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Generalization of Some Integral Inequalities for Arithmetic Harmonically Convex Functions

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ABSTRACT

In this study, by using an integral identity, Hölder integral inequality and modulus properties we obtain some new general inequalities of the Hermite-Hadamard and Bullen type for functions whose derivatives in absolute value at certain power are arithmetically harmonically (AH) convex. In the last part of the article, applications including arithmetic mean, geometric mean, harmonic mean, logarithmic mean and p-logarithmic mean, which are some special means of real numbers, are given by using arithmetic harmonically convex functions.

Keywords: Convex function, Arithmetic-harmonically convex function, Hermite-Hadamard and Bullen type inequality.

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Introduction

Definition 1.1. A function $\mathfrak{R}: I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if the inequality

$$\mathfrak{R}(t\mathbb{mm} + (1-t)\mathbb{m}) \leq t\mathfrak{R}(\mathbb{mm}) + (1-t)\mathfrak{R}(\mathbb{m})$$

valid for all $\mathbb{mm}, \mathbb{m} \in I$ and $t \in [0,1]$. If this inequality reverses, then $\mathfrak{R}$ is said to be concave on interval $I \neq \varnothing$.

Theorem 1.2. (Hermite-Hadamard integral inequality) Let $\mathfrak{R}: I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on I of real numbers and $\mathbb{mm}, \mathbb{m} \in I$ with $\mathbb{mm} < \mathbb{m}$. The following inequality

$$\mathfrak{R}\left(\frac{\mathbb{mm}+\mathbb{m}}{2}\right) \leq \frac{1}{\mathbb{m}-\mathbb{mm}}\int_{\mathbb{mm}}^{\mathbb{m}} \mathfrak{R}(x)dx \leq \frac{\mathfrak{R}(\mathbb{mm})+\mathfrak{R}(\mathbb{m})}{2}. \qquad (1)$$

holds.

Some of inequalities for means can be derived from (1) for appropriate choices of $\mathfrak{R}$. See [1-4], for the results of the generalization and improvement of (1).

Theorem 1.3. (Bullen's inequality) Suppose that $\mathfrak{R}: [\mathbb{mm}, \mathbb{m}] \to \mathbb{R}$ is a convex function on $[\mathbb{mm}, \mathbb{m}]$. Then we get:

$$\begin{aligned}\mathfrak{R}\left(\frac{\mathbb{mm}+\mathbb{m}}{2}\right) &\leq \frac{1}{2}\left[\mathfrak{R}\left(\frac{3\mathbb{mm}+\mathbb{m}}{4}\right)+\mathfrak{R}\left(\frac{\mathbb{mm}+3\mathbb{m}}{4}\right)\right] \\ &\leq \frac{1}{\mathbb{m}-\mathbb{mm}}\int_{\mathbb{mm}}^{\mathbb{m}} \mathfrak{R}(x)dx \\ &\leq \frac{1}{2}\left[\mathfrak{R}\left(\frac{\mathbb{mm}+\mathbb{m}}{2}\right)+\frac{\mathfrak{R}(\mathbb{mm})+\mathfrak{R}(\mathbb{m})}{2}\right] \\ &\leq \frac{\mathfrak{R}(\mathbb{mm})+\mathfrak{R}(\mathbb{m})}{2}. \qquad (2)\end{aligned}$$

Definition 1.4. [5, 6] A function $\mathfrak{R}: I \subset \mathbb{R} \to (0,\infty)$ is said to be AH convex function if for all $\mathbb{mm}, \mathbb{m} \in I$ and $t \in [0,1]$ the equality

$$\mathfrak{R}(t\mathbb{mm} + (1-t)\mathbb{m}) \leq \frac{\mathfrak{R}(\mathbb{mm})\mathfrak{R}(\mathbb{m})}{t\mathfrak{R}(\mathbb{m}) + (1-t)\mathfrak{R}(\mathbb{mm})} \qquad (3)$$

holds.

For further details and proofs on both AH convex functions and other kinds of convexity, we refer the reader to [7-21] and references there in.

To derive main results for AH convex functions, we need the following Lemma 1.5.

