Cumhuriyet Sci. J., 43(4) (2022) 676-683
DOI: https://doi.org/10.17776/csj.1115789

Cumhuriyet Science Journal

| csj.cumhuriyet.edu.tr | Founded: 2002 ISSN: 2587-2680 e-ISSN: 2587-246X Publisher: Sivas Cumhuriyet University

Fixed Point Theorems In 2-Banach Spaces For Non-expansive Type Conditions

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Research Article

History
Received: 12/05/2022
Accepted: 16/11/2022

ABSTRACT

Fixed point theorems had been established and developed under various non-expansive type conditions on different metric spaces. In this paper, we have generalized (ψ,ϕ) - weak contractions, which is the generalizations of F-contraction, (φ,F)-contraction as well as (ψ,ϕ)-contractions. Then we have established some unique common fixed point results for a sequence of mappings for (ψ,ϕ)- weak contractions in 2-Banach spaces. Some basic definitions, properties and examples are given in the introduction and preliminaries part. Some corollaries are also given on the basis on the results.

Keywords: 2-Banach spaces, Coincidence points, Non-decreasing function, Cauchy sequence, Fixed point.

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Introduction and Preliminaries

In 1922, S. Banach first established the Banach contraction principle (BCP) and proved fixed point results in complete metric spaces. Since then lots of fixed point results have been proved in many contractive conditions on various spaces. In 2011, Mujahid Abbas, Talat Nazir and Stojan Radenovic [1] established some common fixed point results for four maps in partially ordered metric spaces. Also in 2018, Seonghoon Cho [2] obtained some fixed point theorems for generalized weakly contractive mappings in metric spaces.

In this paper we have established a unique common fixed point theorem on generalized (ψ,ϕ)- weak contractions for a sequence of mappings in 2-Banach spaces, which is a generalization of the results of Seonghoon Cho [2].

The concept of 2- Banach space has been initiated by S. Gahler [3] and then many authors established fixed point results on this space under different contractive conditions as in other spaces. In 2013, Liu and Chai [4] established fixed point theorem for weakly contractive mappings in generalized metric spaces. Later in 2021, Zhiqun and Guiwen Lv [5] also developed some fixed point results for generalized (ψ,φ)-weak contraction in Branciari- type generalized metric spaces.

We recall some basic definitions, properties and conclusions which are as follows:

Definition 1 [2] Let X be a real linear space and $||.,.||$ be a non-negative real valued function defined on $X\times X$ satisfying the following conditions:

$\|v,w\|=0$, if and only if v and w are linearly dependent in X ;
$\|v,w\|=\|w,v\|$, for all $v,w\in X$;
$\|v,kw\|=|k|\|v,w\|$, $v,w\in X; k\in R$;
$\|v,w+z\|\le\|v,w\|+\|w,z\|$, for all $v,w,z\in X$.

Then $||.,.||$ is called a 2-norm on X and the pair $(X,||.,.||)$ is called a linear 2-normed space.

Note: 2-Norm are non-negative and $\|v,w+kv\|=\|v,w\|$, for all $v,w\in X; k\in R$.

Definition 2 [6,7] A sequence $\{v_n\}$ in a linear 2-normed space $(X,||.,.||)$ is called Cauchy sequence if

$$\lim_{m,n\to\infty}\|v_m-v_n,a\|=0, \text{ for all } a\in X.$$

Definition 3 [6] A sequence $\{v_n\}$ in a linear 2-normed space $(X,||.,.||)$ is said to be convergent in X, if there is a point v in $X\times X$ such that $\lim_{n\to\infty}\|v_n-v,a\|=0$, for all $a\in X$.

Definition 4 [6] A linear 2-normed space X is said to be complete with respect to the 2-norm $||.,.||$ if every Cauchy sequence is convergent to an element of X. Then we call $(X,||.,.||)$ to be a 2- Banach space.

Definition 5 [6] Let X be a 2-Banach space and T be a self-mapping on X. T is said to be continuous at$x\in X$ if for every sequence $\{v_n\}$ in X , $v_n\to v$ as $n\to\infty$ implies $T(x_{n)}\to T(x)$ as $n\to\infty$.

Definition 6 [6] Let F and G be self maps on a set X (i.e, $F, G: X \to X$). If $u = Fw = Gw$ for some $\in X$, then w is called a coincidence point of F and G; and u is called a point of coincidence of F and G.

