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(k,μ)-Paracontact Manifolds and Their Curvature Classification

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ABSTRACT

The aim of this paper is to study (k,μ)-Paracontact metric manifold. We introduce the curvature tensors of a (k,μ)-paracontact metric manifold satisfying the conditions $R\cdot P_* = 0, R\cdot L = 0, R\cdot W_1 = 0, R\cdot W_0 = 0$ and $R\cdot M = 0$. According to these cases, (k,μ)-paracontact manifolds have been characterized such as η-Einstein and Einstein. We get the necessary and sufficient conditions of a (k,μ)-paracontact metric manifold to be η-Einstein. Also, we consider new conclusions of a (k,μ)-paracontact metric manifold contribute to geometry. We think that some interesting results on a (k,μ)-paracontact metric manifold are obtained.

Keywords: (k,μ) −Paracontact manifold, η −Einstein manifold, Riemannian curvature tensor.

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Introduction

In 1985, Kaneyuki and Williams initiated the notion of paracontact geometry [1]. Zamkovoy achieved systematic research on paracontact metric manifolds and some remarkable subclasses named para-Sasakian manifolds [2]. Recently, B. Cappeletti-Montano, I. Küpeli Erken and C. Murathan introduced a new type of paracontact geometry so-called paracontact metric (k,μ) −space, where k and μ are constants [3]. This is known [4] about the contact case $k\le 1$, but in the paracontact case there is no restriction of k.

Zamkovoy studied paracontact metric manifolds and some remarkable subclasses named para-Sasakian manifolds. In particular, many authors have pointed to the importance of paracontact geometry and para-Sasakian geometry in recent years. A normal paracontact metric manifold is a para-Sasakian manifold. An almost paracontact metric manifold is a para-Sasakian manifold if and only if [2]

$$(\nabla_{\beta_1}\phi)\beta_2 = -g(\beta_1,\beta_2)\xi + \eta(\beta_2)\beta_1.$$

As a generalization of locally symmetric spaces, many authors have studied semi-symmetric spaces and in turn their generalizations. A semi-Riemannian manifold (M^{2n+1},g), $n\ge 1$, is said to be semi-symmetric if its curvature tensor R satisfies $R\cdot R = 0$ for all vector fields β_1,β_2 on M^{2n+1}, where $R(\beta_1,\beta_2)$ acts as a derivation on [5,6]. D. Kowalezyk researched some subclass of semi-symmetric manifolds [5].

On the other hand, B. Prasad introduced a pseudo projective curvature tensor on a Riemannian manifold [6]. S. Ivanov, D. Vassilev and S. Zamkovoy studied a tensor invariant characterizing locally the integrable paracontact Hermitian structures which are paracontact conformally equivalent to the flat structure on G(P) [7]. Since then several geometers studied curvature conditions and obtain various important properties [8,9,19].

The object of this paper is to study properties of the some certain curvature tensor in a (k,μ) −paracontact metric manifold we research $R\cdot P_* = 0$, $R\cdot L = 0$, $R\cdot W_1 = 0$, $R\cdot W_0 = 0$ and $R(X,Y)\cdot M = 0$, where R, P_*, L, W_1, W_0 and M denote the Riemannian, pseudo-projective, conharmonic, W_1, W_0 and M −projective curvature tensors of manifold, respectively.

Preliminaries

An $(2n+1)$-dimensional manifold M is called to have a paracontact structure if it admits a (1,1) −tensor field ϕ, a vector field ξ and a 1-form η satisfying the following conditions [1]:

$$(i)\ \ \phi^2\beta_1 = \beta_1 - \eta(\beta_1)\xi,$$

for any vector field $\beta_1 \in \chi(M)$, where $\chi(M)$ the set of all differential vector fields on M,

$$(ii)\ \eta(\xi) = 1, \eta\circ\phi = 0, \phi\xi = 0,$$

an almost paracontact manifold equipped with a pseudo-Riemannian metric g such that

$$g(\phi\beta_1,\phi\beta_2) = -g(\beta_1,\beta_2) + \eta(\beta_1)\eta(\beta_2), \quad g(\beta_1,\xi) = \eta(\beta_1) \tag{1}$$

for all vector fields $\beta_1,\beta_2 \in \chi(M)$. An almost paracontact structure is called a paracontact structure if $g(\beta_1,\phi\beta_2) = d\eta(\beta_1,\beta_2)$ with the associated metric g

[2]. We now define a (1,1) tensor field h by $h = \frac{1}{2} L_\xi \phi$, where L denotes the Lie derivative. Then h is symmetric and satisfies the conditions

$$h\phi = -\phi h, \quad h\xi = 0, \quad Tr.h = Tr.\phi h = 0 \quad (2)$$

If $\tilde{\nabla}$ denotes the Levi-Civita connection of g, then we have the following relation

$$\tilde{\nabla}_{\beta_1} \xi = -\phi\beta_1 + \phi h\beta_1 \quad (3)$$

for any $\beta_1 \in \chi(M)$ [2]. For a paracontact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)$, if ξ is a killing vector field or equivalently, $h = 0$, then it is called a K-paracontact manifold.

An almost paracontact manifold is said to be para-Sasakian if and only if the following condition holds

$$(\tilde{\nabla}_{\beta_1} \phi)\beta_2 = -g(\beta_1, \beta_2)\xi + \eta(\beta_2)\beta_1$$

for all $\beta_1, \beta_2 \in \chi(M)$ [2]. A normal paracontact metric manifold is para-Sasakian and satisfies

$$R(\beta_1, \beta_2)\xi = -(\eta(\beta_2)\beta_1 - \eta(\beta_1)\beta_2) \quad (4)$$

for all $\beta_1, \beta_2 \in \chi(M)$, but this is not a sufficient condition for a para-contact manifold to be para-Sasakian. It is clear that every para-Sasakian manifold is K-paracontact. But the converse is not always true [16].