Lemma 1.5. Let $\mathfrak{R}: I^{\circ} \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° where $\mathbb{mm}, \mathbb{m} \in I^{\circ}$ with $\mathbb{mm} < \mathbb{m}$. If $\mathfrak{R}' \in L[a,b]$, then the following identity holds:

$$\begin{aligned}&I_n(\mathfrak{R}, \mathbb{mm}, \mathbb{m}) \\ &= \sum_{i=0}^{n-1} \frac{\mathbb{m}-m}{2n^2}\left[\int_0^1 (1-2t)\mathfrak{R}'\left(t\frac{(n-i)\mathbb{mm}+i\mathbb{m}}{n}\right.\right. \\ &\left.\left. + (1-t)\frac{(n-i-1)\mathbb{mm}+(i+1)\mathbb{m}}{n}\right)dt\right] \qquad (4)\end{aligned}$$

where

$$\begin{aligned}I_n(\mathfrak{R}, \mathbb{mm}, \mathbb{m}) = \sum_{i=0}^{n-1} \frac{1}{2n}&\left[f\left(\frac{(n-i)\mathbb{mm}+i\mathbb{m}}{n}\right)\right. \\ &\left.+ f\left(\frac{(n-i-1)\mathbb{mm}+(i+1)\mathbb{m}}{n}\right)\right] \\ &- \frac{1}{\mathbb{m}-\mathbb{mm}}\int_{\mathbb{mm}}^{\mathbb{m}} \mathfrak{R}(x)dx.\end{aligned}$$

In this study, we use Hölder integral inequality and (4) in order to provide inequality for functions whose first derivatives in absolute value at certain power are AH-convex.

Throughout this paper, the following notations will be used for nonnegative numbers $\mathbb{m}\mathbb{m},\mathbb{m}$ $(\mathbb{m}>\mathbb{m}\mathbb{m})$:

1. $A:=A(\mathbb{m}\mathbb{m},\mathbb{m})=\frac{\mathbb{m}\mathbb{m}+\mathbb{m}}{2}$, $\mathbb{m}\mathbb{m},\mathbb{m}>0$, (arithmetic mean)
2. $G:=G(\mathbb{m}\mathbb{m},\mathbb{m})=\sqrt{\mathbb{m}\mathbb{m}\mathbb{m}}$, $\mathbb{m}\mathbb{m},\mathbb{m}\geq 0$, (geometric mean)
3. $H:=H(\mathbb{m}\mathbb{m},\mathbb{m})=\frac{2\mathbb{m}\mathbb{m}\mathbb{m}}{\mathbb{m}\mathbb{m}+\mathbb{m}}$, $\mathbb{m}\mathbb{m},\mathbb{m}>0$, (harmonic mean)
4. $L:=L(\mathbb{m}\mathbb{m},\mathbb{m})=\begin{cases}\frac{\mathbb{m}-\mathbb{m}\mathbb{m}}{ln\mathbb{m}-ln\mathbb{m}\mathbb{m}}, & \mathbb{m}\mathbb{m}\neq\mathbb{m}\\ \mathbb{m}\mathbb{m}, & \mathbb{m}\mathbb{m}=\mathbb{m}\end{cases}$; $\mathbb{m}\mathbb{m},\mathbb{m}>0$, (logarithmic mean)
5. $L_p:=L_p(\mathbb{m}\mathbb{m},\mathbb{m})=\begin{cases}\left(\frac{\mathbb{m}^{p+1}-\mathbb{m}\mathbb{m}^{p+1}}{(p+1)(\mathbb{m}-\mathbb{m}\mathbb{m})}\right)^{\frac{1}{p}}, & \mathbb{m}\mathbb{m}\neq\mathbb{m}, p\in\mathbb{R}\backslash\{-1,0\}\\ a, & \mathbb{m}\mathbb{m}=\mathbb{m}\end{cases}$; $\mathbb{m}\mathbb{m},\mathbb{m}>0$ $(p-$logarithmic mean$)$.

L_p is monotonically increasing over $p\in\mathbb{R}$, denoting $L_0=I$ and $L_{-1}=L$. In addition,

$$A_{n,i}=A_{n,i}(\mathbb{m}\mathbb{m},\mathbb{m})=\frac{(n-i)\mathbb{m}\mathbb{m}+i\mathbb{m}}{n}.$$

Main results

Theorem 2.1. Let $\Re: I\subset(0,\infty)\to(0,\infty)$ be a differentiable mapping on I°, and $\mathbb{m}\mathbb{m},\mathbb{m}\in I^{\circ}$ with $\mathbb{m}\mathbb{m}<\mathbb{m}$. If $|\Re'|$ is an AH convex function on $[\mathbb{m}\mathbb{m},\mathbb{m}]$, then the following inequalities hold:

i) If $|\Re'(A_{n,i+1})|-|\Re'(A_{n,i})|\neq 0$, then

$$|I_n(\Re,\mathbb{m}\mathbb{m},\mathbb{m})|\leq\sum_{i=0}^{n-1}\left\{\frac{\mathbb{m}-\mathbb{m}\mathbb{m}}{2n^2}\frac{|\Re'(A_{n,i})||\Re'(A_{n,i+1})|}{\left(|\Re'(A_{n,i+1})|-|\Re'(A_{n,i})|\right)^2}ln\frac{A\left(|\Re'(A_{n,i+1})|,|\Re'(A_{n,i})|\right)}{H\left(|\Re'(A_{n,i+1})|,|\Re'(A_{n,i})|\right)}\right\} \tag{5}$$

ii) If $|\Re'(A_{n,i+1})|-|\Re'(A_{n,i})|=0$, then

$$|I_n(\Re,\mathbb{m}\mathbb{m},\mathbb{m})|\leq\sum_{i=0}^{n-1}\frac{\mathbb{m}-\mathbb{m}\mathbb{m}}{4n^2}|\Re'(A_{n,i})|. \tag{6}$$

