Research Article / Araştırma Makalesi GU J Sci, Part A, 9(2): 173-186 (2022) https://doi.org/10.54287/gujsa.1113567

Gazi University
Journal of Science
PART A: ENGINEERING AND INNOVATION
http://dergipark.org.tr/gujsa

On Approximation Properties of Stancu Type Post-Widder Operators Preserving Exponential Functions

Gülten TORUN[1*]

[1] *Kastamonu University, Faculty of Education, Mathematics and Science Education, Kastamonu, TURKEY*

Keywords

Post-Widder Operators

Stancu Type Post-Widder Pperators

Modulus of Continuity

Voronovskaja Type Theorem

Abstract

In this article, Stancu type Post-Widder operators are introduced, which are a modification of the Post-Widder operators that preserve the functions constant and e^{2ax} for fixed $a > 0$. The uniform convergence of these modified operators for the function f on $[0, \infty)$ is examined and then the convergence rate is investigated with the help of the continuity module. The Voronovskaja type asymptotic formula is obtained to examine the asymptotic behavior of these operators. Finally, numerical examples and graphs are given to show the convergence of Stancu type Post- Widder operators and compared with Post Widder operators.

Cite

Torun, G. (2022). On Approximation Properties of Stancu Type Post-Widder Operators Preserving Exponential Functions. *GU J Sci, Part A*, *9*(2), 173-186.

Author ID (ORCID Number)

G. Torun, 0000-0002-1897-0174

Article Process

Submission Date	07.05.2022
Revision Date	25.05.2022
Accepted Date	27.06.2022
Published Date	29.06.2022

1. INTRODUCTION

Linear positive operators take an important place in approximation theory. These operators are monotonous operators since they convert positive functions to positive functions. This property allows to proving inequalities for positive operators (Hacısalihoğlu & Hacıyev, 1995).

For $n \in \mathbb{N} = \{1,2, \ldots\}$ and $f \in C(0, \infty)$, Widder (1941) examined the Post-Widder operators is defined by

$$P_n(f; x) := \frac{1}{n!}\left(\frac{n}{x}\right)^{n+1} \int_0^\infty t^n e^{-\frac{nt}{x}} f(t)dt, \tag{1}$$

where $x \in (0, \infty)$, and these operators protect only fixed functions. After Widder (1941), Rathore and Sing (1980) defined the operators in the following way

$$P_n^p(f; x) := \frac{1}{(n+p)!}\left(\frac{n}{x}\right)^{n+p+1} \int_0^\infty t^{n+p} e^{-\frac{nt}{x}} f(t)dt, \tag{2}$$

where p be a fixed integer. They created the simultaneous approximation property of the operators (2) and obtained an asymptotic formula. In the case of $p = 0$, the operators (2) reduce to the operators (1). In addition for the $p = -1$ case, the operators (2) was handled by May (1976).

Rempulska and Skorupka (2009) introduced the Post-Widder and Stancu operators preserving test function x^2 in polynomial weighted space. They showed that these operators had better approximation properties than classical Post-Widder and Stancu operators.

*Corresponding Author, e-mail: gtorun@kastamonu.edu.tr

In recent years, the Post-Widder operators preserving test functions x^r for $r \in \mathbb{N}$ have been appropriately modified to get a better approximation. The approximation properties of the modified form of the Post-Widder operators have been studied by Gupta and Agrawal (2019), Gupta and Tachev (2022). In addition, the several linear operators preserving the functions constantly and e^{2ax} for fixed $a > 0$, such as Szász-Mirakyan, Baskakov, Baskakov-Schurer-Szász, Baskakov-Szász-Stancu, Baskakov-Schurer-Szász-Stancu, Post-Widder and Stancu type Szász-Mirakyan-Durrmeyer operators were studied by Acar et al. (2017), Gürel-Yılmaz et al. (2017; 2018), Bodur et al. (2018), Aral et al. (2019), Sofyalıoğlu and Kanat (2019; 2020), Gupta and Maheshwari (2019), Kanat & Sofyalıoğlu (2021), and they examined the approximation properties of these operators. Gupta and Maheshwari (2019) considered a modification of Post-Widder operators preserving the exponential functions. They made a direct estimate and proved the quantitative asymptotic formula for these modified operators. The case $p = -1$ of Post-Widder operators (2) preserving constant and e^{-2ax} for fixed $a > 0$ has been handled by Sofyalıoğlu and Kanat (2020). They investigated the convergence behavior of modified Post-Widder operators and the convergence ratio using different module types. Finally, they compared their newly established operators with the Post-Widder operators which preserve x^r for $r \in \mathbb{N}$.

In this article, the approximation properties of the Stancu type Post-Widder operators that preserve the functions constant and e^{2ax} for fixed $a > 0$ are examined and the Voronovskaja type approximation theorem is given for the asymptotic behavior of these operators.

Several studies were conducted on Voronovskaja type approximation for some operators by Dinlemez Kantar and Ergelen (2019), Cai et al. (2020; 2021a, 2021b), Sofyalıoğlu et al. (2021), Dinlemez Kantar and Yüksel (2022), Torun et al. (2022).

Let be defined the Stancu type Post-Widder operators for $n \in \mathbb{N}$ and $x \in [0, \infty)$ as

$$P_{n,p,\Psi}^{\alpha,\beta}(f;x) = \frac{1}{(n+p)!}\left(\frac{n}{\Psi_n(x)}\right)^{n+p+1} \int_0^\infty t^{n+p}\, e^{\frac{-nt}{\Psi_n(x)}} f\left(\frac{nt+\alpha}{n+\beta}\right) dt, \tag{3}$$

where the real-valued function f is a bounded function over the interval $[0, \infty)$, α and β positive real numbers satisfying $0 \le \alpha \le \beta$, and p is a constant integer such that $p < n$. For $a > 0$, assume that operators (3) preserve the function e^{2ax}. It can be easily seen that the conditions

$$P_{n,p,\Psi}^{\alpha,\beta}(e^{2at};x) = e^{2ax}$$

are satisfied. In this case, it would be

$$\begin{aligned} P_{n,p,\Psi}^{\alpha,\beta}(e^{2at};x) = e^{2ax} &= \frac{1}{(n+p)!}\left(\frac{n}{\Psi_n(x)}\right)^{n+p+1} \int_0^\infty t^{n+p}\, e^{-\frac{nt}{\Psi_n(x)}} e^{2a\frac{nt+\alpha}{n+\beta}} dt \\ &= \frac{e^{\frac{2a\alpha}{n+\beta}}}{(n+p)!}\left(\frac{n}{\Psi_n(x)}\right)^{n+p+1} \int_0^\infty t^{n+p}\, e^{-\frac{(n+\beta)-2a\Psi_n(x)}{(n+\beta)\Psi_n(x)}nt} dt \\ &= e^{\frac{2a\alpha}{n+\beta}}\left(\frac{n+\beta}{n+\beta-2a\Psi_n(x)}\right)^{n+p+1}, \end{aligned}$$

where $\frac{n+\beta}{\Psi_n(x)} > 2a$. With a simple calculation, the function $\Psi_n(x)$ is obtained as follows:

$$\Psi_n(x) = \frac{n+\beta}{2a}\left(1 - \left(e^{\frac{2a(x(n+\beta)-\alpha)}{n+\beta}}\right)^{\frac{-1}{n+p+1}}\right). \tag{4}$$

And it can be easily shown that $\lim_{n\to\infty} \Psi_n(x) = x$.