A paracontact manifold M is said to be η-Einstein if its Ricci tensor S of type (0,2) is of the from $S(\beta_1, \beta_2) = ag(\beta_1, \beta_2) + b\eta(\beta_1)\eta(\beta_2)$,where a, b are smooth functions on M. If $b = 0$, then the manifold is also called Einstein [11].

A paracontact metric manifold is said to be a (k, μ) −paracontact manifold if the curvature tensor $\tilde{R}$ satisfies

$$\tilde{R}(\beta_1, \beta_2)\xi = k[\eta(\beta_2)\beta_1 - \eta(\beta_1)\beta_2] + \mu[\eta(\beta_2)h\beta_1 - \eta(\beta_1)h\beta_2] \quad (5)$$

for all $\beta_1, \beta_2 \in \chi(M)$, where k and μ are real constants.

This class is very wide containing the para-Sasakian manifolds as well as the paracontact metric manifolds satisfying $R(\beta_1, \beta_2)\xi = 0$ [12].

In particular, if $\mu = 0$, then the paracontact metric manifold is called paracontact metric $N(k)$-manifold. Thus, for a paracontact metric $N(k)$-manifold the curvature tensor satisfies the following relation

$$R(\beta_1, \beta_2)\xi = k(\eta(\beta_2)\beta_1 - \eta(\beta_1)\beta_2) \quad (6)$$

for all $\beta_1, \beta_2 \in \chi(M)$. Though the geometric behavior of paracontact metric (k, μ) −spaces is different according as $k < -1$, or $k > -1$, but there are some common results for $k < -1$ and $k > -1$ [3].

Lemma 2.1 There does not exist any paracontact (k, μ) −manifold of dimension greater than 3 with $k > -1$ which is Einstein whereas there exist such manifolds for $k < -1$ [3].

In a paracontact metric (k, μ) −manifold $(M^{2n+1}\phi, \xi, \eta, g)$, $n > 1$, the following relation hold:

$$h^2 = (k+1)\phi^2, \text{for } k \neq -1, \quad (7)$$

$$(\tilde{\nabla}_{\beta_1} \phi)\beta_2 - g(\beta_1 - h\beta_1, \beta_2)\xi + \eta(\beta_2)(\beta_1 - h\beta_1), \quad (8)$$

$$S(\beta_1, \beta_2) = [2(1-n) + n\mu]g(\beta_1, \beta_2) + [2(n-1) + \mu]g(h\beta_1, \beta_2) + [2(n-1) + n(2k-\mu)]\eta(\beta_1)\eta(\beta_2), \quad (9)$$

$$S(\beta_1, \xi) = 2nk\eta(\beta_1), \quad (10)$$

$$Q\beta_2 = [2(1-n) + n\mu]\beta_2 + [2(n-1) + \mu]h\beta_2 + [2(n-1) + n(2k-\mu)]\eta(\beta_2)\xi \quad (11)$$

$$Q\xi = 2nk\xi, g(Q\beta_1, \beta_2) = S(\beta_1, \beta_2), \quad (12)$$

$$Q\phi - \phi Q = 2[2(n-1) + \mu]h\phi \quad (13)$$

for any vector fields β_1, β_2 on M^{2n+1} , where Q and S denotes the Ricci operator and Ricci tensor of (M^{2n+1}, g), respectively [3].

The concept of conharmonic curvature tensor was defined by Y. Ishii [13]. Conharmonic, pseudo-projective, M −projective, W_0-curvature tensor and W_1-curvature tensor of a $(2n+1)$-dimensional Riemannian manifolds are, respectively, defined

$$L(\beta_1, \beta_2) = R(\beta_1, \beta_2)\beta_3 - \frac{1}{2n-1}[S(\beta_2, \beta_3)\beta_1 - S(\beta_1, \beta_3)\beta_2 + g(\beta_2, \beta_3)Q\beta_1 - g(\beta_1, \beta_3)Q\beta_2], \quad (14)$$

$$P_*(\beta_1, \beta_2)\beta_3 = aR(\beta_1, \beta_2)\beta_3 + b[S(\beta_2, \beta_3)\beta_1 - S(\beta_1, \beta_3)\beta_2] - \frac{r}{2n+1}\left(\frac{a}{2n} + b\right)[g(\beta_2, \beta_3)\beta_1 - g(\beta_1, \beta_3)\beta_2], \quad (15)$$

$$M(\beta_1, \beta_2)\beta_3 = R(\beta_1, \beta_2)\beta_3 - \frac{1}{4n}[S(\beta_2, \beta_3)\beta_1 - S(\beta_1, \beta_3)\beta_2 + g(\beta_2, \beta_3)Q\beta_1 - g(\beta_1, \beta_3)Q\beta_2], \quad (16)$$

$$W_0(\beta_1, \beta_2)\beta_3 = R(\beta_1, \beta_2)\beta_3 - \frac{1}{2n}[S(\beta_2, \beta_3)\beta_1 - g(\beta_1, \beta_3)Q\beta_2], \quad (17)$$

$$W_1(\beta_1, \beta_2)\beta_3 = R(\beta_1, \beta_2)\beta_3 + \frac{1}{2n}[S(\beta_2, \beta_3)\beta_1 - S(\beta_1, \beta_3)\beta_2], \quad (18)$$

for all $\beta_1, \beta_2, \beta_3 \in \chi(M)$ [14].