Proof. i) Let $|\Re'(A_{n,i+1})|-|\Re'(A_{n,i})|\neq 0$. From the properties of modulus and the Lemma 1.5, we write

$$\begin{aligned}|I_n(\Re,\mathbb{m}\mathbb{m},\mathbb{m})|&=\left|\sum_{i=0}^{n-1}\frac{\mathbb{m}-\mathbb{m}\mathbb{m}}{2n^2}\left[\int_0^1(1-2t)\Re'(tA_{n,i}+(1-t)A_{n,i+1})dt\right]\right|\\&\leq\sum_{i=0}^{n-1}\frac{\mathbb{m}-\mathbb{m}\mathbb{m}}{2n^2}\left|\left[\int_0^1(1-2t)\Re'(tA_{n,i}+(1-t)A_{n,i+1})dt\right]\right|\\&\leq\sum_{i=0}^{n-1}\frac{\mathbb{m}-\mathbb{m}\mathbb{m}}{2n^2}\left[\int_0^1|1-2t||\Re'(tA_{n,i}+(1-t)A_{n,i+1})|\right]dt.\end{aligned} \tag{7}$$

Since $|\Re'|$ is an AH convex function on $[\mathbb{m}\mathbb{m},\mathbb{m}]$, then we have

$$|\Re'(tA_{n,i}+(1-t)A_{n,i+1})|\leq\frac{|\Re'(A_{n,i})||\Re'(A_{n,i+1})|}{t|\Re'(A_{n,i+1})|+(1-t)|\Re'(A_{n,i})|}$$

If we use the inequality in (7), we obtain

$$\begin{aligned}&|I_n(\Re,\mathbb{m}\mathbb{m},\mathbb{m})|\\&\leq\sum_{i=0}^{n-1}\frac{\mathbb{m}-\mathbb{m}\mathbb{m}}{2n^2}\int_0^1|1-2t|\frac{|\Re'(A_{n,i})||\Re'(A_{n,i+1})|}{t|\Re'(A_{n,i+1})|+(1-t)|\Re'(A_{n,i})|}dt\\&=\sum_{i=0}^{n-1}\frac{\mathbb{m}-\mathbb{m}\mathbb{m}}{2n^2}|\Re'(A_{n,i})||\Re'(A_{n,i+1})|\int_0^1\frac{|1-2t|}{t|\Re'(A_{n,i+1})|+(1-t)|\Re'(A_{n,i})|}dt\end{aligned}$$

$$= \sum_{i=0}^{n-1} \frac{\mathbb{m} - \mathbb{mm}}{2n^2} |\Re'(A_{n,i})||\Re'(A_{n,i+1})| \left[\int_0^{\frac{1}{2}} \frac{1-2t}{t|\Re'(A_{n,i+1})| + (1-t)|\Re'(A_{n,i})|} dt + \int_{\frac{1}{2}}^{1} \frac{2t-1}{t|\Re'(A_{n,i+1})| + (1-t)|\Re'(A_{n,i})|} dt \right]. \quad (8)$$

By changing variable as $u = t|\Re'(A_{n,i+1})| + (1-t)|\Re'(A_{n,i})|$ in the last two integrals, it is easily seen that

$$\int_0^{\frac{1}{2}} \frac{1-2t}{t|\Re'(A_{n,i+1})| + (1-t)|\Re'(A_{n,i})|} dt = \frac{1}{\left(|\Re'(A_{n,i+1})| - |\Re'(A_{n,i})|\right)^2} \Big[|\Re'(A_{n,i+1})| - |\Re'(A_{n,i})| + \left(|\Re'(A_{n,i+1})| + |\Re'(A_{n,i})|\right) \ln \frac{|\Re'(A_{n,i+1})| + |\Re'(A_{n,i})|}{2|\Re'(A_{n,i})|} \Big] \quad (9)$$

$$\int_{\frac{1}{2}}^{1} \frac{2t-1}{t|\Re'(A_{n,i+1})| + (1-t)|\Re'(A_{n,i})|} dt = \frac{1}{\left(|\Re'(A_{n,i+1})| - |\Re'(A_{n,i})|\right)^2} \times \left[\left(|\Re'(A_{n,i+1})| + |\Re'(A_{n,i})|\right) ln \frac{|\Re'(A_{n,i+1})| + |\Re'(A_{n,i})|}{2|\Re'(A_{n,i+1})|} - \left(|\Re'(A_{n,i+1})| - |\Re'(A_{n,i})|\right) \right]. \quad (10)$$