If the function $\Psi_n(x)$ given in (4) is replaced in (3), the Stancu type Post-Widder operators take the form

$$P_{n,p}^{\alpha,\beta}(f;x) := P_{n,p,\Psi}^{\alpha,\beta}(f;x)$$

$$= \frac{1}{(n+p)!}\left(\frac{2an}{(n+\beta)\left(1-e^{\frac{-2a(x(n+\beta)-\alpha)}{(n+\beta)(n+p+1)}}\right)}\right)^{n+p+1} \int_0^\infty t^{n+p}\, e^{\frac{-2ant}{(n+\beta)\left(1-e^{\frac{-2a(x(n+\beta)-\alpha)}{(n+\beta)(n+p+1)}}\right)}} f\left(\frac{nt+\alpha}{n+\beta}\right) dt. \quad (5)$$

2. SOME PRELIMINARY RESULTS

In this section, several lemmas and their results necessary to prove the main theorem are given.

Lemma 2.1 Let $\Psi_n(x)$ be function given in (4). The Stancu type Post-Widder operators (3) give the following equations:

$$P_{n,p,\Psi}^{\alpha,\beta}\left(e^{\phi t};x\right) = \mathrm{e}^{\frac{\phi\alpha}{n+\beta}}\left(1-\frac{\phi\Psi_n(x)}{n+\beta}\right)^{-(n+p+1)}, \ \ \phi \in \mathbb{R}. \quad (6)$$

Proof: Let $f\ (t) = e^{\phi t},\ \phi \in \mathbb{R}$. From the operators (3), the following equation is given by

$$P_{n,p,\Psi}^{\alpha,\beta}\left(e^{\phi t};x\right) = \frac{1}{(n+p)!}\left(\frac{n}{\Psi_n(x)}\right)^{n+p+1} \int_0^\infty t^{n+p}\, e^{-\frac{nt}{\Psi_n(x)}} e^{\phi\frac{nt+\alpha}{n+\beta}} dt$$

$$= \frac{1}{(n+p)!}\left(\frac{n}{\Psi_n(x)}\right)^{n+p+1} e^{\frac{\phi\alpha}{n+\beta}} \int_0^\infty t^{n+p}\, e^{-\left(\frac{1}{\Psi_n(x)}-\frac{\phi}{n+\beta}\right)nt} dt.$$

By substituting the variable $\left(\frac{1}{\Psi_n(x)}-\frac{\phi}{n+\beta}\right)nt = u$ in the above integral and then using the gamma function,

$$P_{n,p,\Psi}^{\alpha,\beta}\left(e^{\phi t};x\right) = \frac{e^{\frac{\phi\alpha}{n+\beta}}}{(n+p)!}\left(\frac{n}{\Psi_n(x)}\left(\frac{n+\beta-\phi\Psi_n(x)}{\Psi_n(x)(n+\beta)}n\right)^{-1}\right)^{n+p+1} \int_0^\infty u^{n+p}\, e^{-u} du$$

$$= \frac{e^{\frac{\phi\alpha}{n+\beta}}}{(n+p)!}\left(\frac{n+\beta-\phi\Psi_n(x)}{n+\beta}\right)^{-(n+p+1)} \Gamma\,(n+p+1) = e^{\frac{\phi\alpha}{n+\beta}}\left(1-\frac{\phi\Psi_n(x)}{n+\beta}\right)^{-(n+p+1)}$$

is obtained.

Lemma 2.2 Let $e_j(t) = t^j\ ,\ j = 0,1,2,3,4$. The moments of the Stancu type Post-Widder operators (3) are obtained as follows:

$(\boldsymbol{i})$ $\quad P_{n,p,\Psi}^{\alpha,\beta}(e_0(t);x) = 1,$

$(\boldsymbol{ii})$ $\quad P_{n,p,\Psi}^{\alpha,\beta}(e_1(t);x) = \frac{n+p+1}{n+\beta}\Psi_n(x) + \frac{\alpha}{n+\beta},$

$(\boldsymbol{iii})$ $\quad P_{n,p,\Psi}^{\alpha,\beta}(e_2(t);x) = \frac{(n+p+2)(n+p+1)}{(n+\beta)^2}\Psi_n^2(x) + \frac{2\alpha(n+p+1)}{(n+\beta)^2}\Psi_n(x) + \frac{\alpha^2}{(n+\beta)^2},$

$(\boldsymbol{iv})$ $\quad P_{n,p,\Psi}^{\alpha,\beta}(e_3(t);x) = \frac{(n+p+3)(n+p+2)(n+p+1)}{(n+\beta)^3}\Psi_n^3(x) + \frac{3\alpha(n+p+2)(n+p+1)}{(n+\beta)^3}\Psi_n^2(x)$

$$+\frac{3\alpha^2(n+p+1)}{(n+\beta)^3}\Psi_n(x) + \frac{\alpha^3}{(n+\beta)^3},$$

$(\boldsymbol{v})$ $\quad P_{n,p,\Psi}^{\alpha,\beta}(e_4(t);x) = \frac{(n+p+4)(n+p+3)(n+p+2)(n+p+1)}{(n+\beta)^4}\Psi_n^4(x) + \frac{4\alpha(n+p+3)(n+p+2)(n+p+1)}{(n+\beta)^4}\Psi_n^3(x)$

$$+\frac{6\alpha^2(n+p+2)(n+p+1)}{(n+\beta)^4}\Psi_n^2(x)+\frac{4\alpha^3(n+p+1)}{(n+\beta)^4}\Psi_n(x)+\frac{\alpha^4}{(n+\beta)^4}.$$

Proof: $(\boldsymbol{i})$ Taking $e_j(t)=t^j,\ j=0$ in operators $P_{n,p,\Psi}^{\alpha,\beta}(e_j(t);x)$, the following equation can be obtained by

$$P_{n,p,\Psi}^{\alpha,\beta}(e_0(t);x)=\frac{1}{(n+p)!}\left(\frac{n}{\Psi_n(x)}\right)^{n+p+1}\int_0^\infty t^{n+p}\,e^{-\frac{nt}{\Psi_n(x)}}\,dt.$$

Substituting $\frac{n}{\Psi_n(x)}t=u$ and then from the gamma function, it is easily found that

$$P_{n,p,\Psi}^{\alpha,\beta}(e_0(t);x)=\frac{1}{(n+p)!}\left(\frac{n}{\Psi_n(x)}\right)^{n+p+1}\left(\frac{\Psi_n(x)}{n}\right)^{n+p+1}\int_0^\infty u^{n+p}\,e^{-u}\,du=\frac{1}{(n+p)!}\Gamma(\mathrm{n}+\mathrm{p}+1)=1$$

$(\boldsymbol{ii})$ The operators $P_{n,p,\Psi}^{\alpha,\beta}(e_j(t);x)$ for $j=1$ are yielded as follows:

$$P_{n,p,\Psi}^{\alpha,\beta}(e_1(t);x)=\frac{1}{(n+p)!}\left(\frac{n}{\Psi_n(x)}\right)^{n+p+1}\int_0^\infty t^{n+p}\,e^{-\frac{nt}{\Psi_n(x)}}\left(\frac{nt+\alpha}{n+\beta}\right)dt$$

$$=\frac{n}{(n+p)!(n+\beta)}\left(\frac{n}{\Psi_n(x)}\right)^{n+p+1}\int_0^\infty t^{n+p+1}\,e^{-\frac{nt}{\Psi_n(x)}}\,dt+\frac{\alpha}{(n+p)!(n+\beta)}\left(\frac{n}{\Psi_n(x)}\right)^{n+p+1}\int_0^\infty t^{n+p}\,e^{-\frac{nt}{\Psi_n(x)}}dt.$$