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In this part, we will give the major results for this paper.

Let M be $(2n+1)$ −dimensional (k,μ) −paracontact metric manifold and we denote conharmonic curvature tensor by L, then from (14), we have for later

$$L(\beta_1,\beta_2)\xi = \frac{k}{2n-1}[\eta(\beta_1)\beta_2 - \eta(\beta_2)\beta_1] + \mu[\eta(\beta_2)h\beta_1 - \eta(\beta_1)h\beta_2] - \frac{1}{2n-1}[\eta(\beta_2)Q\beta_1 - \eta(\beta_1)Q\beta_2]. \quad (19)$$

Putting $\beta_1 = \xi$, in (19)

$$L(\xi,\beta_2)\xi = \frac{k}{2n-1}[\beta_2 - \eta(\beta_2)\xi] - \mu h\beta_2 - \frac{1}{2n-1}[2nk\eta(\beta_2)\xi - Q\beta_2]. \quad (20)$$

In (15), choosing $\beta_3 = \xi$ and using (5), we obtain

$$P_*(\beta_1,\beta_2)\xi = \left[ak + 2nkb - \frac{r}{2n+1}\left(\frac{a}{2n} + b\right)\right](\eta(\beta_2)\beta_1 - \eta(\beta_1)\beta_2) + a\mu(\eta(\beta_2)h\beta_1 - \eta(\beta_1)h\beta_2). \quad (21)$$

In (21), it follows

$$P_*(\xi,\beta_2)\xi = [ak + 2nkb - \frac{r}{2n+1}(\frac{a}{2n} + b)](\eta(\beta_2)\xi - \beta_2) - a\mu h\beta_2. \quad (22)$$

In the same way, putting $\beta_3 = \xi$ in (16) and using (5), we have

$$M(\beta_1,\beta_2)\xi = \frac{k}{2}(\eta(\beta_2)\beta_1 - \eta(\beta_1)\beta_2 + \mu(\eta(\beta_2)h\beta_1 - \eta(\beta_1)h\beta_2 - \frac{1}{4n}(\eta(\beta_2)Q\beta_1 - \eta(\beta_1)Q\beta_2). \quad (23)$$

Using $\beta_1 = \xi$ in (23), we get

$$M(\xi,\beta_2)\xi = \frac{1}{4n}Q\beta_2 - \frac{k\beta_2}{2} - \mu h\beta_2. \quad (24)$$

In (17), choosing $\beta_3 = \xi$, we obtain

$$W_0(\beta_1,\beta_2)\xi = \frac{1}{2n}\eta(\beta_1)Q\beta_2 - k\eta(\beta_1)\beta_2 + \mu(\eta(\beta_2)h\beta_1 - \eta(\beta_1)h\beta_2). \quad (25)$$

and

$$W_0(\xi,\beta_2)\xi = \frac{1}{2n}Q\beta_2 - k\beta_2 - \mu h\beta_2. \quad (26)$$

In (18), choosing $\beta_3 = \xi$ and using (5), we obtain

$$W_1(\beta_1,\beta_2)\xi = 2k(\eta(\beta_2)\beta_1 - \eta(\beta_1)\beta_2) + \mu(\eta(\beta_2)h\beta_1 - \eta(\beta_1)h\beta_2). \quad (27)$$

Setting $\beta_1 = \xi$ in (27), we get

$$W_1(\xi,\beta_2)\xi = 2k(\eta(\beta_2)\xi - \beta_2) - \mu h\beta_2. \quad (28)$$

From (5), we can derive

$$R(\xi,\beta_2)\beta_3 = k(g(\beta_2,\beta_3)\xi - \eta(\beta_3)\beta_2) + \mu(g(h\beta_2,\beta_3)\xi - \eta(\beta_3)h\beta_2), \quad (29)$$

Choosing $\beta_3 = \xi$, in (29)

$$R(\xi,\beta_2)\xi = k(\eta(\beta_2)\xi - \beta_2) - \mu h\beta_2. \quad (30)$$

Theorem 3.1 Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a (k,μ)-paracontact space. Then M is a conharmonic semi-symmetric if and only if M is an η −Einstein manifold.

Proof. Suppose that M is a conharmonic semi-symmetric. This implies that

$$(R(\beta_1,\beta_2)L)(\beta_3,\beta_4)\beta_5 = R(\beta_1,\beta_2)L(\beta_3,\beta_4)\beta_5 - L(R(\beta_1,\beta_2)\beta_3,\beta_4)\beta_5 - L(\beta_3,R(\beta_1,\beta_2)\beta_4)\beta_5 - L(\beta_3,\beta_4)R(\beta_1,\beta_2)\beta_5 = 0, \quad (31)$$

for any $\beta_1,\beta_2,\beta_3,\beta_4,\beta_5 \in \chi(M)$. Taking $\beta_1=\beta_5=\xi$ in (31), making use of (19), (29) and (30), for $B=-\frac{1}{2n-1}$, we have

$$(R(\xi,\beta_2)L)(\beta_3,\beta_4)\xi = R(\xi,\beta_2)(Bk(\eta(\beta_4)\beta_3-\eta(\beta_3)\beta_4)+\mu(\eta(\beta_4)h\beta_3-\eta(\beta_3)h\beta_4)+B(\eta(\beta_4)Q\beta_3-\eta(\beta_3)Q\beta_4)- L(k(g(\beta_2,\beta_3)\xi-\eta(\beta_3)\beta_2)+\mu(g(h\beta_2,\beta_3)\xi-\eta(\beta_3)h\beta_2,\beta_4)\xi - L(\beta_3,k(g(\beta_2,\beta_4)\xi-\eta(\beta_4)\beta_2)+\mu(g(h\beta_2,\beta_4)\xi- \eta(\beta_4)h\beta_2)\xi - L(\beta_3,\beta_4)(k(\eta(\beta_2)\xi-\beta_2)-\mu h\beta_2)=0. \quad (32)$$

