each tile is counted, and a histogram is created for this tessellation.

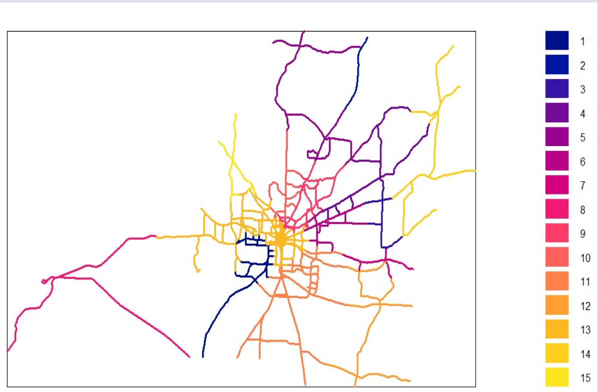

Figure 9: Voronoi tessellation on a linear network in Leon County

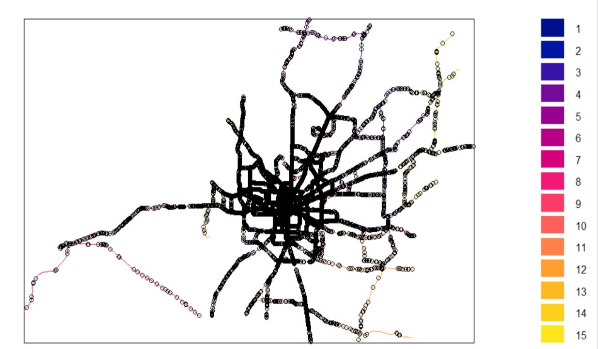

Figure 10: Voronoi tessellation of traffic accidents in Leon County

To get the expected number of points in each tile, the total length of each tile is computed and shown in Table 2's first row. Afterward, the homogeneous intensity is assumed and λ_0 is estimated as the total number of points over the total length of the linear network

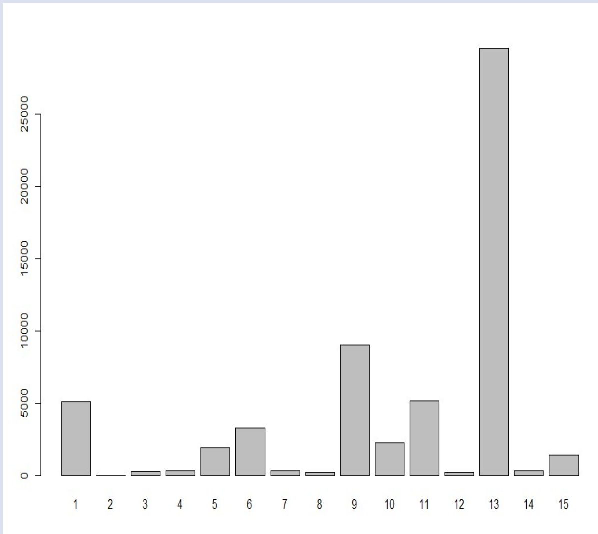

Figure 11: Observed event counts

(λ_0=0.07585275). Finally, the expected value in each tile is computed as λ_0 times the total length in each tile which is shown in the second row of Table 2.

$$\chi^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i} \tag{5}$$

Table 1: Observed counts of events in Leon County

Tile	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Observed	5126	42	280	373	1939	3308	336	254	9069	2301	5197	221	29528	348	1448

Table 2: Total length and expected points in each tile in Leon County

Tile	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Length	46113.4	10135.48	18208.63	46408.23	63609.3	32589	16190.31	56148.9	57209.4	28667.7	114416.5	52673.3	166410	56182.2	23011.64
Expected	3497.83	768.8	1381.17	3520.19	4824.94	2471.96	1228.08	4259.05	4339.49	2174.52	8678.81	3995.41	12622.66	4261.57	1745.5

The null hypothesis is whether these events follow complete spatial randomness. To test this hypothesis, Equation (5) is used to compute the χ^2 test statistic as 47969.96. There are 15 tiles and one unknown parameter to estimate. The degree of freedom for the χ^2 test employed is 14. Therefore, the 0.05 critical value of χ^2 distribution is $X_{14}^2 = 23.69$. The p-value for this test is computed as $P(X^2 > 47969.96) = 0$. The null hypothesis is rejected. This suggests that these events do not follow complete spatial randomness.

In situations where the total length of each tile is difficult to compute, an approach to get the expected number of points is to use Monte Carlo Approximation. To create a long-run average for each tile to approximate the "expected'' counts, *n* number of uniformly random points repeatedly generate M random uniform points on the linear network (where M is the total number of events we have in our data) and for each repetition count the number of events in each tile. Averaging the event counts in each tile over sufficiently many repetitions leads to estimates of Ei which may be used in equation (5) to compute χ2. Applying this procedure to our data (which has M = 59770 events on the network in Figure 9) and using 10000 repetitions leads to the estimates of E_i given in Table 3. Using these values in equation (5) produces the test statistic χ2 = 47967.39, which is very close to the earlier value and leads to the same conclusion.

Ten thousand simulations are executed for randomly generated 59770 uniform events on the linear network and the expected average number of points in each tile are shown in Table 3.

Table 3: Average expected counts of events in Leon County

Tile	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Expected	3497.25	769.11	1381.5	3519.6	4825.29	2471.78	1227.95	4259.59	4340.39	2174.26	8678.78	3995.66	12622.75	4260.67	1745.41

Equation (5) computes χ^2 test statistic as 47967.39. There are 15 tiles and one unknown parameter to estimate. The degree of freedom for the χ^2 test employed is 14. Therefore, the critical value of χ^2 distribution is $\chi^2 = 23.69$. Under a significance level of 0.05, the p-value for this test is computed as $P(\chi^2 > 47967.39) = 0$. The null hypothesis is rejected. This suggests that these events do not follow complete spatial randomness.

Conclusion

Spatial point processes are everywhere from gold mines to tree species and from river streams to road maps. Therefore, testing the randomness of events in these study areas is the first natural step. Quadrat analysis, the nearest neighbor distance, and the K function are common methods to test complete spatial randomness on plane spaces. However, when the study area is a linear network such as a fault line, river stream, and road map, Quadrat analysis is misleading. Therefore, in this paper, Voronoi tessellation is used for testing complete spatial randomness on a linear network.

The proposed method is an upgrade to the quadrat analysis. The road map of Leon County, Florida, USA, has been used for a real data analysis purpose. As in quadrat analysis, a weakness of this approach is that there is not an optimal number of tiles. However, we have seen applied from 5 to 20 tiles and see no significant difference. Hence, 15 tiles have been used in this analysis, with the critical value of χ^2 distribution is $\chi^2 = 23.69$ obtained. Based on a significance level of 0.05, the p-value for this test is computed as $P(\chi^2 > 47967.39) = 0$. It has been observed that there is no spatial randomness and can be further investigation of analysis of the data. The purpose of the paper is to find a quick randomness test on linear networks and the Voronoi tessellation is an easy and quick approach to test complete spatial randomness on linear networks.

Conflict of interest

There are no conflicts of interest in this work.

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