Example 1 Let $X = R^3$. Define 2-norm on X as

$$\|v, w\| = 0, \quad \text{if and if } v, w \text{ are linearly dependent;}$$
$$= |\begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}|, \quad \text{where,} \quad v = (v_1, v_2, v_3),$$
$$w = (w_1, w_2, w_3) \in R^3,$$

Then (X, ‖., .‖) is called a 2-Normed on X.

Example 2 [8]. Let Q_n denote the set of all real polynomials of degree $\leq n$ on the interval $[0,1]$. For usual addition and scalar multiplication, $Q_n$is a linear vector space over the real numbers. Let $v_1, v_2, v_3, \dots$ be distinct fixed points in $[0,1]$ and define the following 2-norm on Q_n as

$$\|F, G\| = \sum_{k=0}^{2n} |F(v_{k})| G(v_k), \quad \text{whenever } F \text{ and } G \text{ are linearly independent;}$$
$$= 0 \quad \text{if and only if } F \text{ and } G \text{ are linearly dependent.}$$

Then $(Q_n, ||.,.||)$ is a 2-Banach space.

In 2013, Liu and Chai [4] established some fixed point theorems in generalized metric spaces and 2018, Seonghoon Cho[2] obtained some fixed point results in a metric space using a generalised weakly contractive mapping.

Inspiring the results of Liu and Chai [4] and Seonghoon Cho[2] (c.f. Theorem 2.2) we have established the following results.

Main Results

To establish our main results we introduce two classes of functions Ψ and Φ which are given below:
$\Psi = \{\psi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:
(i) ψ is monotonic non-decreasing;
(ii) $\lim_{t \to r} \psi(t) > 0 \; for \; r > 0 \; and \; \lim_{t \to 0+} \psi(t) = 0$;
(iii) $\psi(t) = 0 \; if \; and \; only \; if \; t = 0.\}$
$\Phi = \{\varphi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:
(i) $\lim_{t \to r} inf \varphi(t) > 0 \; for \; r > 0$;
(ii) $\varphi(t_n) \to 0$ implies $t_n \to 0$;
(iii) $\varphi(t) = 0 \; if \; and \; only \; if \; t = 0$.

Theorem 1 Let $(X, ||.,.||)$ be a 2-Banach space and $\{T_i\}_{i=1}^{\infty}$ be a sequence of self maps on X satisfying the following conditions:

$$\psi\{\|T_i x - T_j y, a\| + \phi(T_i x) + \phi(T_j y)\} \leq \psi\{Z^{(1)}(x, y, T_i, T_j, \phi)\} - \varphi\{Z^{(2)}(x, y, T_i, T_j, \phi)\}, \; \forall x, y \in X, \; \psi \in \Psi, \varphi \in \Phi; \quad (1)$$
where,

$$Z^{(1)}(x, y, T_i, T_j, \phi) = \max\Big\{\|x - y, a\| + \phi(x) + \phi(y), \|x - T_i x, a\| + \phi(x) + \phi(T_i x), \; \|y - T_j y, a\| + \phi(y) + \phi(T_j y), \; \tfrac{1}{2}\{\|x - T_j y, a\| + \phi(x) + \phi(T_j y) + \; \|y - T_i x, a\| + \phi(y) + \phi(T_i x)\Big\} \quad (2)$$

$$Z^{(2)}(x, y, T_i, T_j, \phi) = \max\{\|x - y, a\| + \phi(x) + \phi(y), \|x - T_i x, a\| + \phi(x) + \phi(T_i x), \; \|y - T_j y, a\| + \phi(y) + \phi(T_j y)\} \quad (3)$$

And $\phi: X \to [0, \infty)$ *is a lower semi continuous function*.
Then there exists a unique $z \in X$ such that $z = T_i z$, ∀ i = 1,2,3, ... and $\phi(z) = 0$.
Proof: Let $x_0 \in X$ be a point and we define a sequence $\{x_n\}_{n=1}^{\infty} \subset X$ by

$$x_{n+1} = T_i x_n \; \forall \; i = 1,2,3, \dots$$

Consider two cases:
Case-I: Let $\{x_n\}$ be periodic.
subcase-I.I: If for some $n \in N$, $x_n = x_{n+1}$, then $x_n = T_i x_n$ and hence x_n is a fixed point of T_i $\forall i =$ 1,2,3,...
subcase-I.II: If for some $n \in N$, $x_n = x_{n+p}$, for p = 2,3, .., then $T_i^{p-1} x_n$ is a fixed point of T_i.
For $p = 2$, $x_n = x_{n+2}$, then $T_i x_n$ is a fixed point of T_i i.e., $T_i(x_n) = T_i(T_i(x_n))$, i.e., $x_{n+1} = x_{n+2}$
If not then, $\|x_{n+1} - x_{n+2}, a\| > 0$, $\forall \; a \in X$.
From (2) we have,