Taking into account (19), (20), (29) and inner product both sides of (32) by $\beta_5 \in \chi(M)$

$$kg(L(\beta_3,\beta_4)\beta_2,\beta_5)+\mu g(L(\beta_3,\beta_4)f\beta_2,\beta_5)+k\mu\big(\eta(\beta_4)\eta(\beta_5)g(\beta_2,h\beta_3)-\eta(\beta_3)\eta(\beta_5)g(\beta_2,h\beta_4)\big)+\mu^2(1+ k)\big(\eta(\beta_4)\eta(\beta_5)g(\beta_2,\beta_3)-\eta(\beta_3)\eta(\beta_5)g(\beta_2,\beta_4)\big)+B\mu\big(\eta(\beta_4)\eta(\beta_5)S(\beta_2,h\beta_3)-\eta(\beta_3)\eta(\beta_5)S(\beta_2,h\beta_4)\big)+ Bk\big(\eta(\beta_4)\eta(\beta_5)S(\beta_2,\beta_3)-\eta(\beta_3)\eta(\beta_5)S(\beta_2,\beta_4)\big)+2nk\mu B\big(\eta(\beta_3)\eta(\beta_5)g(h\beta_2,\beta_4)-g(h\beta_2,\beta_3)\eta(\beta_4)\eta(\beta_5)\big)+ 2nk^2B\big(g(\beta_2,\beta_4)\eta(\beta_3)\eta(\beta_5)-g(\beta_2,\beta_3)\eta(\beta_4)\eta(\beta_5)\big)+Bk\big(g(\beta_2,\beta_3)S(\beta_4,\beta_5)-g(\beta_2,\beta_4)S(\beta_3,\beta_5)\big)+ Bk^2\big(g(h\beta_2,\beta_3)S(\beta_4,\beta_5)+g(h\beta_2,\beta_4)S(\beta_3,\beta_5)\big)+\mu^2\big(g(h\beta_2,\beta_3)g(h\beta_4,\beta_5)-g(h\beta_2,\beta_4)g(h\beta_3,\beta_5)\big)+ k\mu\big(g(\beta_2,\beta_3)g(h\beta_4,\beta_5)+g(\beta_2,\beta_4)g(h\beta_3,\beta_5)\big)=0. \quad (33)$$

Putting (7), (10), (14) and choosing $\beta_4=\beta_2=e_i,\xi$, in (33), $1\le i\le n$, for orthonormal basis of $\chi(M)$, we arrive

$$k(1-B)S(\beta_3,\beta_5)+\mu(1-B)S(\beta_3,h\beta_5)+(Bkr+2n(1+k)[2(n-1)+\mu]+\mu^2(1+k)-2nk^2B)g(\beta_3,\beta_5)+ (k\mu B-2nk\mu)g(\beta_3,h\beta_5)+\big(\mu^2(1+k)(2n+1)-Bkr-2n\mu B(1+k)[2(n-1)+\mu]+2nk^2B(2n+ 1)\big)\eta(\beta_3)\eta(\beta_5)=0. \quad (34)$$

Using (7) and replacing $h\beta_5$ of β_5 in (34), we get

$$k(1-B)S(\beta_3,h\beta_5)+\mu(1-B)(1+k)S(\beta_3,\beta_5)-2nk\mu(1+k)(1-B)\eta(\beta_2)\eta(\beta_3)+(Bkr+2n(1+k)[2(n-1)+\mu]+ \mu^2(1+k)-2nk^2B)g(\beta_3,h\beta_5)+(1+k)(k\mu B-2nk\mu)g(\beta_3,\beta_5)-(1+k)(k\mu B-2nk\mu)\eta(\beta_3)\eta(\beta_5)=0. \quad (35)$$

From (34), (35) and also using (9), for the sake of brevity we set

$p_1=\frac{2nk}{2n-1}$,

$p_2=\frac{2n\mu}{2n-1}$,

$p_3=\left(-\frac{kr}{2n-1}+2n(1+k)[2(n-1)+\mu]+\mu^2(1+k)+\frac{2nk^2}{2n-1}\right)$,

$p_4=\left(-\frac{k\mu}{2n-1}-2nk\mu\right)$,

$p_5=(\mu^2(1+k)(2n+1)+\frac{kr}{2n-1}+\frac{2n\mu}{2n-1}(1+k)[2(n-1)+\mu]-\frac{2nk^2}{2n-1}(2n+1)$,

and

$q_1=(p_4p_2(1+k)-p_3p_1)[2(n-1)+\mu]+(p_4p_1-p_3p_2)[2(1-n)+n\mu]$,

$q_2=\big(p_1^2-p_2^2(1+k)\big)[2(n-1)+\mu]+(p_4p_1-p_3p_2)$,

$q_3=(p_4p_2-p_3p_2)[2(n-1)+n(2k-\mu)]-\big(p_1p_5+2nkp_2^2(1+k)+p_4p_2(1+k)\big)[2(n-1)+\mu]$,

we conclude

$$q_2S(\beta_3,\beta_5)=q_1g(\beta_3,\beta_5)+q_3\eta(\beta_3)\eta(\beta_5).$$

So, M is an η −Einstein manifold. Conversely, let $M^{2n+1}(\varphi,\xi,\eta,g)$ be an η −Einstein manifold, i.e. $q_2S(\beta_3,\beta_5)= q_1g(\beta_3,\beta_5)+q_3\eta(\beta_3)\eta(\beta_5)$, then from equations (35), (34), (33), (32) and (31) we obtain M is a conharmonic semi-symmetric.