From (i), the value of the second sum on the right side of the above equation is $\frac{\alpha}{n+\beta}$. In the first integral on the right side of the above equation, the variable $\frac{n}{\Psi_n(x)}t=u$ is changed. And then using the gamma function, the following equation is obtained:

$$P_{n,p,\Psi}^{\alpha,\beta}(e_1(t);x)=\frac{n+p+1}{n+\beta}\Psi_n(x)+\frac{\alpha}{n+\beta}.$$

$(\boldsymbol{iii})$ For $j=2$, the operators $P_{n,p,\Psi}^{\alpha,\beta}(e_j(t);x)$ are written as follows:

$$P_{n,p,\Psi}^{\alpha,\beta}(e_2(t);x)=\frac{1}{(n+p)!}\left(\frac{n}{\Psi_n(x)}\right)^{n+p+1}\int_0^\infty t^{n+p}\,e^{-\frac{nt}{\Psi_n(x)}}\left(\frac{nt+\alpha}{n+\beta}\right)^2dt$$

$$=\frac{n^2}{(n+p)!(n+\beta)^2}\left(\frac{n}{\Psi_n(x)}\right)^{n+p+1}\int_0^\infty t^{n+p+2}\,e^{-\frac{nt}{\Psi_n(x)}}\,dt+\frac{2n\alpha}{(n+p)!(n+\beta)^2}\left(\frac{n}{\Psi_n(x)}\right)^{n+p+1}\int_0^\infty t^{n+p+1}\,e^{-\frac{nt}{\Psi_n(x)}}dt$$

$$\left.+\frac{\alpha^2}{(n+p)!(n+\beta)^2}\left(\frac{n}{\Psi_n(x)}\right)^{n+p+1}\int_0^\infty t^{n+p}\,e^{-\frac{nt}{\Psi_n(x)}}\,dt\right).$$

In the first integral on the right side of the above equation, the variable $\frac{n}{\Psi_n(x)}t=u$ is changed and then the gamma function is used. From (i) and (ii),

$$P_{n,p,\Psi}^{\alpha,\beta}(e_2(t);x)=\frac{(n+p+2)(n+p+1)}{(n+\beta)^2}\Psi_n^2(x)+\frac{2\alpha(n+p+1)}{(n+\beta)^2}\Psi_n(x)+\frac{\alpha^2}{(n+\beta)^2}.$$

Similarly, the equations (iv) and (v) are proved.

Corollary 2.3 Let $\varphi_x^j(t)=(t-x)^j$, $j=0,1,2,4$. The central moments of Stancu type Post-Widder operators (3) are bellowed

$$P_{n,p,\Psi}^{\alpha,\beta}(\varphi_x^0(t);x)=1,$$

$$P_{n,p,\Psi}^{\alpha,\beta}(\varphi_x^1(t);x)=\frac{(n+p+1)\Psi_n(x)+\alpha}{n+\beta}-x,$$

$$P_{n,p,\Psi}^{\alpha,\beta}(\varphi_x^2(t);x)=\frac{(n+p+2)(n+p+1)\Psi_n^2(x)+2\alpha(n+p+1)\Psi_n(x)+\alpha^2}{(n+\beta)^2}-2x\frac{(n+p+1)\Psi_n(x)+\alpha}{n+\beta}+x^2,$$

$$P_{n,p,\Psi}^{\alpha,\beta}(\varphi_x^4(t);x)=\frac{(n+p+4)(n+p+3)(n+p+2)(n+p+1)\Psi_n^4(x)+4\alpha(n+p+3)(n+p+2)(n+p+1)\Psi_n^3(x)}{(n+\beta)^4}$$

$$+\frac{6\alpha^2(n+p+2)(n+p+1)\Psi_n^2(x)+4\alpha^3(n+p+1)\Psi_n(x)+\alpha^4}{(n+\beta)^4}$$

$$-4x\left(\frac{(n+p+3)(n+p+2)(n+p+1)\Psi_n^3(x)+3\alpha(n+p+2)(n+p+1)\Psi_n^2(x)+3\alpha^2(n+p+1)\Psi_n(x)+\alpha^3}{(n+\beta)^3}\right)$$

$$+6x^2\left(\frac{(n+p+2)(n+p+1)\Psi_n^2(x)+2\alpha(n+p+1)\Psi_n(x)+\alpha^2}{(n+\beta)^2}\right)-4x^3\left(\frac{(n+p+1)\Psi_n(x)+\alpha}{n+\beta}\right)+x^4.$$

In addition, considering the equation $\Psi_n(x)$ defined in (4), the following limits are obtained:

$$(\boldsymbol{i})\quad \lim_{n\to\infty} nP_{n,p,\Psi}^{\alpha,\beta}(t-x;x)=-ax^2 \tag{7}$$

$$(\boldsymbol{ii})\quad \lim_{n\to\infty} nP_{n,p,\Psi}^{\alpha,\beta}((t-x)^2;x)=x^2 \tag{8}$$

$$(\boldsymbol{iii})\quad \lim_{n\to\infty} n^2P_{n,p,\Psi}^{\alpha,\beta}((t-x)^4;x)=3x^4 \tag{9}$$

3. THE UNIFORM CONVERGENCE OF THE OPERATORS $P_{n,p,\Psi}^{\alpha,\beta}$

Boyanov and Veselinov (1970) showed uniform convergence of the sequence of linear positive operators. In the following theorem, the uniform convergence of the Stancu type Post-Widder operators (3) for the function f on $[0,\infty)$ is investigated. Let the subspace of all continuous and real-valued functions on $[0,\infty)$ be denoted by $C^*[0,\infty)$ with the property that $\lim_{x\to\infty} f(x)$ exists and finite, given with the uniform norm.

Theorem 3.1 If the sequence of the Stancu type Post-Widder operators (3) satisfy

$$\lim_{n\to\infty} P_{n,p,\Psi}^{\alpha,\beta}(e^{-\nu t};x)=e^{-\nu x},\ \ \nu=0,1,2 \tag{10}$$

uniformly in $[0,\infty)$, then for each $f\in C^*[0,\infty)$

$$\lim_{n\to\infty} P_{n,p,\Psi}^{\alpha,\beta}(f;x)=f(x) \tag{11}$$

uniformly in $[0,\infty)$.