$Z^{(1)}(x_n, x_{n+1}, T_i, T_j, \phi)$

= max{ $||x_n - x_{n+1}, a|| + \phi(x_n) + \phi(\boldsymbol{x_{n+1}})$, $\|x_n - x_{n+1}, a\| + \phi(x_n) + \phi(x_{n+1})$, $\|x_{n+1} - x_{n+2}, a\| + \phi(x_{n+1}) + \phi(x_{n+2})$, $\frac{1}{2}\{\|x_n - x_{n+1}, a\| + \phi(x_n) + \phi(x_{n+1})$, $\|x_{n+1} - x_{n+1}, a\| + \phi(x_{n+1}) + \phi(x_{n+1})\}\}$

= $\{\|x_n - x_{n+1}, a\| + \phi(x_n) + \phi(x_{n+1})\}$, $\quad [\ as\ \ x_n = x_{n+2}]$

and

$Z^{(2)}(x_n, x_{n+1}, T_i, T_j, \phi)$= max { $||x_n - x_{n+1}, a|| + \phi(\boldsymbol{x_n}) + \phi(\boldsymbol{x_{n+1}})$, $\|x_n - x_{n+1}, a\| + \phi(x_n) + \phi(x_{n+1})$, $\|x_{n+1} - x_{n+2}, a\| + \phi(x_{n+1}) + \phi(x_{n+2})$,**}**

$$=\{\|x_n - x_{n+1}, a\| + \phi(x_n) + \phi(x_{n+1})\} \qquad [as\ \ x_n = x_{n+2}]$$

Therefore, by (1) we obtain

$\psi\{\|x_{n+1} - x_{n+2}, a\| + \phi(x_{n+1}) + \phi(x_{n+2})\}$
$=\psi\ \{\|T_i x_n - T_j x_{n+1}, a\| + \phi(T_i x_{x_n}) + \phi(T_j x_{n+1})\}$
$\leq \psi\{\|x_n - x_{n+1}, a\| + \phi(x_n) + \phi(x_{n+1})\}$ - $\phi\{\|x_n - x_{n+1}, a\| + \phi(x_n) + \phi(x_{n+1})\}$,

which gives $\psi\{\|x_{n+1} - x_n, a\| + \phi(x_{n+1}) + \phi(x_n)\} < \psi\ \{\|x_n - x_{n+1}, a\| + \phi(x_n) + \phi(x_{n+1})\}$, which is a contradiction.

Hence, $T_i(T_i x_n) = T_i x_n$ i.e., $T_i x_n$ is a fixed point of T_i, $\forall\ i = 1, 2, 3\ldots\ldots$.
Therefore the statement is true for $p = 2$.
Assume that the statement is true for $p = m$, $m \geq 2$, i.e., $T_i^{m-1} x_n$ is a fixed point of T_i.

$$Then \qquad T_i^m x_n = T_i^{m-1} x_n \tag{4}$$

Applying T_i on both sides of (4), we get

$$T_i(T_i^m x_n) = T_i(T_i^{m-1} x_n) \quad \text{i.e.,} \quad T_i(T_i^m x_n) = T_i^m x_n.$$

Hence, $T_i^m x_n$ is a fixed point of T_i. Therefore the statement is true for $p = m + 1$.
Thus by the Principle of Mathematical Induction if $x_n = x_{n+p}$, then x_n is a fixed point of T_i, for $p = 1,2,3,\ldots$
Case-II: Assume $x_n \neq x_{n+1}$, for all $n \in N$. Now from (2) we have

$Z^{(1)}\ x_{n-1}, x_n, T_i, T_j, \phi)$
= max{$\|x_{n-1} - x_n, a\| + \phi\ (x_{n-1}) + \phi(x_n)$, $\|x_{n-1} - x_n, a\| + \phi(x_{n-1}) + \phi(x_n)$, $\|x_n - x_{n+1}, a\| + \phi(x_n) + \phi(x_{n+1})$,