Theorem 3.2 Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a (k,μ)-paracontact space. Then M is a pseudo-projective semi-symmetric if and only if M is an Einstein manifold.

Proof. Assume that M is a pseudo-projective semi-symmetric. This yields to

$$(R(\beta_1,\beta_2)P_*)(\beta_3,\beta_4)\beta_5=R(\beta_1,\beta_2)P_*(\beta_3,\beta_4)\beta_5-P_*(R(\beta_1,\beta_2)\beta_3,\beta_4)\beta_5-P_*(\beta_3,R(\beta_1,\beta_2)\beta_4)\beta_5- P_*(\beta_3,\beta_4)R(\beta_1,\beta_2)\beta_5=0, \quad (36)$$

for any $\beta_1,\beta_2,\beta_3,\beta_4,\beta_5\in\chi(M)$. Taking $\beta_1=\beta_5=\xi$ in (36) and using (21), (29), (30), for $A=[ak+2nkb-\frac{r}{2n+1}(\frac{a}{2n}+b)]$, we obtain

$$(R(\xi,\beta_2)P_*)(\beta_3,\beta_4)\xi = R(\xi,\beta_2)(A(\eta(\beta_4)\beta_3-\eta(\beta_3)\beta_4)+a\mu(\eta(\beta_4)h\beta_3-\eta(\beta_3)h\beta_4)-P_*(k(g(\beta_2,\beta_3)\xi-\eta(\beta_3)\beta_2)+\mu(g(h\beta_2,\beta_3)\xi-\eta(\beta_3)h\beta_2),\beta_4)\xi-P_*(\beta_3,kg(\beta_2,\beta_4)\xi-\eta(\beta_4)\beta_2)+\mu(g(h\beta_2,\beta_4)\xi-\eta(\beta_4)h\beta_2)\xi-P_*(\beta_3,\beta_4)k(\eta(\beta_2)\xi-\beta_2)-\mu h\beta_2)=0. \quad (37)$$

Again, taking into account that (21), (22), (29) in (37), we get

$$kP_*(\beta_3,\beta_4)\beta_2+\mu P_*(\beta_3,\beta_4)h\beta_2+ak\mu(\eta(\beta_4)g(\beta_2,h\beta_3)\xi-\eta(\beta_3)g(\beta_2,h\beta_4)\xi)+a\mu^2(1+k)(\eta(\beta_4)g(\beta_2,\beta_3)\xi-\eta(\beta_3)g(\beta_2,\beta_4)\xi)+Ak(g(\beta_2,\beta_3)\beta_4-g(\beta_2,\beta_4)\beta_3)+A\mu(g(h\beta_2,\beta_3)\beta_4)-g(h\beta_2,\beta_4)\beta_3)+a\mu^2(g(h\beta_2,\beta_3)h\beta_4-g(h\beta_2,\beta_4)h\beta_3)+ak\mu(g(\beta_2,\beta_3)h\beta_4-g(\beta_2,\beta_4)h\beta_3)=0. \quad (38)$$

Putting $\beta_3=\xi$, using (7), (21) and inner product both sides of in (38) by $\xi\in\chi(M)$, we get

$$bkS(\beta_2,\beta_4)+b\mu S(\beta_4,h\beta_2)-2nk^2bg(\beta_2,\beta_4)-2nkb\mu g(\beta_4,h\beta_2)=0 \quad (39)$$

Replacing $h\beta_4$ of β_4 in (39) and making use of (7), we have

$$bkS(\beta_2,h\beta_4)+b\mu(1+k)S(\beta_2,\beta_4)-2nkb\mu(1+k)\eta(\beta_2)\eta(\beta_4)-2nk^2bg(\beta_2,h\beta_4)-2nkb\mu(1+k)g(\beta_2,\beta_4)+2nkb\mu(1+k)\eta(\beta_2)\eta(\beta_4)=0. \quad (40)$$

From (39) and (40), we obtain

$$S(\beta_2,\beta_4)=2nkg(\beta_2,\beta_4).$$

Thus, M is an Einstein manifold. Conversely, let $M^{2n+1}(\varphi,\xi,\eta,g)$ be an Einstein manifold, i.e., $S(\beta_2,\beta_4)=2nkg(\beta_2,\beta_4)$, then from equations (40), (39), (38), (37) and (36), we arrive M is a pseudo-projective semi-symmetric. This implies that

$$\mu=2(k+1-\frac{1}{n}).$$

Theorem 3.3 Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a (k,μ)-paracontact space. Then M is a M −projective semi-symmetric if and only if M is an Einstein manifold.