By substituting the equalities (9) and (10) in (8), we have

$$|I_n(\Re, \mathbb{mm}, \mathbb{m})| \leq \sum_{i=0}^{n-1} \left\{ \frac{\mathbb{m} - \mathbb{mm}}{2n^2} \frac{|\Re'(A_{n,i})||\Re'(A_{n,i+1})|}{\left(|\Re'(A_{n,i+1})| - |\Re'(A_{n,i})|\right)^2} ln \frac{A\left(|\Re'(A_{n,i+1})|, |\Re'(A_{n,i})|\right)}{H\left(|\Re'(A_{n,i+1})|, |\Re'(A_{n,i})|\right)} \right\},$$

which is the desired result.

ii) Let $|\Re'(A_{n,i+1})| - |\Re'(A_{n,i})| = 0$. Then, substituting $|\Re'(A_{n,i+1})| = |\Re'(A_{n,i})|$ in the inequality (8), we obtain

$$|I_n(\Re, \mathbb{mm}, \mathbb{m})| \leq \sum_{i=0}^{n-1} \frac{\mathbb{m} - \mathbb{mm}}{4n^2} |\Re'(A_{n,i+1})|.$$

Remark 2.2. Using the arithmetic harmonically convexity of the function $|\Re'|$ in the Theorem 2.1, we get

i) If $|\Re'(A_{n,i+1})| - |\Re'(A_{n,i})| \neq 0$, then

$$|I_n(\Re, \mathbb{mm}, \mathbb{m})| \leq \sum_{i=0}^{n-1} \frac{\mathbb{m} - \mathbb{mm}}{2n^2} \frac{[(n-i-1)|\Re'(\mathbb{mm})| + (i+1)|\Re'(\mathbb{m})|][(n-i)|\Re'(\mathbb{m})| + i|\Re'(\mathbb{mm})|]}{(|\Re'(\mathbb{m})| + |\Re'(\mathbb{mm})|)^2} \times ln \frac{[(2n-2i-1)|\Re'(\mathbb{m})| + (2i+1)|\Re'(\mathbb{mm})|]^2}{4[(n-i)|\Re'(\mathbb{m})| + i|\Re'(\mathbb{mm})|][(n-i-1)|\Re'(\mathbb{m})| + (i+1)|\Re'(\mathbb{mm})|]},$$

ii) If $|\Re'(A_{n,i+1})| - |\Re'(A_{n,i})| = 0$, then

$$|I_n(\Re, \mathbb{mm}, \mathbb{m})| \leq \sum_{i=0}^{n-1} \frac{\mathbb{m} - \mathbb{mm}}{2\mathrm{n}} \frac{|\Re'(\mathbb{mm})||\Re'(\mathbb{m})|}{(n-i-1)|\Re'(\mathbb{m})| + (i+1)|\Re'(\mathbb{mm})|}.$$

Corollary 2.3. By choosing $\mathrm{n} = 1$ in Remark 2.2, we obtain the following inequalities:

i) If $|\Re'(\mathbb{mm})| - |\Re'(\mathbb{m})| \neq 0$, then

$$\left| \frac{\Re(\mathbb{mm}) + \Re(\mathbb{m})}{2} - \frac{1}{\mathbb{m} - \mathbb{mm}} \int_{\mathbb{mm}}^{\mathbb{m}} \Re(x) dx \right| \leq \frac{\mathbb{m} - \mathbb{mm}}{2} \frac{|\Re'(\mathbb{mm})||\Re'(\mathbb{m})|}{(|\Re'(\mathbb{mm})| + |\Re'(\mathbb{m})|)^2} \ln \frac{[|\Re'(\mathbb{mm})| + |\Re'(\mathbb{m})|]^2}{4|\Re'(\mathbb{mm})||\Re'(\mathbb{m})|} = \frac{\mathbb{m} - \mathbb{mm}}{8} \frac{H(|\Re'(\mathbb{mm})|, |\Re'(\mathbb{m})|)}{A(|\Re'(\mathbb{mm})|, |\Re'(\mathbb{m})|)} \ln \frac{A(|\Re'(\mathbb{mm})|, |\Re'(\mathbb{m})|)}{H(|\Re'(\mathbb{mm})|, |\Re'(\mathbb{m})|)},$$

ii) If $|\Re'(\mathbb{mm})| - |\Re'(\mathbb{m})| = 0$, then

$$\left| \frac{\Re(\mathbb{mm}) + \Re(\mathbb{m})}{2} - \frac{1}{\mathbb{m} - \mathbb{mm}} \int_{\mathbb{mm}}^{\mathbb{m}} \Re(x) dx \right| \leq \frac{\mathbb{m} - \mathbb{mm}}{4} ||\Re'(\mathbb{m})||.$$