$\frac{1}{2}\ \|x_{n-1} - x_n, a\| + \phi(x_{n-1}) + \phi(x_n) + \|x_n - x_n, a\| + \phi(x_n) + \phi(x_n)\}$
=max {$\|x_{n-1} - x_n, a\| + \phi\ (x_{n-1}) + \phi(x_n)$, $\|x_n - x_{n+1}, a\| + \phi(x_n) +$

$$\phi(x_{n+1})\} \tag{5}$$

And from (3) we

$Z^{(2)}\ (x_{n-1}, x_n, T_i, T_j, \phi)$=max {$\|x_{n-1} - x_n, a\| + \phi\ (x_{n-1}) + \phi(x_n)$, $\|x_{n-1} - x_n, a\| + \phi(x_{n-1}) + \phi(x_n)$, $\|x_n - x_{n+1}, a\| + \phi(x_n) + \phi(x_{n+1})\}$

= max{$\|x_{n-1} - x_n, a\| + \phi\ (x_{n-1}) + \phi(x_n)$, $\|x_n - x_{n+1}, a\| + \phi(x_n) + \phi(x_{n+1})\}$

If $\|x_{n-1} - x_n, a\| + \phi\ (x_{n-1}) + \phi(x_n) \leq \|x_n - x_{n+1}, a\| + \phi(x_n) + \phi(x_{n+1})$,

Then from (1) we get

$\psi\{\|x_n - x_{n+1}, a\| + \phi(x_n) + \phi(x_{n+1})\} \leq \psi\{\|x_n - x_{n+1}, a\| + \phi(x_n) + \phi(x_{n+1})\} - \varphi\{\|x_n - x_{n+1}, a\| + \phi(x_n) + \phi(x_{n+1})\}$,

Which gives $\varphi\{\|x_n - x_{n+1}, a\| + \phi(x_n) + \phi(x_{n+1})\} = 0$.
By definition of φ function we have

$$\left\|x_n - x_{n+1}, a\right\| + \phi(x_n) + \phi(x_{n+1}) = 0.$$

Hence $x_{n+1} = x_n$ and $\phi(x_{n+1}) = \phi(x_n) = 0$, which is a contradiction.
Therefore,

$$\|x_n - x_{n+1}, a\| + \phi(x_n) + \phi(x_{n+1}) < \|x_{n-1} - x_n, a\| + \phi(x_{n-1}) + \phi(x_n), \ \forall \ n = 1,2,3, \dots \tag{7}$$

and

$$Z^{(1)}(x_{n-1}, x_n, T_i, T_j, \phi) = \left\|x_{n-1} - x_n, a\right\| + \phi(x_{n-1}) + \phi(x_n) \tag{8}$$

and

$$Z^{(2)}(x_n, x_{n+1}, T_i, T_j, \phi) = \left\|x_{n-1} - x_n, a\right\| + \phi(x_{n-1}) + \phi(x_n) \tag{9}$$

From (1) we have

$$\psi\left\|x_n - x_{n+1}, a\right\| + \phi(x_n) + \phi(x_{n+1}) \leq \psi\{\left\|x_{n-1} - x_n, a\right\| + \phi(x_{n-1}) + \phi(x_n)\} - \varphi\{\left\|x_{n-1} - x_n, a\right\| + \phi(x_{n-1}) + \phi(x_n)\} \tag{10}$$

From (7) the sequence $\{\left\|x_{n-1} - x_n, a\right\| + \phi(x_{n-1}) + \phi(x_n)\}$ is monotonic decreasing and bounded below and hence convergent.
Let

$$\lim_{n\to\infty}\{\left\|x_{n-1} - x_n, a\right\| + \phi(x_{n-1}) + \phi(x_n)\} = r$$

and $\lim_{r\to\infty} \psi\{\left\|x_{n-1} - x_n, a\right\| + \phi(x_{n-1}) + \phi(x_n)\} = r^*$, where, $r, r^* \geq 0$.