Proof. Suppose that M is a M −projective semi-symmetric. This implies that

$$(R(\beta_1,\beta_2)M)(\beta_3,\beta_4)\beta_5=R(\beta_1,\beta_2)M(\beta_3,\beta_4)\beta_5-M(R(\beta_1,\beta_2)\beta_3,\beta_4)\beta_5-M(\beta_3,R(\beta_1,\beta_2)\beta_4)\beta_5-M(\beta_3,\beta_4)R(\beta_1,\beta_2)\beta_5=0, \quad (41)$$

for any $\beta_1,\beta_2,\beta_3,\beta_4,\beta_5\in\chi(M)$. Setting $\beta_1=\beta_5=\xi$ in (41) and making use of (23), (29), (30), for $A=\frac{k}{2}$, $B=-\frac{1}{4n}$, we obtain

$$(R(\xi,\beta_2)M)(\beta_3,\beta_4)\xi=R(\xi,\beta_2)(A(\eta(\beta_4)\beta_3-\eta(\beta_3)\beta_4)+\mu(\eta(\beta_4)h\beta_3-\eta(\beta_3)h\beta_4)+B(\eta(\beta_4)Q\beta_3-\eta(\beta_3)Q\beta_4)-M(k(g(\beta_2,\beta_3)\xi-\eta(\beta_3)\beta_2)+\mu(g(h\beta_2,\beta_3)\xi-\eta(\beta_3)h\beta_2),\beta_4)\xi-M\big(\beta_3,k(g(\beta_2,\beta_4)\xi-\eta(\beta_4)\beta_2)+\mu(g(h\beta_2,\beta_4)\xi-\eta(\beta_4)h\beta_2)\big)\xi-M(\beta_3,\beta_4)(k(\eta(\beta_2)\xi-\beta_2)-\mu h\beta_2)=0. \quad (42)$$

Inner product both sides of (42) by $\beta_5\in\chi(M)$, using of (23), (24) and (29), we get

$$kg(M(\beta_3,\beta_4)\beta_2,\beta_5)+\mu g(M(\beta_3,\beta_4)h\beta_2,\beta_5)+Ak\big(\eta(\beta_4)\eta(\beta_5)g(\beta_2,h\beta_3)-\eta(\beta_3)\eta(\beta_5)g(\beta_2,\beta_4)\big)+\mu^2(1+k)\big(\eta(\beta_4)\eta(\beta_5)g(\beta_3,\beta_2)-\eta(\beta_3)\eta(\beta_5)g(\beta_2,\beta_4)\big)+A\mu\big(\eta(\beta_4)\eta(\beta_5)g(h\beta_2,\beta_3)-\eta(\beta_5)\eta(\beta_3)g(h\beta_2,\beta_4)\big)+k\mu\big(\eta(\beta_4)\eta(\beta_5)g(h\beta_2,\beta_3)-\eta(\beta_5)\eta(\beta_3)g(h\beta_2,\beta_4)\big)+Bk\big(\eta(\beta_4)\eta(\beta_5)S(\beta_2,\beta_3)-\eta(\beta_5)\eta(\beta_3)S(\beta_2,\beta_4)\big)+\mu B\big(\eta(\beta_4)\eta(\beta_5)S(h\beta_2,\beta_3)-\eta(\beta_5)\eta(\beta_3)S(h\beta_2,\beta_4)\big)+ak\big(g(\beta_2,\beta_3)g(\beta_5,\beta_4)-g(\beta_2,\beta_4)g(\beta_3,\beta_5)\big)+k\mu\big(g(\beta_3,\beta_2)g(h\beta_4,\beta_5)-g(\beta_2,\beta_4)g(h\beta_3,\beta_5)\big)+\mu^2\big(g(h\beta_4,\beta_5)g(h\beta_2,\beta_3)-g(h\beta_2,\beta_4)g(h\beta_3,\beta_5)\big)+A\mu\big(g(\beta_5,\beta_4)g(h\beta_2,\beta_3)-g(\beta_3,\beta_5)g(\beta_2,h\beta_4)\big)=0. \quad (43)$$

Making use of (7), (16) and choosing $\beta_3=\beta_5=e_i,\xi,\ 1\le i\le n$, for orthonormal basis of $\chi(M)$ in (43), we have

$$kS(\beta_4,\beta_2)+\mu S(\beta_4,h\beta_2)-2nk^2g(\beta_4,\beta_2)-2nk\mu g(\beta_4,h\beta_2)=0. \quad (44)$$

Replacing $h\beta_2$ of β_2 in (44) and taking into account (7), we get

$$kS(\beta_4, h\beta_2) + \mu(1+k)S(\beta_4, \beta_2) - 2nk^2 g(\beta_4, h\beta_2) - 2nk\mu(1+k)g(\beta_4, \beta_2) = 0. \quad (45)$$

From (44), (45) and by using (9), we set

$$S(\beta_4, \beta_2) = 2nkg(\beta_4, \beta_2),$$

This tell us M is an Einstein manifold. Conversely, let $M^{2n+1}(\varphi, \xi, \eta, g)$ be an Einstein manifold, i.e., $S(\beta_4, \beta_2) = 2nkg(\beta_4, \beta_2)$, then from equations (45), (44), (43), (42) and (41), we get M is a M −projective semi-symmetric.

Theorem 3.4 Let $M^{2n+1}(\phi, \xi, \eta, g)$ be a (k, μ)-paracontact space. Then M is a W_0-semi-symmetric if and only if M is an η −Einstein manifold.