Proof: For $\nu=0$, it becomes that $\lim_{n\to\infty} P_{n,p,\Psi}^{\alpha,\beta}(1;x)=1$ from (i) of Lemma 2.2. Now the equation (6), and $\Psi_n(x)$ defined with (4) will be used to prove the images of $f(t)=e^{-\nu t}$ for $\nu=1,2$ respectively,

$$P_{n,p,\Psi}^{\alpha,\beta}(e^{-t};x)=e^{-\frac{\alpha}{n+\beta}}\left(1+\frac{1}{2a}\left(1-e^{\frac{-2a(x(n+\beta)-\alpha)}{(n+\beta)(n+p+1)}}\right)\right)^{-(n+p+1)} \tag{12}$$

and

$$P_{n,p,\Psi}^{\alpha,\beta}(e^{-2t};x)=e^{-\frac{2\alpha}{n+\beta}}\left(1+\frac{1}{a}\left(1-e^{\frac{-2a(x(n+\beta)-\alpha)}{(n+\beta)(n+p+1)}}\right)\right)^{-(n+p+1)}. \tag{13}$$

Using the software Maple to calculate the right side of the equation (12), the following equation is obtained:

$$P_{n,p,\Psi}^{\alpha,\beta}(e^{-t};x)=e^{-x}+\frac{(2a+1)e^{-x}x^2}{2n}+\frac{(2a+1)\left(3(2a+1)e^{-x}x^4-8(a+1)e^{-x}x^3-12(p+1)e^{-x}x^2-24\alpha e^{-x}x\right)}{24n^2}+O\left(\frac{1}{n^3}\right). \tag{14}$$

Similarly, again using Maple to calculate the right side of the equation (13),

$$P_{n,p,\Psi}^{\alpha,\beta}(e^{-2t};x)=e^{-2x}+\frac{2(a+1)e^{-2x}x^2}{n}+\frac{(a+1)\left(6(a+1)e^{-2x}x^4-4(a+2)e^{-2x}x^3-6(p+1)e^{-2x}x^2-12\alpha e^{-2x}x\right)}{3n^2}+O\left(\frac{1}{n^3}\right) \tag{15}$$

is found. Thus, $\lim_{n\to\infty} P_{n,p,\Psi}^{\alpha,\beta}(e^{-\nu t};x)=e^{-\nu x}$, $\nu=0,1,2$ in the interval $[0;\infty)$. That is,

$$\lim_{n\to\infty} P_{n,p,\Psi}^{\alpha,\beta}(f;x)=f$$

for any $f\in C^*[0;\infty)$. This indicates that the sequence $\left\{P_{n,p,\Psi}^{\alpha,\beta}f\right\}$ uniformly converges in the interval $[0,\infty)$ for any $f\in C^*[0,\infty)$.

After Boyanov and Veselinov (1970), Holhoş (2010) studied the uniform convergence of a sequence of linear positive operators and obtained the following theorem.

Theorem 3.2 If $\{P_n\}$ is a sequence of linear positive operators from $C^*[0,\infty)$, to $C^*[0,\infty)$, then for each $f\in C^*[0,\infty)$, the following inequality is satisfied:

$$\|P_n(f;x)-f(x)\|_{[0,\infty)}\le\|f\|_{[0,\infty)}a_n+(2+a_n)\omega^*\left(f,\sqrt{a_n+2b_n+c_n}\right),$$

where a_n, b_n and c_n are defined as follows:

$$\|P_n(1;x)-1\|_{[0,\infty)}=a_n,$$

$$\|P_n(e^{-t};x)-e^{-x}\|_{[0,\infty)}=b_n,$$

$$\|P_n(e^{-2t};x)-e^{-2x}\|_{[0,\infty)}=c_n$$

and they approach zero as n goes to infinity. In addition, the modulus of continuity is expressed by

$$\omega^*(f,\gamma)=sup_{|e^{-t}-e^{-x}|\le\gamma;\,x,t\ge0}\ |f(t)-f(x)| \tag{16}$$

and this modulus has to property:

$$|f(t)-f(x)|\le\left(1+\frac{1}{\gamma^2}(e^{-t}-e^{-x})^2\right)\omega^*(f,\gamma),\quad \gamma>0 \tag{17}$$

The main result of the uniform convergence of a sequence of linear positive operators is given by the following theorem.

Theorem 3.3 Let $\left\{P_{n,p,\Psi}^{\alpha,\beta}f\right\}$ be a sequence of linear positive operators $P_{n,p,\Psi}^{\alpha,\beta}$: $C^*[0,\infty) \to C^*[0,\infty)$. For every function $f \in C^*[0;\infty)$, the following inequality is satisfied:

$$\left\|P_{n,p,\Psi}^{\alpha,\beta}(f;x)-f(x)\right\|_{[0,\infty)} \leq 2\omega^*\left(f,\sqrt{2\sigma_n+\mu_n}\right), \tag{18}$$

where the modulus of continuity ω^* which is defined in (16) and

$$\left\|P_{n,p,\Psi}^{\alpha,\beta}(e^{-t};x)-e^{-x}\right\|_{[0,\infty)} = \sigma_n,$$

$$\left\|P_{n,p,\Psi}^{\alpha,\beta}(e^{-2t};x)-e^{-2x}\right\|_{[0,\infty)} = \mu_n,$$

Here σ_n and μ_n tend to zero as n goes to infinity and the sequence $\left\{P_{n,p,\Psi}^{\alpha,\beta} f\right\}$ uniformly converges to f.

Proof: From (i) of Lemma 2.2,

$$\rho_n = \left\|P_{n,p,\Psi}^{\alpha,\beta}(1;x)-1\right\|_{[0,\infty)} = 0$$

is obtained. To calculate σ_n and μ_n, the equalities (14) and (15) are taken respectively

$$\sigma_n = \left\|P_{n,p,\Psi}^{\alpha,\beta}(e^{-t};x)-e^{-x}\right\|_{[0,\infty)} = sup_{x\in[0;\infty)}\left|P_{n,p,\Psi}^{\alpha,\beta}(e^{-t};x)-e^{-x}\right|$$

$$= sup_{x\in[0;\infty)}\left|\frac{(2a+1)e^{-x}x^2}{2n}+\frac{(2a+1)\left(3(2a+1)e^{-x}x^4-8(a+1)e^{-x}x^3-12(p+1)e^{-x}x^2-24\alpha e^{-x}x\right)}{24n^2}+O\left(\frac{1}{n^3}\right)\right|$$

$$\leq \frac{2(2a+1)}{ne^2}+\frac{2a+1}{n^2}\left(\frac{32(2a+1)}{e^4}-\frac{9(a+1)}{e^3}-\frac{p+1}{e^2}-\frac{\alpha}{e}\right)+O\left(\frac{1}{n^3}\right),$$

$$\mu_n = \left\|P_{n,p,\Psi}^{\alpha,\beta}(e^{-2t};x)-e^{-2x}\right\|_{[0,\infty)} = sup_{x\in[0;\infty)}\left|P_{n,p,\Psi}^{\alpha,\beta}(e^{-2t};x)-e^{-2x}\right|$$

$$= sup_{x\in[0;\infty)}\left|\frac{2(a+1)e^{-2x}x^2}{n}+\frac{(a+1)\left(6(a+1)e^{-2x}x^4-4(a+2)e^{-2x}x^3-6(p+1)e^{-2x}x^2-12\alpha e^{-2x}x\right)}{3n^2}+O\left(\frac{1}{n^3}\right)\right|$$

$$\leq \frac{2(a+1)}{ne^2}+\frac{a+1}{n^2}\left(\frac{32(a+1)}{e^4}-\frac{\frac{9}{2}(a+2)}{e^3}-\frac{2(p+1)}{e^2}-\frac{2\alpha}{e}\right)+O\left(\frac{1}{n^3}\right)$$

As a consequence, σ_n and μ_n tend to zero as n goes to infinity. Thus, the Theorem is proved.

4. APPROXIMATION PROPERTIES OF THE OPERATORS $P_{n,p,\Psi}^{\alpha,\beta}$

In this section, firstly, the convergence rate is examined with the help of the continuity module.