Corollary 2.4. By choosing $\mathrm{n} = 2$ in Remark 2.2, we get the following Bullen type inequalities:

i) If $|\Re'(A_{n,i+1})| - |\Re'(A_{n,i})| \neq 0$ for all $i = 0,1$, then

$$\left|\frac{1}{2}\left[\frac{\Re(\mathfrak{m}\mathfrak{m})+\Re(\mathfrak{m})}{2}+\Re\left(\frac{\mathfrak{m}\mathfrak{m}+\mathfrak{m}}{2}\right)\right]-\frac{1}{\mathfrak{m}-\mathfrak{m}\mathfrak{m}}\int_{\mathfrak{m}\mathfrak{m}}^{\mathfrak{m}}\Re(x)dx\right| \le \frac{\mathfrak{m}-\mathfrak{m}\mathfrak{m}}{8}\frac{|\Re'(\mathfrak{m}\mathfrak{m})|H(|\Re'(\mathfrak{m}\mathfrak{m})|,|\Re'(\mathfrak{m})|)}{\left[\left|\Re'\left(\frac{\mathfrak{m}\mathfrak{m}+\mathfrak{m}}{2}\right)\right|-|\Re'(\mathfrak{m}\mathfrak{m})|\right]^2}$$

$$\times\ln\frac{A[H(|\Re'(\mathfrak{m}\mathfrak{m})|,|\Re'(\mathfrak{m})|),|\Re'(\mathfrak{m}\mathfrak{m})|]}{H\left(\left|\Re'\left(\frac{\mathfrak{m}\mathfrak{m}+\mathfrak{m}}{2}\right)\right|,|\Re'(\mathfrak{m}\mathfrak{m})|\right)}+\frac{\mathfrak{m}-\mathfrak{m}\mathfrak{m}}{8}\frac{|\Re'(\mathfrak{m})|H(|\Re'(\mathfrak{m}\mathfrak{m})|,|\Re'(\mathfrak{m})|)}{\left[|\Re'(\mathfrak{m})|-\left|\Re'\left(\frac{\mathfrak{m}\mathfrak{m}+\mathfrak{m}}{2}\right)\right|\right]^2}\ln\frac{A[H(|\Re'(\mathfrak{m}\mathfrak{m})|,|\Re'(\mathfrak{m})|),|\Re'(\mathfrak{m})|]}{H\left(\left|\Re'\left(\frac{\mathfrak{m}\mathfrak{m}+\mathfrak{m}}{2}\right)\right|,|\Re'(\mathfrak{m})|\right)}$$

ii) If $|f'(A_{n,i+1})|-|f'(A_{n,i})|=0$ for all $i=0,1$, then

$$\left|\frac{1}{2}\left[\frac{\Re(\mathfrak{m}\mathfrak{m})+\Re(\mathfrak{m})}{2}+\Re\left(\frac{\mathfrak{m}\mathfrak{m}+\mathfrak{m}}{2}\right)\right]-\frac{1}{\mathfrak{m}-\mathfrak{m}\mathfrak{m}}\int_{\mathfrak{m}\mathfrak{m}}^{\mathfrak{m}}\Re(x)dx\right| \le \frac{\mathfrak{m}-\mathfrak{m}\mathfrak{m}}{8}A[H(|\Re'(\mathfrak{m}\mathfrak{m})|,|\Re'(\mathfrak{m})|),|\Re'(\mathfrak{m}\mathfrak{m})|].$$

Theorem 2.5. Let $\Re: I\subset(0,\infty)\to(0,\infty)$ be a differentiable mapping on I°, and $\mathfrak{m}\mathfrak{m},\mathfrak{m}\in I^\circ$ with $\mathfrak{m}\mathfrak{m}<\mathfrak{m}$. If $|\Re'|^q$ is an AH convex function on $[\mathfrak{m}\mathfrak{m},\mathfrak{m}]$ for some fixed $q>1$, then the following inequalities hold:

i) If $|\Re'(A_{n,i+1})|^q-|\Re'(A_{n,i})|^q\neq 0$, then

$$|I_n(\Re,\mathfrak{m}\mathfrak{m},\mathfrak{m})|\le\sum_{i=0}^{n-1}\frac{b-a}{2n^2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\frac{|\Re'(A_{n,i})||\Re'(A_{n,i+1})|}{L^{\frac{1}{q}}(|\Re'(A_{n,i})|^q,|\Re'(A_{n,i+1})|^q)},\tag{11}$$