Proof. Assume that M is a W_0-semi-symmetric. This means that

$$(R(\beta_1, \beta_2)W_0)(\beta_3, \beta_4, \beta_5) = R(\beta_1, \beta_2)W_0(\beta_3, \beta_4)\beta_5 - W_0(R(\beta_1, \beta_2)\beta_3, \beta_4)\beta_5 - W_0(\beta_2, R(\beta_1, \beta_2)\beta_4)\beta_5 - W_0(\beta_3, \beta_4)R(\beta_1, \beta_2)\beta_5 = 0, \quad (46)$$

for any $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5 \in \chi(M)$. Setting $\beta_1 = \beta_5 = \xi$ in (46) and making use of (25), (29), (30), for $A = -\frac{1}{2n}$, we obtain

$$(R(\xi, \beta_2)W_0)(\beta_3, \beta_4)\xi = R(\xi, \beta_2)\big(-A\eta(\beta_3)Q\beta_4 - k\eta(\beta_3)\beta_4 + \mu(\eta(\beta_4)h\beta_3 - \eta(\beta_3)h\beta_4)\big) - W_0(k(g(\beta_2, \beta_3)\xi - \eta(\beta_3)\beta_4) + \mu(g(h\beta_2, \beta_3)\xi - \eta(\beta_3)h\beta_2, \beta_4)\xi - W_0\big(\beta_3, k(g(\beta_2, \beta_4)\xi - \eta(\beta_4)\beta_2) + \mu(g(h\beta_2, \beta_4)\xi - \eta(\beta_4)h\beta_2)\big)\xi - W_0(\beta_3, \beta_4)(k(\eta(\beta_2)\xi - \beta_2) - \mu h\beta_2) = 0. \quad (47)$$

Using (25), (26), (29) and inner product both sides of (47) by $\beta_5 \in \chi(M)$, we get

$$kg(W_0(\beta_3, \beta_4)\beta_2, \beta_5) + \mu g(W_0(\beta_3, \beta_4)h\beta_2, \beta_5) + k\mu\big(\eta(\beta_4)\eta(\beta_5)g(\beta_2, h\beta_3) - \eta(\beta_3)\eta(\beta_5)g(\beta_2, h\beta_4)\big) + \mu^2(1+k)\big(\eta(\beta_4)\eta(\beta_5)g(\beta_2, \beta_3) - \eta(\beta_3)\eta(\beta_5)g(\beta_2, \beta_4)\big) + 2nkA\big(k\eta(\beta_3)\eta(\beta_4)g(\beta_2, \beta_5) - \mu\eta(\beta_4)\eta(\beta_3)g(h\beta_2, \beta_5)\big) + Ak\big(S(\beta_4, \beta_5)g(\beta_2, \beta_3) - \eta(\beta_3)\eta(\beta_5)S(\beta_2, \beta_4)\big) + k^2\big(g(\beta_2, \beta_3)g(\beta_5, \beta_4) - g(\beta_2, \beta_4)g(\beta_3, \beta_5)\big) + k\mu\big(g(\beta_2, \beta_3)g(h\beta_4, \beta_5) - g(\beta_2, \beta_4)g(h\beta_3, \beta_5)\big) + A\mu\big(g(h\beta_2, \beta_3)S(\beta_4, \beta_5) - g(h\beta_2, \beta_4)S(\beta_3, \beta_5)\big) + k\mu\big(g(h\beta_2, \beta_3)g(\beta_4, \beta_5) + g(h\beta_2, \beta_4)g(\beta_3, \beta_5)\big) + \mu^2\big(g(h\beta_2, \beta_3)g(h\beta_4, \beta_5) - g(h\beta_2, \beta_4)g(h\beta_3, \beta_5)\big) - A\mu\big(S(h\beta_2, \beta_4)\eta(\beta_3)\eta(\beta_5) + \eta(\beta_3)\eta(\beta_4)S(h\beta_2, \beta_5)\big) - kA\big(S(\beta_2, \beta_5)\eta(\beta_3)\eta(\beta_4) + S(\beta_3, \beta_5)g(\beta_2, \beta_4)\big) - k\big(\eta(\beta_5)\eta(\beta_3)g(\beta_2, \beta_4) - \mu\eta(\beta_5)\eta(\beta_3)g(\beta_2, h\beta_4)\big) = 0. \quad (48)$$

Making use of (7), (17) and choosing $\beta_2 = \beta_4 = e_i,\ \xi,\ 1 \le i \le n$, for orthonormal basis of $\chi(M)$ in (48), we have

$$k\big(1 - A(2n+1)\big)S(\beta_3, \beta_5) + \mu S(\beta_3, h\beta_5) + \big(kAr + 2n\mu A(1+k)[2(n-1)+\mu] - 2nk^2 + \mu^2(1+k)\big)g(\beta_3, \beta_4) + k\mu(1-2n)g(\beta_3, h\beta_5) + (-k^2(2n+1) - Akr - \mu^2(1+k)(2n+1)(-k^2(2n+1) - Akr - \mu^2(1+k)(2n+1)\eta(\beta_3)\eta(\beta_5) = 0. \quad (49)$$

Replacing $h\beta_5$ of β_5 in (49) and taking into account (7), it follows

$$k\big(1 - A(2n+1)\big)S(\beta_3, h\beta_5) + \mu(1+k)S(\beta_3, \beta_5) - 2nk\mu(1+k)\eta(\beta_3)\eta(\beta_5) + (kAr + 2n\mu A(1+k)[2(n-1)+\mu] - 2nk^2 + \mu^2(1+k)g(\beta_3, h\beta_5) + k\mu(1+k)(1-2n)g(\beta_3, \beta_5) - k\mu(1+k)(1-2n)\eta(\beta_3)\eta(\beta_5) = 0. \quad (50)$$