Let $C_B[0,\infty)$ be the class of all bounded and uniform continuous functions f on $[0,\infty)$ with the norm

$\|f\|_{C_B} = sup_{x\in[0;\infty)} |f(x)|$. For $\varepsilon > 0$, the Peetre K-functional is defined as

$$K_2(f,\varepsilon) := inf_{g\in C_B^2[0,\infty)}\left[\|f-g\|+\varepsilon\|g''\|\right],$$

where $C_B^2[0,\infty) := \{g \in C_B[0,\infty): g', g'' \in C_B[0,\infty)\}$.

The first-order modulus of continuity of $f \in C_B[0,\infty)$ is defined as follows:

$$\omega(f,\varepsilon):=sup_{0<h\leq\varepsilon}\, sup_{x,x+h\epsilon[0,\infty)}|\, f\,(x+h)-f\,(x)|.$$

The second-order modulus of continuity of $f\in C_B[0,\infty)$ is given by

$$\omega_2\big(f,\sqrt{\varepsilon}\big):=sup_{0<h\leq\sqrt{\varepsilon}}\, sup_{x,x+h,x+2h\epsilon[0,\infty)}|\, f\,(x+2h)-2f\,(x+h)+f\,(x)|.$$

In Theorem 2.4 given by DeVore and Lorentz (1993) is proved that there exists an absolute constant $C>0$ such that

$$K_2(f,\varepsilon)\leq C\omega_2\big(f,\sqrt{\varepsilon}\big). \tag{19}$$

Lemma 4.1 For $f\in\ C_B[0,\infty)$, the following inequality is obtained:

$$\left|P_{n,p,\Psi}^{\alpha,\beta}(f\,;x)\right|\leq\|\,f\|. \tag{20}$$

Proof:

$$\left|P_{n,p,\Psi}^{\alpha,\beta}(f\,;x)\right|=\left|P_{n,p}^{\alpha,\beta}\,(f\,;x)\right|$$

$$\leq\frac{1}{(n+p)!}\left(\frac{2an}{(n+\beta)\left(1-e^{\frac{-2a(x(n+\beta)-\alpha)}{(n+\beta)(n+p+1)}}\right)}\right)^{n+p+1}\int_0^\infty t^{n+p}\,e^{\frac{-2ant}{(n+\beta)\left(1-e^{\frac{-2a(x(n+\beta)-\alpha)}{(n+\beta)(n+p+1)}}\right)}}\left|f\left(\frac{nt+\alpha}{n+\beta}\right)\right|dt.$$

$$\leq\|\,f\,\|\,P_{n,p,\Psi}^{\alpha,\beta}(1;x)=\|\,f\|.$$

Theorem 4.2 For $f\in\ C_B[0,\infty)$, there exists a positive constant L, such that

$$\left|P_{n,p,\Psi}^{\alpha,\beta}(f\,;x)-f(x)\right|\leq L\omega_2\left(f,\sqrt{\frac{\xi_{n,p,\Psi}^{\alpha,\beta}}{8}}\right)+\omega\left(f,\left|\frac{n+p+1}{2a}\left(1-e^{\frac{-2a(x(n+\beta)-\alpha)}{(n+\beta)(n+p+1)}}\right)+\frac{\alpha}{n+\beta}-x\right|\right),$$

where

$$\xi_{n,p,\Psi}^{\alpha,\beta}=\frac{(n+p+1)(2n+2p+3)}{(n+\beta)^2}\Psi_n^2(x)+4\left(\frac{\alpha(n+p+1)}{(n+\beta)^2}-\frac{x(n+p+1)}{n+\beta}\right)\Psi_n(x)+\frac{2\alpha^2}{(n+\beta)^2}-\frac{4x\alpha}{n+\beta}+2x^2.$$

Proof: Let the auxiliary operators $\tilde{P}_{n,p,\Psi}^{\alpha,\beta}$ from $C_B[0,\infty)$ to $C_B[0,\infty)$ be defined as

$$\tilde{P}_{n,p,\Psi}^{\alpha,\beta}(g;x)=P_{n,p,\Psi}^{\alpha,\beta}(\,g\,;x)+g(x)-g\left(\frac{n+p+1}{2a}\left(1-e^{\frac{-2a(x(n+\beta)-\alpha)}{(n+\beta)(n+p+1)}}\right)+\frac{\alpha}{n+\beta}\right). \tag{21}$$

Using the equalities (i) and (ii) of Lemma 2.2 and the linearity of the operators $\tilde{P}_{n,p,\Psi}^{\alpha,\beta}(g;x)$,

$$\tilde{P}_{n,p,\Psi}^{\alpha,\beta}(t-x;x)=0 \tag{22}$$

is obtained. Using Taylor expansion for $g\in C_B^2[0,\infty)$, it can be written as

$$g(t)=g(x)+g'(x)(t-x)+\int_x^t(t-u)g''(u)du. \tag{23}$$

Applying the auxiliary operators (21) to both sides of equation (23) and using (22), it is obtained

$$\tilde{P}_{n,p,\Psi}^{\alpha,\beta}(g;x) = g(x) + \tilde{P}_{n,p,\Psi}^{\alpha,\beta}\left(\int_x^t (t-u)g''(u)du\,; x\right).$$

$$\left|\tilde{P}_{n,p,\Psi}^{\alpha,\beta}(g;x) - g(x)\right| \leq \left|P_{n,p,\Psi}^{\alpha,\beta}\left(\int_x^t (t-u)g''(u)du\,; x\right)\right| + \left|\int_x^{P_{n,p,\Psi}^{\alpha,\beta}(e_1;x)} \left(P_{n,p,\Psi}^{\alpha,\beta}(e_1;x) - u\right) g''(u)du\right|$$

$$\leq \frac{\|g''\|}{2} P_{n,p,\Psi}^{\alpha,\beta}((t-x)^2;x) + \frac{\|g''\|}{2}\left(\frac{n+p+1}{2a}\left(1 - e^{\frac{-2a(x(n+\beta)-\alpha)}{(n+\beta)(n+p+1)}}\right) + \frac{\alpha}{n+\beta} - x\right)^2$$

$$\leq \frac{\|g''\|}{2}\left(\left(P_{n,p,\Psi}^{\alpha,\beta}((t-x)^2;x)\right) + \left(P_{n,p,\Psi}^{\alpha,\beta}(t-x;x)\right)^2\right) = \frac{\|g''\|}{2}\ \xi_{n,p,\Psi}^{\alpha,\beta}\,, \qquad (24)$$

where

$$\xi_{n,p,\Psi}^{\alpha,\beta} = \frac{(n+p+1)(2n+2p+3)}{(n+\beta)^2}\Psi_n^2(x) + 4\left(\frac{\alpha(n+p+1)}{(n+\beta)^2} - \frac{x(n+p+1)}{n+\beta}\right)\Psi_n(x) + \frac{2\alpha^2}{(n+\beta)^2} - \frac{4x\alpha}{n+\beta} + 2x^2.$$

Taking the norm of the auxiliary operators (21) and using Lemma (4.1), the following inequality is obtained:

$$\left\|\tilde{P}_{n,p,\Psi}^{\alpha,\beta}(f;x)\right\| \leq 3\|f\|,\ f \in C_B[0,\infty). \qquad (25)$$