Claim: r=0.
If $r > 0$, then taking lower limit as $n \to \infty$ on both sides of (10), we have

$$\lim_{n\to\infty} \psi\{\|x_n - x_{n+1}, a\| + \phi(x_n) + \phi x_{n+1}\} \leq \lim_{n\to\infty} \psi\{\|x_{n-1} - x_n, a\| + \phi(x_{n-1}) + \phi(x_n)\} - \lim_{n\to\infty} \inf \varphi\{\|x_{n-1} - x_n, a\| + \phi(x_{n-1}) + \phi(x_n)\}$$

or, $\lim_{n\to\infty} \inf \varphi\{\|x_{n-1} - x_n, a\| + \phi(x_{n-1}) + \phi(x_n)\} \leq 0$, which is a contradiction as $\lim_{n\to\infty}\{\|x_{n-1} - x_n, a\| + \phi(x_{n-1}) + \phi(x_n)\} = r > 0 \Rightarrow \lim_{n\to\infty} \inf \varphi\{\|x_{n-1} - x_n, a\| + \phi(x_{n-1}) + \phi(x_n)\} \leq 0$.

Therefore $\lim_{n\to\infty}\{\|x_{n-1} - x_n, a\| + \phi(x_{n-1}) + \phi(x_n)\} = 0$, which gives $\|x_{n-1} - x_n, a\| = 0$, and $\lim_{n\to\infty} \phi(x_n) = 0$.

Now we prove that the sequence $\{x_n\}$ is Cauchy in X.
If $\{x_n\}$ is not Cauchy, then there exist $\epsilon > 0$ and two subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with $m(k)$ is the smallest index such that $m(k) > n(k) > k$, implies

$$\|x_{m(k)-1} - x_{n(k)}, a\| \geq \varepsilon,$$

and

$$\|x_{m(k)-1} - x_{n(k)}, a\| < \varepsilon.$$

Now,

$$\begin{aligned} \varepsilon &\leq \|x_{m(k)} - x_{n(k),a}\| + \phi(x_{m(k)}) + \phi(x_{n(k)}) \\ &\leq \|x_{m(k)} - x_{m(k)-1}, a\| + \|x_{m(k)-1} - x_{n(k),a}\| + \phi(x_{m(k)}) + \phi(x_{n(k)}) \\ &< \|x_{m(k)} - x_{m(k)-1}, a\| + e + \phi(x_{m(k)}) + \phi(x_{n(k)}) \end{aligned}$$

Limiting as $k \to \infty$ we have

$$\lim_{k\to\infty}\{\|x_{m(k)} - x_{n(k)}, a\| + \phi(x_{m(k)}) + \phi(x_{n(k)})\} = \varepsilon.$$

From (2) we have,

$$Z^{(1)}(x_{n(k)}, x_{m(k)}, T_i, T_j, \phi)$$
$$= max\{\|x_{n(k)} - x_{m(k)}, a\| + \phi(x_{n(k)}) + \phi(x_{m(k)}), \|x_{n(k)} - T_i x_{n(k)}, a\|$$
$$\phi(x_{n(k)}) + \varphi(T_i x_{n(k)}), \|x_{m(k)} - T_j x_{m(k)}, a\| + \phi(x_{m(k)}) + \phi(T_j x_{m(k)}),$$
$$\frac{1}{2}\{\|x_{n(k)} - T_j x_{m(k)}, a\| + \phi(x_{n(k)}) + \phi(T_j x_{m(k)}), \|x_{m(k)} - T_i x_{n(k)}, a\| +$$
$$\phi(x_{m(k)}) + \phi(T_i x_{n(k)})\}$$

$$= max\{\|x_{n(k)} - x_{m(k)}, a\| + \phi(x_{n(k)}) + \phi(x_{m(k)}), \|x_{n(k)} - x_{n(k)+1}, a\| +$$
$$\phi(x_{n(k)}) + \phi(x_{n(k)+1}), \|x_{m(k)} - x_{m(k)+1}, a\| + \phi(x_{m(k)}) + \phi(x_{m(k)+1}),$$
$$\frac{1}{2}\{\|x_{n(k)} - x_{m(k)+1}, a\| + \phi(x_{n(k)}) + \phi(T x_{m(k)+1}), \|x_{m(k)} - x_{n(k)+1}, a\| +$$
$$\phi(x_{m(k)}) + \phi(x_{n(k)+1})\}$$