From (49), (50) and by using (9), for the sake of brevity we set

$p_1 = k\left(2 + \frac{1}{2n}\right)$,

$p_2 = \left(-\frac{kr}{2n} - \mu(1+k)[2(n-1)+\mu] - 2nk^2 + \mu^2(1+k)\right)$,

$p_3 = k\mu(1-2n)$,

$p_4 = (-k^2(2n+1) + \frac{kr}{n} - \mu^2(1+k)(2n+1) - k^2(2n+1) - \mu^2(1+k)(2n+1)$,

and

$q_1 = (p_3\mu(1+k) - p_1p_2)[2(n-1)+\mu] + (p_1p_3 - p_2\mu)[2(1-n) + n\mu]$,

$q_2 = \big(p_1^2 - \mu^2(1+k)\big)[2(n-1)+\mu] + (p_1p_3 - p_2\mu)$,

$q_3 = (p_1p_3 - p_2\mu)[2(n-1) + n(2k-\mu)] - \big(p_1p_4 + 2nk\mu^2(1+k) + p_3\mu(1+k)\big)[2(n-1)+\mu]$,

we have

$$q_2S(\beta_3, \beta_5) = q_1g(\beta_3, \beta_5) + q_3\eta(\beta_3)\eta(\beta_5).$$

Thus, M is an η −Einstein manifold. Conversely, let $M^{2n+1}(\varphi,\xi,\eta,g)$ be an η −Einstein manifold, i.e., $q_2 S(\beta_3,\beta_5) = q_1 g(\beta_3,\beta_5) + q_3\eta(\beta_3)\eta(\beta_5)$, then from equations (50), (49), (48), (47) and (46) we obtain M is a W_0-semi-symmetric.

Theorem 3.5 Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a (k,μ)-paracontact space. Then M is a W_1-semi-symmetric if and only if M is an Einstein manifold.

Proof. Suppose that M is a W_1-semi-symmetric. This means that

$$(R(\beta_1,\beta_2)W_1)(\beta_3,\beta_4,\beta_5) = R(\beta_1,\beta_2)W_1(\beta_3,\beta_4)\beta_5 - W_1(R(\beta_1,\beta_2)\beta_3,\beta_4)\beta_5 - W_1(\beta_3,R(\beta_1,\beta_2)\beta_4)\beta_5 - W_1(\beta_3,\beta_4)R(\beta_1,\beta_2)\beta_5 = 0, \quad (51)$$

for any $\beta_1,\beta_2,\beta_3,\beta_4,\beta_5 \in \chi(M)$. Setting $\beta_1 = \beta_5 = \xi$ in (51) and making use of (27), (29) and (30), we obtain

$$(R(\xi,\beta_2)W_1)(\beta_3,\beta_4)\xi = R(\xi,\beta_2)\big(2k(\eta(\beta_4)\beta_3 - \eta(\beta_3)\beta_4) + \mu(\eta(\beta_4)h\beta_3 - \eta(\beta_3)h\beta_4)\big) - W_1(k(g(\beta_2,\beta_3)\xi - \eta(\beta_3)\beta_2) + \mu(g(h\beta_2,\beta_3)\xi - \eta(\beta_3)h\beta_2),\beta_4)\xi - W_1\big(\beta_3,k(g(\beta_2,\beta_4)\xi - \eta(\beta_4)\beta_2) + \mu(g(h\beta_2,\beta_4)\xi - \eta(\beta_4)h\beta_2)\big)\xi - W_1(\beta_3,\beta_4)(k(\eta(\beta_2)\xi - \beta_2) - \mu h\beta_2) = 0. \quad (52)$$

Using (27) and (29), we get

$$kW_1(\beta_3,\beta_4)\beta_2 + \mu W_1(\beta_3,\beta_4)h\beta_2 + k\mu(\eta(\beta_4)g(\beta_2,h\beta_3)\xi - \eta(\beta_3)g(\beta_2,h\beta_4)\xi) + \mu^2(1+k)(\eta(\beta_4)g(\beta_2,\beta_3)\xi - \eta(\beta_3)g(\beta_2,\beta_4)\xi) + 2k^2(g(\beta_2,\beta_3)\beta_4 - g(\beta_2,\beta_4)\beta_3) + k\mu(g(\beta_2,\beta_3)h\beta_4 - g(\beta_2,\beta_4)h\beta_3) + 2k\mu(g(h\beta_2,\beta_3)\beta_4 - g(h\beta_2,\beta_4)\beta_3) + \mu^2(g(h\beta_2,\beta_3)h\beta_4 + g(h\beta_2,\beta_4)h\beta_3) = 0. \quad (53)$$

Making use of (10), (18) and choosing $\beta_3 = \xi$ and inner product both sides of in (53) by $\xi \in \chi(M)$, we have

$$kS(\beta_4,\beta_2) + \mu S(\beta_4,h\beta_2) - 2nk^2 g(\beta_4,\beta_2) - 2nk\mu g(h\beta_2,\beta_4) = 0. \quad (54)$$

Replacing $h\beta_2$ of β_2 in (54) and by using (7), we get

$$kS(\beta_4,h\beta_2) + \mu(1+k)S(\beta_4,h\beta_2) - 2nk^2 g(\beta_4,h\beta_2) - 2nk\mu(1+k)g(\beta_4,\beta_2) = 0. \quad (55)$$

From (54) and (55), we obtain
$S(\beta_2,\beta_4) = 2nkg(\beta_2,\beta_4)$.

So, M is an Einstein manifold. Conversely, let $M^{2n+1}(\varphi,\xi,\eta,g)$ be an Einstein manifold, i.e., $S(\beta_2,\beta_4) = 2nkg(\beta_2,\beta_4)$, then from equations (55), (54), (53), (52) and (51) we get M is a W_1-semi-symmetric.

Conflicts of interest

There are no conflicts of interest in this work.

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