ii) If $|\Re'(A_{n,i+1})|^q-|\Re'(A_{n,i})|^q=0$, then

$$|I_n(\Re,\mathfrak{m}\mathfrak{m},\mathfrak{m})|\le\sum_{i=0}^{n-1}\frac{\mathfrak{m}-\mathfrak{m}\mathfrak{m}}{2n^2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}|\Re'(A_{n,i+1})|.\tag{12}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. i) Let $|\Re'(A_{n,i+1})|^q-|\Re'(A_{n,i})|^q\neq 0$. From the properties of modulus and the Lemma 1.5, we write

$$|I_n(\Re,\mathfrak{m}\mathfrak{m},\mathfrak{m})|\le\sum_{i=0}^{n-1}\frac{\mathfrak{m}-\mathfrak{m}\mathfrak{m}}{2n^2}\left[\int_0^1|1-2t||\Re'(tA_{n,i}+(1-t)A_{n,i+1})|\right]dt.\tag{13}$$

Since $|\Re'|^q$ is an AH convex function on $[\mathfrak{m}\mathfrak{m},\mathfrak{m}]$, the following inequality

$$|\Re'(tA_{n,i}+(1-t)A_{n,i+1})|^q\le\frac{|\Re'(A_{n,i})|^q|\Re'(A_{n,i+1})|^q}{t|\Re'(A_{n,i+1})|^q+(1-t)|\Re'(A_{n,i})|^q}\tag{14}$$

holds. If we use the inequality in (13) and consider the Hölder integral inequality, we get

$$|I_n(\Re,\mathfrak{m}\mathfrak{m},\mathfrak{m})|\le\sum_{i=0}^{n-1}\frac{\mathfrak{m}-\mathfrak{m}\mathfrak{m}}{2n^2}\left(\int_0^1|1-2t|^p dt\right)^{\frac{1}{p}}\left(\int_0^1|f'(tA_{n,i}+(1-t)A_{n,i+1})|^q dt\right)^{\frac{1}{q}}$$

$$\le\sum_{i=0}^{n-1}\frac{\mathfrak{m}-\mathfrak{m}\mathfrak{m}}{2n^2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\int_0^1\frac{|f'(A_{n,i})|^q|f'(A_{n,i+1})|^q}{t|f'(A_{n,i+1})|^q+(1-t)|f'(A_{n,i})|^q}dt\right)^{\frac{1}{q}}\tag{15}$$

$$=\sum_{i=0}^{n-1}\frac{\mathfrak{m}-\mathfrak{m}\mathfrak{m}}{2n^2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\frac{|f'(A_{n,i})||f'(A_{n,i+1})|}{L^{\frac{1}{q}}(|f'(A_{n,i})|^q,|f'(A_{n,i+1})|^q)},\tag{16}$$

where

$$\int_0^1|1-2t|^p dt=\frac{1}{p+1}$$

$$\int_0^1\frac{1}{t|\Re'(A_{n,i+1})|^q+(1-t)|\Re'(A_{n,i})|^q}dt=L^{-1}(|\Re'(A_{n,i})|^q,|\Re'(A_{n,i+1})|^q).$$

ii) Let $\left|\Re'\left(A_{n,i+1}\right)\right|^q - \left|\Re'\left(A_{n,i}\right)\right|^q = 0$. Then, substituting $\left|\Re'\left(A_{n,i+1}\right)\right|^q = \left|\Re'\left(A_{n,i}\right)\right|^q$ in the inequality (15), we have

$$|I_n(\Re, \mathbb{m}\mathbb{m}, \mathbb{m})| \leq \sum_{i=0}^{n-1} \frac{\mathbb{m} - \mathbb{m}\mathbb{m}}{2n^2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left|\Re'\left(A_{n,i+1}\right)\right|.$$

Remark 2.6. By using the arithmetic harmonically convexity of the function $|\Re'|^q$ in (16), we get the following for $\left|\Re'\left(A_{n,i+1}\right)\right|^q - \left|\Re'\left(A_{n,i}\right)\right|^q \neq 0$,

$$|I_n(\Re, \mathbb{m}\mathbb{m}, \mathbb{m})| \leq \sum_{i=0}^{n-1} \frac{\mathbb{m} - \mathbb{m}\mathbb{m}}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left|\Re'\left(A_{n,i}\right)\right|\left|\Re'\left(A_{n,i+1}\right)\right| \left(n \frac{\begin{array}{c} ln[(n-i)|\Re'(\mathbb{m})|^q + i|\Re'(\mathbb{m}\mathbb{m})|^q] \\ -ln[(n-i-1)|\Re'(\mathbb{m}\mathbb{m})|^q + (i+1)|\Re'(\mathbb{m})|^q] \end{array}}{|\Re'(\mathbb{m})|^q - |\Re'(\mathbb{m}\mathbb{m})|^q}\right)^{\frac{1}{q}}.$$