Using the operators (21) and the inequalities (24) and (25), for every $g \in C_B^2[0,\infty)$, it can be written as

$$\left|P_{n,p,\Psi}^{\alpha,\beta}(f;x) - f(x)\right| \leq \left|\tilde{P}_{n,p,\Psi}^{\alpha,\beta}(f-g;x) - (f-g)(x)\right|$$

$$+\left|f\left(\frac{n+p+1}{2a}\left(1 - e^{\frac{-2a(x(n+\beta)-\alpha)}{(n+\beta)(n+p+1)}}\right) + \frac{\alpha}{n+\beta}\right) - f(x)\right| + \left|\tilde{P}_{n,p,\Psi}^{\alpha,\beta}(g;x) - g(x)\right|$$

$$\leq 4\|f-g\| + \frac{\|g''\|}{2}\xi_{n,p,\Psi}^{\alpha,\beta} + \left|f\left(\frac{n+p+1}{2a}\left(1 - e^{\frac{-2a(x(n+\beta)-\alpha)}{(n+\beta)(n+p+1)}}\right) + \frac{\alpha}{n+\beta}\right) - f(x)\right|. \quad (26)$$

If the infimum on the right side of (26) over all the function $g \in C_B^2[0,\infty)$ is taken, then

$$\left|P_{n,p,\Psi}^{\alpha,\beta}(f;x) - f(x)\right| \leq 4K_2\left(f, \frac{\xi_{n,p,\Psi}^{\alpha,\beta}}{8}\right) + \omega\left(f, \left|\frac{n+p+1}{2a}\left(1 - e^{\frac{-2a(x(n+\beta)-\alpha)}{(n+\beta)(n+p+1)}}\right) + \frac{\alpha}{n+\beta} - x\right|\right)$$

$$\leq L\omega_2\left(f, \sqrt{\frac{\xi_{n,p,\Psi}^{\alpha,\beta}}{8}}\right) + \omega\left(f, \left|\frac{n+p+1}{2a}\left(1 - e^{\frac{-2a(x(n+\beta)-\alpha)}{(n+\beta)(n+p+1)}}\right) + \frac{\alpha}{n+\beta} - x\right|\right),$$

where L is a positive constant. Thus, the Theorem is proved.

Remark 4.3 Since $\lim_{n\to\infty} \xi_{n,p,\Psi}^{\alpha,\beta} = 0$ and $\lim_{n\to\infty}\left(\frac{n+p+1}{2a}\left(1 - e^{\frac{-2a(x(n+\beta)-\alpha)}{(n+\beta)(n+p+1)}}\right) + \frac{\alpha}{n+\beta} - x\right) = 0$, these limits guarantees a rate of pointwise convergence of the operators $P_{n,p,\Psi}^{\alpha,\beta}(f;x)$ to $f\ (x)$.

Finally, the Voronovskaja-type theorem is given to examine the asymptotic behavior of the Stancu type Post-Widder operators (3).

Theorem 4.4 For each $f \in C^*[0,\infty)$ and $x \in [0,\infty)$, the following inequality holds:

$$\left|n\left(P_{n,p,\Psi}^{\alpha,\beta}(f;x) - f\ (x)\right) + ax^2 f\,'(x) - \frac{x^2}{2} f\,''(x)\right| \leq |u_n(x)||\,f\,'(x)| + \frac{1}{2}|v_n(x)||\ f\,''(x)|$$

$$+2\left(v_n(x)+x^2+w_n(x)\right)\omega^*\left(f'',n^{-\frac{1}{2}}\right),$$

where f', f'' exists in $C^*[0,\infty)$, and

$$u_n(x)=nP_{n,p,\Psi}^{\alpha,\beta}(t-x;x)+ax^2,$$

$$v_n(x)=nP_{n,p,\Psi}^{\alpha,\beta}((t-x)^2;x)-x^2,$$

$$w_n(x)=\left(n^2P_{n,p,\Psi}^{\alpha,\beta}((e^{-t}-e^{-x})^4;x)\right)^{\frac{1}{2}}\left(n^2P_{n,p,\Psi}^{\alpha,\beta}((t-x)^4;x)\right)^{\frac{1}{2}}.$$

Proof: By Taylor's formula for a function f, the following equation can be written:

$$f(t)=f(x)+f'(x)(t-x)+\frac{f''(x)}{2}(t-x)^2+r(t,x)(t-x)^2, \tag{27}$$

where Peano form of the remainder $r(t,x)$ is defined by

$$r(t,x):=\frac{f''(\tau)-f''(x)}{2}\,,\quad x<\tau<t$$

and the limit value of the remainder term $r(t,x)$ is 0 as t approaches x. By applying the operators $P_{n,p,\Psi}^{\alpha,\beta}(f;x)$ on both sides of the equation in (27), the following equality is obtained:

$$P_{n,p,\Psi}^{\alpha,\beta}(f;x)-f(x)=f'(x)P_{n,p,\Psi}^{\alpha,\beta}(t-x;x)+\frac{f''(x)}{2}P_{n,p,\Psi}^{\alpha,\beta}((t-x)^2;x)+P_{n,p,\Psi}^{\alpha,\beta}(r(t,x)(t-x)^2;x).$$

Taking into account the Corollary (2.3), the following inequality can be written:

$$\left|n\left(P_{n,p,\Psi}^{\alpha,\beta}(f;x)-f(x)\right)+ax^2f'(x)-\frac{x^2}{2}f''(x)\right|\le\left|nP_{n,p,\Psi}^{\alpha,\beta}(t-x;x)+ax^2\right||f'(x)|$$
$$+\frac{1}{2}\left|nP_{n,p,\Psi}^{\alpha,\beta}((t-x)^2;x)-x^2\right||f''(x)|+\left|nP_{n,p,\Psi}^{\alpha,\beta}(r(t,x)(t-x)^2;x)\right|.$$

Let $u_n(x)=nP_{n,p,\Psi}^{\alpha,\beta}(t-x;x)+ax^2$, and $v_n(x)=nP_{n,p,\Psi}^{\alpha,\beta}((t-x)^2;x)-x^2$. Then

$$\left|n\left(P_{n,p,\Psi}^{\alpha,\beta}(f;x)-f(x)\right)+ax^2f'(x)-\frac{x^2}{2}f''(x)\right|$$
$$\le|u_n(x)||f'(x)|+\frac{1}{2}|v_n(x)||f''(x)|+nP_{n,p,\Psi}^{\alpha,\beta}(|r(t,x)|(t-x)^2;x). \tag{28}$$

From (7) and (8), $u_n(x)$ and $v_n(x)$ approach zero, as n goes to infinity at any point $x\in[0,\infty)$. To calculate the term $|r(t,x)|$ in the inequality (28), from (17),

$$|r(t,x)|\le\left(1+\frac{(e^{-t}-e^{-x})^2}{\gamma^2}\right)\omega^*(f'',\gamma)\,,\quad \gamma>0$$

can be written and here the modulus of continuity $\omega^*(f,\gamma)$ is defined in (16). Moreover,

$$|r(t,x)|\le\begin{cases}2\omega^*(f'',\gamma) & , \quad |e^{-t}-e^{-x}|\le\gamma\\ 2\frac{(e^{-t}-e^{-x})^2}{\gamma^2}\omega^*(f'',\gamma), & |e^{-t}-e^{-x}|>\gamma\end{cases},$$

and thus $|r(t,x)| \leq 2\left(1+\frac{(e^{-t}-e^{-x})^2}{\gamma^2}\right)\omega^*(f'',\gamma)$.