Taking limit as $k \to \infty$ on both sides of (15) and using (11) and (14) we have

$$\lim_{k\to\infty} Z^{(1)}(x_{n(k)}, x_{m(k)}, T_i, T_j, \phi) = \varepsilon.$$

Also it follows from (3) that

$$Z^{(2)}(x_n(k), x_{m(k)}, T_i, T_j, \phi)$$
$$= max\{\|x_n(k) - x_{m(k)}, a\| + \phi(x_n(k)) + \phi(x_m(k)), \|x_{n(k)} - T_i x_n(k), a\| +$$
$$\phi(x_{n(k)}) + \phi(T_i x_{n(k)}), \|x_{m(k)} - T_j x_{m(k)}, a\| + \phi(x_{m(k)}) + \phi(T_j x_{m(k)})\}$$
$$= max\{\|x_{n(k)} - x_{m(k)}, a\| + \phi(x_n(k)) + \phi(x_{m(k)}), \|x_{n(k)} - x_{n(k)+1}, a\| +$$
$$\phi(x_{n(k)}) + \phi(x_{n(k)+1}), \|x_{m(k)} - x_{m(k)+1}, a\| + \phi(x_{m(k)}) + \phi(x_{m(k)+1})\}$$

Taking limit as $k \to \infty$ on both sides of (17) and using (11) and (14) we have

$$\lim_{k\to\infty} Z^{(2)}(x_{n(k)}, x_{m(k)}, T_i, T_j, \phi) = \varepsilon.$$

Also from (1) we have,

$$\psi\{\|x_{n(k)+1} - x_{m(k)+1}, a\| + \phi(x_{n(k)+1}) + \phi(x_{m(k)+1})\}$$
$$\leq \psi\{Z^{(1)}(x_{n(k)}, x_{m(k)}, T_i, \phi)\} - \varphi\{Z^{(2)}(x_{n(k)}, x_{m(k)}, T_i, \phi)\}$$

Taking lower limit as $k \to \infty$ on both sides of (19) we have

$$\lim_{n\to\infty} \psi\left\{\|x_{n(k)+1} - x_{m(k)+1}, a\| + \phi(x_{n(k)+1}) + \phi(x_{m(k)+1}\right\}$$
$$\leq \lim_{n\to\infty} \psi\{Z^{(1)}(x_{n(k)}, x_{m(k)}, T_i, T_j, \phi)\} - \lim_{n\to\infty} inf \varphi\{Z^{(2)}(x_{n(k)}, x_{m(k)}, T_i, T_j, \phi)\},$$

[using (14) and (16)]
Which gives $\lim_{k\to\infty} Z^{(2)}(x_{n(k)}, x_{m(k)}, T_i, T_j, \phi) \leq 0$ which is a contradiction as

$$\lim_{k\to\infty} Z^{(2)}(x_{n(k)}, x_{m(k)}, T_i, T_j, \phi) = \varepsilon.$$

Hence $\{x_n\}$ is a Cauchy in X.
Since X is complete, there exist $z \in X$ such that $\lim_{n\to\infty} x_n = z$.
Since ϕ is lower semi-continuous, $\phi(z) \leq \lim_{n\to\infty} inf(x_n) \leq \lim_{n\to\infty} \phi(x_n) = 0$, which gives $\phi(z) = 0$.
Claim: $T_i(z) = z, \quad \forall\ i = 1,2,3, ...$
Now from (2) we have

$$Z^{(2)}(x_{n(k)}, x_{m(k)}, T_i, T_j, \phi)$$

$$= max\{\|x_n - z, a\| + \phi(x_n) + \phi(z), \|x_n - T_i x_n, a\| + \phi(x_n) + \phi(T_i x_n), \|z - T_j z, a\| + \phi(z) + \phi(T_j z), \tfrac{1}{2}\{\|x_n - T_j z, a\| + \phi(x_n) + \phi(T_j z) +, \|z - T_j x_n, a\| + \phi(z) + \phi(T_i x_n)\}\} \quad (20)$$

Taking lower as $n \to \infty$ on both side of (20) we have

$$\lim_{n\to\infty} Z^{(1)}(x_n, z, T_i, T_j, \phi) = \|z - T_j z, a\| + \phi(T_j z) \quad (21)$$

Also from (3)

$$Z^{(2)}(x_n, z, T_i, T_j, \phi) = \max\{\|x_n - z, a\| + \phi(x_n) + \phi(z), \|x_n - T_j z, a\| + \phi(x_n) + \phi(T_j x_n), \|z - T_j z, a\| + \phi(T_j z) + \phi(z)\} \quad (22)$$

Limiting as $n \to \infty$ on the both side of (22) we have

$$\lim_{n\to\infty} Z^{(2)}(x_n, z, T_i, T_j, \phi) = \|z - T_j z, a\| + \phi(T_j z) \quad (23)$$