Corollary 2.7. By choosing $n = 1$ in Remark 2.6, we obtain the following inequalities

i) If $|\Re'(\mathbb{m}\mathbb{m})|^q - |\Re'(\mathbb{m})|^q \neq 0$, then

$$\left|\frac{\Re(\mathbb{m}\mathbb{m}) + \Re(\mathbb{m})}{2} - \frac{1}{\mathbb{m} - \mathbb{m}\mathbb{m}}\int_{\mathbb{m}\mathbb{m}}^{\mathbb{m}} \Re(x)dx\right| \leq \frac{\mathbb{m} - \mathbb{m}\mathbb{m}}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \frac{G^2(|\Re'(\mathbb{m}\mathbb{m})|, |\Re'(\mathbb{m})|)}{L^{\frac{1}{q}}(|\Re'(\mathbb{m}\mathbb{m})|^q, |\Re'(\mathbb{m})|^q)}.$$

ii) If $|\Re'(\mathbb{m}\mathbb{m})|^q - |\Re'(\mathbb{m})|^q = 0$, then

$$\left|\frac{\Re(\mathbb{m}\mathbb{m}) + \Re(\mathbb{m})}{2} - \frac{1}{\mathbb{m} - \mathbb{m}\mathbb{m}}\int_{\mathbb{m}\mathbb{m}}^{\mathbb{m}} \Re(x)dx\right| \leq \frac{\mathbb{m} - \mathbb{m}\mathbb{m}}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}} |\Re'(\mathbb{m})|.$$

Corollary 2.8. By choosing $n = 2$ in Remark 2.6, we obtain the following Bullen type inequalities:

i) If $\left|\Re'\left(A_{2,i+1}\right)\right|^q - \left|\Re'\left(A_{2,i}\right)\right|^q \neq 0$ for all $i = 0,1$, then

$$\left|\frac{1}{2}\left[\frac{\Re(\mathbb{m}\mathbb{m}) + \Re(\mathbb{m})}{2} + \Re\left(\frac{\mathbb{m}\mathbb{m} + \mathbb{m}}{2}\right)\right] - \frac{1}{\mathbb{m} - \mathbb{m}\mathbb{m}}\int_{\mathbb{m}\mathbb{m}}^{\mathbb{m}} \Re(x)dx\right|$$
$$\leq \frac{\mathbb{m} - \mathbb{m}\mathbb{m}}{8}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\frac{G^2\left(|\Re'(\mathbb{m}\mathbb{m})|, \left|\Re'\left(\frac{\mathbb{m}\mathbb{m} + \mathbb{m}}{2}\right)\right|\right)}{L^{\frac{1}{q}}\left(\left|\Re'\left(\frac{\mathbb{m}\mathbb{m} + \mathbb{m}}{2}\right)\right|^q, |\Re'(\mathbb{m}\mathbb{m})|^q\right)} + \frac{G^2\left(\left|\Re'\left(\frac{\mathbb{m}\mathbb{m} + \mathbb{m}}{2}\right)\right|, |\Re'(\mathbb{m})|\right)}{L^{\frac{1}{q}}\left(\left|\Re'\left(\frac{\mathbb{m}\mathbb{m} + \mathbb{m}}{2}\right)\right|^q, |\Re'(\mathbb{m})|^q\right)}\right],$$

ii) If $\left|f'\left(A_{2,i+1}\right)\right|^q - \left|f'\left(A_{2,i}\right)\right|^q = 0$ for all $i = 0,1$, then

$$\left|\frac{1}{2}\left[\frac{\Re(\mathbb{m}\mathbb{m}) + \Re(\mathbb{m})}{2} + \Re\left(\frac{\mathbb{m}\mathbb{m} + \mathbb{m}}{2}\right)\right] - \frac{1}{\mathbb{m} - \mathbb{m}\mathbb{m}}\int_{\mathbb{m}\mathbb{m}}^{\mathbb{m}} \Re(x)dx\right| \leq \frac{\mathbb{m} - \mathbb{m}\mathbb{m}}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}} A\left(\left|\Re'\left(\frac{\mathbb{m}\mathbb{m} + \mathbb{m}}{2}\right)\right|, |\Re'(\mathbb{m})|\right).$$

Applications for special means

$\Re(x) = x^p, x > 0$ is an AH convex function for $p \in (-1,0)$ [5]. Using this function we have the following propositions:

Proposition 3.1. Let $0 < a < b$ and $p \in (-1,0)$. Then we get:

$$\frac{1}{p+1}\left|\sum_{i=0}^{n-1}\frac{1}{n}A\left(\left(A_{n,i}\right)^{p+1}, \left(A_{n,i+1}\right)^{p+1}\right) - L_{p+1}^{p+1}(\mathbb{m}\mathbb{m}, \mathbb{m})\right| \leq \sum_{i=0}^{n-1}\left\{\frac{\mathbb{m} - \mathbb{m}\mathbb{m}}{2n^2}\frac{\left(A_{n,i}\right)^p\left(A_{n,i+1}\right)^p}{\left[\left(A_{n,i+1}\right)^p - \left(A_{n,i}\right)^p\right]^2} ln\frac{A\left(\left(A_{n,i+1}\right)^p, \left(A_{n,i}\right)^p\right)}{H\left(\left(A_{n,i+1}\right)^p, \left(A_{n,i}\right)^p\right)}\right\}.$$

Proof. For $p \in (-1,0)$, the function $f(x) = \frac{x^{p+1}}{p+1}, x > 0$ is AH convex. Therefore, the assertion follows from (5) in Theorem 2.1, for $\Re: (0, \infty) \to \mathbb{R}$, $\Re(x) = \frac{x^{p+1}}{p+1}$.

Corollary 3.2. If we take $n = 1$ in Proposition 3.1, we get:

$$\left|A\left(\left(A_{1,0}\right)^{p+1},\left(A_{1,1}\right)^{p+1}\right)-L_{p+1}^{p+1}(\mathbb{m},\mathbb{n})\right|\leq\frac{\mathbb{n}-\mathbb{m}}{2}\frac{\left(A_{1,0}\right)^{p}\left(A_{1,1}\right)^{p}}{\left[\left(A_{1,1}\right)^{p}-\left(A_{1,0}\right)^{p}\right]^{2}}\ln\frac{A\left(\left(A_{1,1}\right)^{p},\left(A_{1,0}\right)^{p}\right)}{H\left(\left(A_{1,1}\right)^{p},\left(A_{1,0}\right)^{p}\right)},$$

that is,

$$\frac{1}{p+1}\left|A\left(a^{p+1},b^{p+1}\right)-L_{p+1}^{p+1}(\mathbb{m},\mathbb{n})\right|\leq\frac{\mathbb{n}-\mathbb{m}}{2}\frac{\mathbb{m}^{p}\mathbb{n}^{p}}{\left[\mathbb{m}^{p}-\mathbb{n}^{p}\right]^{2}}\ln\frac{A(\mathbb{m}^{p},\mathbb{n}^{p})}{H(\mathbb{m}^{p},\mathbb{n}^{p})}.$$

Proposition 3.3. Let $\mathbb{m},\mathbb{n}\in(0,\infty)$ with $\mathbb{m}<\mathbb{n}$, $q>1$ and $m\in(-1,0)$. Then, we have:

$$\frac{q}{q+m}\left|\sum_{i=0}^{n-1}\frac{1}{n}A\left(\left(A_{n,i}\right)^{\frac{q}{m+q}},\left(A_{n,i+1}\right)^{\frac{q}{m+q}}\right)-L_{\frac{m}{q}+1}^{\frac{m}{q}+1}(\mathbb{m},\mathbb{n})\right|\leq\sum_{i=0}^{n-1}\frac{\mathbb{n}-\mathbb{m}}{2n^{2}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\frac{\left(A_{n,i}\right)^{\frac{m}{q}}\left(A_{n,i+1}\right)^{\frac{m}{q}}}{L^{\frac{1}{q}}\left(\left(A_{n,i}\right)^{m},\left(A_{n,i+1}\right)^{m}\right)}.$$

Proof. The assertion follows from (11) in Theorem 2.5. Let $\Re(x)=\frac{q}{m+q}x^{\frac{m}{q}+1}$, $x\in(0,\infty)$. Then $|\Re'(x)|^{q}=x^{m}$ is AH convex on $(0,\infty)$ and the result follows from Theorem 2.5.

Corollary 3.4. Taking $\mathrm{n}=1$ in Proposition 3.3, we get:

$$\frac{q}{q+m}\left|A\left(a^{\frac{q}{m+q}},b^{\frac{q}{m+q}}\right)-L_{\frac{m}{q}+1}^{\frac{m}{q}+1}(\mathbb{m},\mathbb{n})\right|\leq\frac{\mathbb{n}-\mathbb{m}}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\frac{\mathbb{m}^{\frac{m}{q}}\mathbb{n}^{\frac{m}{q}}}{L^{\frac{1}{q}}(\mathbb{m}^{m},\mathbb{n}^{m})}.$$

Conclusion

In this paper, by using the definition of arithmetically-harmonically functions and some simple mathematical inequalities, we obtained some new inequalities related to Hermite-Hadamard and Bullen type.

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Conflicts of interest

There are no conflicts of interest in this work.

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