The Cauchy Schwartz inequality is applied to the last term of the sum on the right side of (28) and

$\gamma^2 = n^{-1}$ is chosen.

$$
\begin{aligned}
& nP_{n,p,\Psi}^{\alpha,\beta}(|r(t,x)|(t-x)^2;x) \\
& \quad \leq 2n\omega^*\left(f'',n^{-\frac{1}{2}}\right)\left(P_{n,p,\Psi}^{\alpha,\beta}((t-x)^2;x)+nP_{n,p,\Psi}^{\alpha,\beta}((e^{-t}-e^{-x})^2(t-x)^2;x)\right) \\
& \quad \leq 2\omega^*\left(f'',n^{-\frac{1}{2}}\right)\left(nP_{n,p,\Psi}^{\alpha,\beta}((t-x)^2;x)+\left(n^2P_{n,p,\Psi}^{\alpha,\beta}((e^{-t}-e^{-x})^4;x)\right)^{\frac{1}{2}}\left(n^2P_{n,p,\Psi}^{\alpha,\beta}((t-x)^4;x)\right)^{\frac{1}{2}}\right) \\
& \quad \leq 2\big(v_n(x)+x^2+w_n(x)\big)\omega^*\left(f'',n^{-\frac{1}{2}}\right),
\end{aligned}
$$

where $w_n(x) = \left(n^2P_{n,p,\Psi}^{\alpha,\beta}((e^{-t}-e^{-x})^4;x)\right)^{\frac{1}{2}}\left(n^2P_{n,p,\Psi}^{\alpha,\beta}((t-x)^4;x)\right)^{\frac{1}{2}}$.

Thus, the Voronovskaja type asymptotic formula is obtained.

Remark 4.5 Using the software Maple, the following equation is obtained:

$$\lim_{n\to\infty} n^2P_{n,p,\Psi}^{\alpha,\beta}((e^{-t}-e^{-x})^4;x) = 3e^{-4x}x^4.$$

A result of Theorem 4.4, from the equation (9) and Remark 4.5 can be given as follows :

Corollary 4.6 Let $f, f'' \in C^*[0,\infty)$. Thus

$$\lim_{n\to\infty} n\left(P_{n,p,\Psi}^{\alpha,\beta}(f;x) - f(x)\right) = -ax^2 f'(x) + \frac{x^2}{2} f''(x)$$

holds for any $x \in [0,\infty)$.

5. SOME GRAPHICAL ANALYSIS

In this section, the graphs below show the convergence of the Stancu type Post-Widder operators to the considered function $f(x) = x^3\, e^{-3x}$ for different values of n, p, a, α and β (Figure 1).

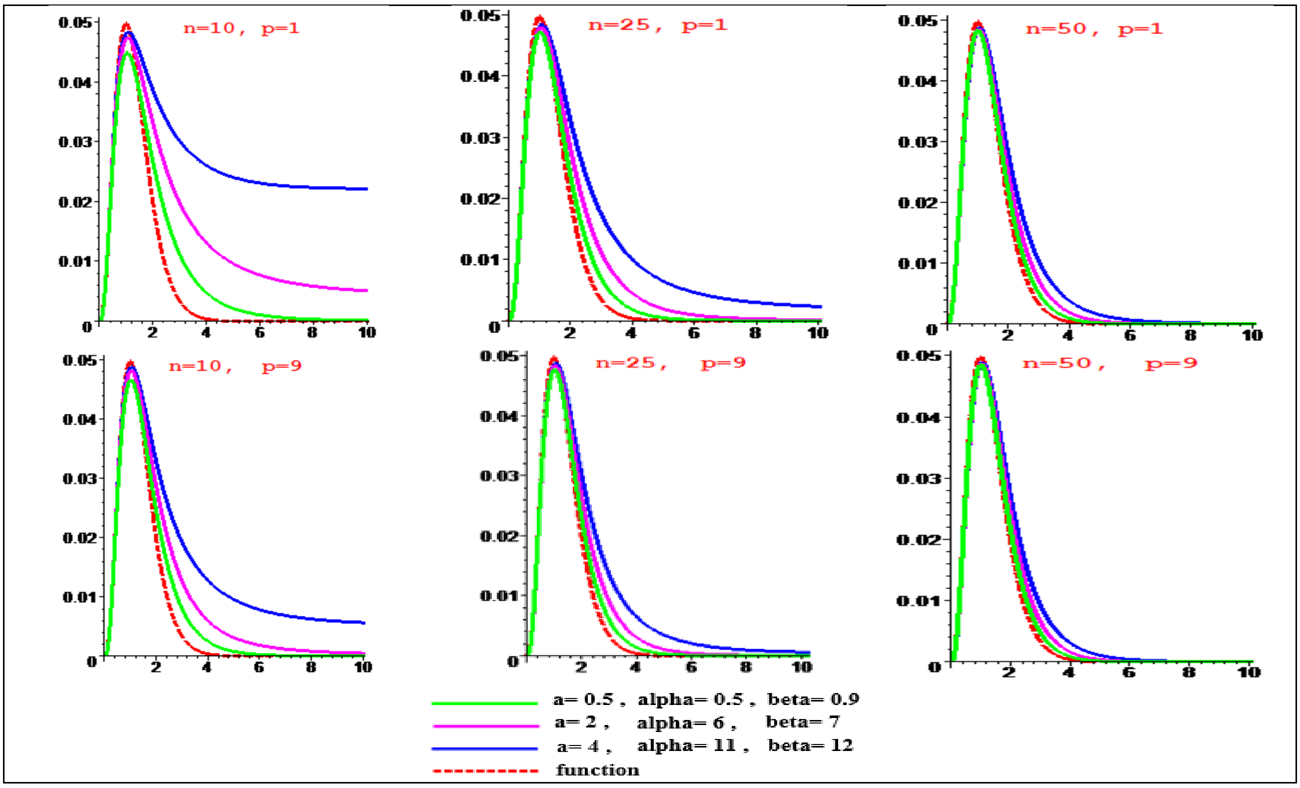

Figure 1. *Convergence of* $P_{n,p,\Psi}^{\alpha,\beta}(f;x)$ *for different values of* n, p, a, α *and* β

The graph below shows the convergence of the Post-Widder Operators $P_{n,\theta}^{*}(f;x)$ and the Stancu type Post-Widder Operators $P_{n,p,\Psi}^{\alpha,\beta}(f;x)$ to the function $f(x) = x^3\, e^{-3x}$ for $n = 50$, $a = 4$, and different values of p, α and β (Figure 2).