Now from (1), we have

$$\begin{aligned} &\psi\{\|x_{n+1} - T_j z, a\| + \phi(x_{n+1}) + \phi(T_j z)\} \\ &= \psi\{\|T_i x_n - T_j z, a\| + \phi(T_i x_n) + \phi(T_j z)\} \\ &\le \psi\{Z^{(1)}(x_n, z, T_i, T_j, \phi)\} - \varphi\{Z^{(2)}(x_n, z, T_i, T_j, \phi)\} \end{aligned} \quad (24)$$

Taking lower limit as $n \to \infty$ on both side of (24) we have

$$\begin{aligned} &\lim_{n\to\infty} \psi\{\|x_{n+1} - T_j z, a\| + \phi(x_{n+1}) + \phi(T_j z)\} \\ &= \lim_{n\to\infty} \psi\{\|T_i x_n - T_j z, a\| + \phi(T_i x_n) + \phi(T_j z)\} \\ &\le \lim_{n\to\infty} \psi\{Z^{(1)}(x_n, z, T_i, T_j, \phi)\} - \lim_{n\to\infty} inf\ \varphi\{Z^{(2)}(x_n, z, T_i, T_j, \phi)\} \end{aligned}$$

Or, $\lim_{n\to\infty} \psi\{\|x_{n+1} - T_j z, a\| + \phi(x_{n+1}) + \phi(T_j z)\} \le \lim_{n\to\infty} \psi\{Z^{(1)}(x_n, z, T_i, T_j, \phi)\} - \lim_{n\to\infty} inf\ \varphi\{Z^{(2)}(x_n, z, T_i, T_j, \phi)\}$,

[As $\lim_{n\to\infty} Z^{(1)}(x_n, z, T_i, T_j, \phi) = \|z - T_j z, a\| + \phi(T_j z) = \lim_{n\to\infty} \{\|x_{n+1} - T_j z, a\| + \phi(x_{n+1}) + \phi(T_j z)\}$]

which gives $\lim_{n\to\infty} inf\ \psi\{Z^{(2)}(x_n, z, T_i, T_j, \phi)\} \le 0$, which is a contradiction.

[As $\lim_{n\to\infty} \psi\{Z^{(2)}(x_n, z, T_i, T_j, \phi)\} > 0$ implies $\lim_{n\to\infty} inf\ \varphi\{Z^{(2)}(x_n, z, T_i, T_j, \phi)\} > 0$.]
Hence $T_i(z) = z \ \ \forall\, i = 1,2,3, \ldots$

Uniqueness: To prove the uniqueness , let us assume that u and z be two fixed points of T_i i.e., $T_i z = z$ and $T_i u = u$ with $\phi(z) = 0$, $\phi(u) = 0$ such that $z \ne u$.
Then from (2) we have

$$\begin{aligned} Z^{(1)}(z, u, T_i, T_j, \phi) &= \max\Big\{\|z - u, a\| + \phi(z) + \phi(u), \|z - T_j u, a\| + \phi(u) + \phi(T_j u), \tfrac{1}{2}\{\|u - T_j u, a\| + \phi(T_j z) + \phi(u)\}\Big\} \\ &= \|z - u, a\| + \phi(z) + \phi(u) \\ &= \|z - u, a\| \end{aligned} \quad (25)$$

And from (3) we have

$$\begin{aligned} Z^{(2)}(z, u, T_i, T_j, \phi) &= \max\{\|z - u, a\| + \phi(z) + \phi(u), \|z - T_j u, a\| + \phi(z) + \phi(T_j u), \|u - T_j z, a\| + \phi(T_j z) + \phi(u)\} \\ &= \|z - u, a\| + \phi(z) + \phi(u) \\ &= \|z - u, a\| \end{aligned} \quad (26)$$

Also from (1) we have

$$\psi\{\|z-u,a\|+\phi(z)+\phi(u)\}=\psi\{\|T_i z-T_j u,a\|\}+\phi(T_i z)+\phi(T_j u)\}$$
$$\leq \psi\{Z^{(1)}(z,u,T_i,T_j,\phi)\}-\varphi\{Z^{(2)}(z,u,T_i,T_j,\phi)\}$$
$$=\psi\{\|z-u,a\|\}-\varphi\{\|z-u,a\|\},$$

which gives $\varphi\{\|z-u,a\|\}\leq 0$, which is a contradiction as $\varphi\{\|z-u,a\|\}>0$.
Hence $z=u$.
This completes the proof.