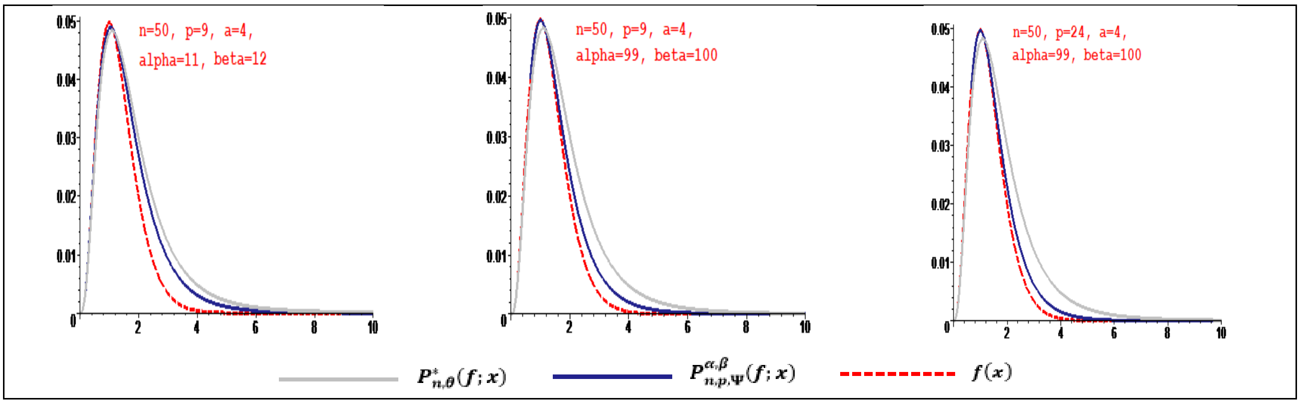

Figure 2. *Convergence of* $P_{n,\theta}^{*}(f;x)$ *and* $P_{n,p,\Psi}^{\alpha,\beta}(f;x)$ *to* $f(x)$ *for* $n = 50$ *and* $a = 4$

ACKNOWLEDGMENT

The author would like to thank the referee for the careful reading of this paper.

CONFLICTS OF INTEREST

The author declares no conflict of interest.

REFERENCES

Acar, T., Aral, A., & Gonska, H. (2017). On Szász-Mirakyan operators preserving $e^{2ax}, a > 0$. *Mediterranean Journal of Mathematics*, *14*(6), 1-14. doi:10.1007/s00009-016-0804-7

Aral, A., Inoan, D., & Raşa, I. (2019). Approximation properties of Szász-Mirakyan operators preserving exponential functions. *Positivity*, *23*, 233-246. doi:10.1007/s11117-018-0604-3

Bodur, M., Gürel-Yılmaz, Ö., & Aral, A. (2018). Approximation by Baskakov-Szász-Stancu operators preserving exponential functions. *Constructive Mathematical Analysis*, *1*(1), 1-8. doi:10.33205/cma.450708

Boyanov, B. D., & Veselinov, V. M. (1970). A note on the approximation of functions in an infinite interval by linear positive operators. *Bull Math Soc Sci Math Roum*, *14*(62), 9-13.

Cai, Q.-B., Dinlemez Kantar, Ü., & Çekim, B. (2020). Approximation properties for the genuine modified Bernstein-Durrmeyer-Stancu operators. Applied Mathematics-A Journal of Chinese Universities, 35(4), 468-478. doi:10.1007/s11766-020-3918-y

Cai, Q.-B., Sofyalıoğlu, M., Kanat, K., & Çekim, B. (2021a). Some approximation results for the new modification of Bernstein-Beta operators. *AIMS Mathematics*, *7*(2): 1831-1844. doi:10.3934/math.2022105

Cai, Q.-B., Torun, G., & Dinlemez Kantar, Ü. (2021b). Approximation properties of generalized λ–Bernstein–Stancu-type operators. *Journal of Mathematics*, *2021*, 5590439. doi:10.1155/2021/5590439

DeVore, R. A., & Lorentz, G. G. (1993). *Constructive Approximation*. (Grundlehren der mathematischen Wissenschaften, Vol. 303) Springer, Berlin.

Dinlemez Kantar, Ü., & Ergelen, G. (2019). A voronovskaja-type theorem for a kind of Durrmeyer-Bernstein-Stancu operators. *Gazi University Journal of Science*, *32*(4), 1228-1236. doi:10.35378/gujs.513478

Dinlemez Kantar, Ü., & Yüksel, İ. (2022). Investigating (p, q)-hybrid Durrmeyer-type operators in terms of their approximation properties. *Gazi University Journal of Science Part A: Engineering and Innovation*, *9*(1), 1-11. doi:10.54287/gujsa.1029633

Gupta, V., & Agrawal, D. (2019). Convergence by modified Post-Widder operators. *RACSAM*, *113*(2), 1475-1486. doi:10.1007/s13398-018-0562-4

Gupta, V., & Maheshwari, P. (2019). Approximation with certain Post-Widder operators. *Publications de L'Institut Mathématique*, *105*(119), 131-136. doi:10.2298/PIM1919131G

Gupta, V., & Tachev, G. (2022). Some results on Post-Widder operators preserving test function x^r. *Kragujevac Journal of Mathematics*, *46*(1), 149-165. doi:10.46793/KgJMat2201.149G

Gürel-Yılmaz, Ö., Gupta, V., & Aral, A. (2017). On Baskakov operators preserving the exponential function. *Journal of Numerical Analysis and Approximation Theory*, *46*(2), 150-161.

Gürel-Yılmaz, Ö., Bodur, M., & Aral, A. (2018). On approximation properties of Baskakov-Schurer-Szász operators preserving exponential functions. *Filomat*, *32*(15), 5433-5440. doi:10.2298/FIL1815433Y

Hacısalihoğlu, H., & Hacıyev, A. (1995). *Lineer Pozitif Operatör Dizilerin Yakınsaklığı*. Ankara University F.F., Ankara.

Holhoş, A. (2010). The rate of approximation of functions in an infinite interval by positive linear operators. *Stud Univ Babes-Bolyai Math.*, *55*(2). 133-142.

Kanat, K., & Sofyalıoğlu, M. (2021). On Stancu type Szász-Mirakyan-Durrmeyer operators preserving $e^{2ax}, a > 0$. *Gazi University Journal of Science*, *34*(1), 196-209. doi:10.35378/gujs.691419

May, C. P. (1976). Saturation and inverse theorems for combinations of a class of exponential type operators. *Canadian Journal of Mathematics*, *28*(6), 1224-1250. doi:10.4153/CJM-1976-123-8

Rathore, R. K. S., & Singh, O. P. (1980). On convergence of derivatives of Post-Widder operators. *Indian J. Pure Appl. Math.*, *11*(5), 547-561.

Rempulska, L., & Skorupka, M. (2009). On approximation by Post-Widder and Stancu operators preserving x^2. *Kyungpook Mathematical Journal*, *49*(1), 57-65. doi:10.5666/KMJ.2009.49.1.057

Sofyalıoğlu, M., & Kanat, K. (2019). Approximation properties of generalized Baskakov–Schurer–Szasz–Stancu operators preserving $e^{2ax}, a > 0$. *Journal of Inequalities and Applications and Applications*, *2019*(1), 112. doi:10.1186/s13660-019-2062-2

Sofyalıoğlu, M., & Kanat, K. (2020). Approximation properties of the Post-Widder operators preserving $e^{2ax}, a > 0$. *Mathematical Methods in the Applied Sciences*, *43*(7), 4272-4285. doi:10.1002/mma.6192

Sofyalıoğlu, M., Kanat, K., & Çekim, B. (2021). Parametric generalization of the Meyer-König-Zeller operators. *Chaos, Solitons & Fractals*, *152*, 111417. doi:10.1016/j.chaos.2021.111417

Torun, G., Mercan Boyraz, M., & Dinlemez Kantar, Ü. (2022). Investigation of the Asymptotic Behavior of Generalized Baskakov-Durrmeyer-Stancu Type Operators. *Cumhuriyet Science Journal*, *43*(1), 98-104. doi:10.17776/csj.1030637

Widder, D. V. (1941). *The Laplace Transform*. (Princeton Mathematical Series, 6) Princeton University Press, Princeton, New Jersey.