Corollary 1 Let $(X,||.,.||)$ be a 2-Banach space and $\{T_i\}_{i=1}^{\infty}$ be a sequence of self maps on X satisfying the following conditions:

$$\psi\{\|T_i^k x-T_j^k y\ ,a\|+\phi(T_i^k x)+\phi(T_j^k y)\}$$
$$\leq \psi\{Z^{(1)}\ (x,y,T_i^k,T_j^k,\phi)\}-\varphi\{Z^{(2)}\ (x,y,T_i^k,T_j^k,\phi)\},\quad \forall\, x,y\in X,$$
$$\psi\in\Psi,\phi\in\Phi;$$

where,

$$Z^{(1)}\ (x,y,T_i^k,T_j^k,\phi)$$
$$=\max\Big\{\|x-y,a\|+\phi(x)+\phi(y),\|x-T_i^k x,a\|+\phi(x)\ +\phi(T_i^k x),\ \ \|y-T_j^k\, y,a\|+\phi(y)$$
$$+\phi(T_j^k y),\ \frac{1}{2}\{\|x-T_j^k y,a\|+\phi(x)+\varphi(T_j^k y)+\ \|y-T_i^k x,a\|+\phi(y)$$
$$+\phi(T_i^k x)\Big\}$$

$$Z^{(2)}\ (x,y,T_i^k,T_j^k,\phi)$$
$$=\max\{\|x-y,a\|+\phi(x)+\phi(y),\|x-T_i^k x,a\|+\phi(x)$$
$$+\phi(T_i^k x),\ \ \|y-T_j^k\, y,a\|+\phi(y)+\phi(T_j^k\, y)\ \}$$

and $\phi: X\to[0,\infty)$ is a lower semi continuous function.
Then there exists a unique $z\in X$ such that $z=T_i\,z$, $\forall\, i=1,2,3,\ldots$ and $\phi(z)=0$.

Proof: Let $S_i=T_i^k$. Then by Theorem 2.1, the sequence $\{S_i\}_{i=1}^{\infty}$ have a unique fixed point, say $z\in X$. $T_i^k z=S_i z=z$. Then $\phi(z)=\phi(T_i^k z)=\phi(S_i z)=0$.
Since $T_i^{k+1}z=T_i\,z$, then $S_i(T_i\,z)=T_i^k(T_i\,z)$, so $T_i\,z$ is a fixed point of S_i.
By the uniqueness of the fixed point of S_i, $T_i\,z=z$, $\forall\, i=1,2,3,\ldots$

Corollary 2 Let $(X,||.,.||)$ be a 2-Banach space and T_1,T_2 be two self maps on X satisfying the following conditions:
$\psi\{\|T_1 x-T_2\,y\,,a\|+\phi(T_1 x)+\phi(T_2\,y\,)\}\leq\psi\{Z^{(1)}\ (x,y,T_1,T_2\,,\phi)\}-\varphi\{Z^{(2)}\,(x,y,T_i,T_2\,,\phi)\}$, $\forall\, x,y\in X$, $\psi\in\Psi,\phi\in\Phi$;

where,

$$Z^{(1)}\ (x,y,T_1,T_2\,,\phi)$$
$$=\max\Big\{\|x-y,a\|+\phi(x)+\phi(y),\|x-T_1 x,a\|+\phi(x)+\phi(T_1 x),\ \ \|y-T_2\,y,a\|+\phi(y)+$$
$$\phi(T_2\,y),\ \tfrac{1}{2}\{\|x-T_2 y,a\|+\phi(x)+\phi(T_2 y)+\ \|y-T_1 x,a\|+\phi(y)+\phi(T_1 x)\Big\}$$

$$Z^{(2)}\ (x,y,T_1,T_2\,,\phi)=\max\{\|x-y,a\|+\phi(x)+\phi(y),\|x-T_1 x,a\|+\phi(x)\ +\phi(T_1 x),\ \|y-T_2 y,a\|+\phi(y)+\phi(T_2\,y)\ \}$$

And $\phi: X\to[0,\infty)$ *is a lower semi continuous function.*
Then there exists a unique $z\in X$ such that $T_1 z=T_2 z=z$ with $\phi(z)=0$.

Acknowledgment

The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestion to improve the paper.

Conflicts of interest

There are no conflicts of interest in this work.

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