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A Note on Fractional Midpoint Type Inequalities for Co-ordinated (s1, s2)-Convex Functions

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ABSTRACT

In the present paper, some Hermite-Hadamard type inequalities in the case of differentiable co-ordinated (s_1, s_2)-convex functions are investigated. Namely, the generalizations of the midpoint type inequalities in the case of differentiable co-ordinated (s_1, s_2)-convex functions in the second sense on the rectangle from the plain are established. In addition to this, it is presented several inequalities to the case of Riemann-Liouville fractional integrals and k-Riemann-Liouville fractional integrals by choosing the special cases of our obtained main results.

Keywords: Midpoint inequality, Co-ordinated (s_1, s_2)-convex function, Generalized fractional integrals.

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Introduction

The theory of inequality is a considerable topic and remains an interesting research area with numerous number of applications in many mathematical fields. Additionally, convex functions have also an important place in the theory of inequality. One of the most famous inequalities for the case of convex functions is the Hermite-Hadamard inequality. Therefore, many mathematicians have established Hermite-Hadamard-type inequalities and related inequalities such as trapezoid, midpoint, and Simpson's inequality. Furthermore, Fractional calculus has increased interest owing to the its demonstrated applications in a range of the inequality theory on convex functions in recent years. Because of the importance of fractional calculus, mathematicians have investigated distinct fractional integral inequalities.

The inequalities, established by C. Hermite and J. Hadamard for convex functions, are significant topic in the literature. These inequalities state that if F : I→R is a convex function on the interval I of real numbers and a, b∈I with a<b, then the following double inequality holds:

$$F\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b F(x)dx \leq \frac{F(a)+F(b)}{2}. \qquad (1)$$

In recent years, remarkable number of papers have been investigated to trapezoid and midpoint type inequalities which give bounds for the right-hand side and left-hand side of the inequality (1), respectively. For instance, Dragomir and Agarwal first established to trapezoid inequalities in the case of convex functions in the [1] and Kirmacı first investigated to midpoint inequalities to the case of convex functions in the [2]. Iqbal et al. presented some fractional midpoint type inequalities for convex functions in [3]. On the other hand, Dragomir established Hermite-Hadamard inequalities in the case of co-ordinated convex mappings in [4]. The midpoint and trapezoid type inequalities for co-ordinated convex functions were presented in the papers [5] and [6], respectively. Moreover, some fractional midpoint type inequalities for co-ordinated convex functions were presented in the paper [7].

In [8], Sarikaya and Ertuğral first investigated new fractional integrals which are called generalized fractional integrals. Moreover, several trapezoids and midpoint type inequalities for generalized fractional integrals are proved by the authors. In addition to these, Turkay et al. described the generalized fractional integrals in the case of functions with two variables. These authors investigated Hermite-Hadamard inequalities for this kind of fractional integrals in [9]. For more information about these type of inequalities, we refer to [10-12].

Premiliaries & Generalized Fractional and Double Fractional Integrals

In this section, some definitions and notations which are used frequently in main section are presented.

Definition 1. A function $F\colon \Delta \to \mathbb{R}$ will be called s-convex in the second sense on Δ if the following inequality

$$F(tx+(1-t)\,z,ty+(1-t)\,w)\le t^sF(x,y)+(1-t)^sF(z,w)$$

is valid for all $s,t\in[0,1]$ and $(x,y),(z,w)\in\Delta$.
Modifications to the case of convex and s −convex functions on Δ, which are also known as co-ordinated convex and co-ordinated s-convex functions on Δ, respectively, were introduced by Dragomir [5], Sarikaya [6] and Latif [13].
In the paper [6,14], clasical definition in the case of co-ordinated s-convex functions in the second sense can be stated as follows:
Definition 2. A function $F:\Delta\to\mathbb{R}^2$ is called co-ordinated s-convex in the second sense on Δ if the following inequality holds

$$\begin{aligned}F(tx+(1-t)\,z,\lambda y+(1-\lambda)\,w)&\le t^s\lambda^sF(x,y)+(1-t)^s\lambda^sF(z,y)\\&+t^s(1-\lambda)^sF(x,w)+(1-t)^s(1-t)^sF(z,w)\end{aligned}\tag{2}$$

for all $t,\lambda\in[0,1]$

and $(x,y),(z,w)\in\Delta$, and for fixed $s\in(0,1]$.
Let us consider $s=1$ in inequality (2). Then, the function F is said to be co-ordinated convex on Δ. If the inequality (2) is in reversed order, then F will be called a co-ordinated s- concave in the second sense on Δ.
Definition 3. A function $F:\Delta\to\mathbb{R}^2$ is said to be co-ordinated (s_1,s_2)-convex in the second sense on Δ if the following inequality

$$\begin{aligned}F(tx+(1-t)\,z,\lambda y+(1-\lambda)\,w)&\le t^{s_1}\lambda^{s_2}F(x,y)+(1-t)^{s_1}\lambda^{s_2}F(z,y)\\&+t^{s_1}(1-\lambda)^{s_2}F(x,w)+(1-t)^{s_1}(1-t)^{s_2}F(z,w)\end{aligned}$$

is valid for all $t,\lambda\in[0,1]$ and $(x,y),(z,w)\in\Delta$, and for fixed $s_1,s_2\in(0,1]$.
The well-known gamma and beta are defined as follows: For $0<x,y<\infty$, and $x,y\in\mathbb{R}$,

$$\Gamma(x)\ :=\int_0^\infty t^{x-1}e^{-t}\,dt$$

and

$$\mathfrak{B}(x,y)\ :=\int_0^1 t^{x-1}(1-t)^{y-1}\,dt=\frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}.$$

The generalized fractional integrals were introduced by Sarikaya and Ertugral as follows:
Definition 4. [8] Let us define a function $\varphi:[0,\infty)\to[0,\infty)$ satisfying the following condition:

$$\int_0^1\frac{\varphi(\eta)}{\eta}d\eta<\infty.$$

Let us consider the following left-sided and right-sided generalized fractional integral operators

$$_{a+}I_\varphi F(x)=\int_a^x\frac{\varphi(x-\eta)}{x-\eta}F(\eta)d\eta,\quad x>a\tag{3}$$

and

$$_{b-}I_\varphi F(x)\ =\int_x^b\frac{\varphi(\eta-x)}{\eta-x}F(\eta)d\eta\,,\quad x<b,\tag{4}$$

respectively.
Some shapes of fractional integrals, namely, Riemann-Liouville fractional integral, k-Riemann-Liouville fractional integral, conformable fractional integral, Hadamard fractional integrals, Katugampola fractional integrals, etc are generalized as the most significant feature of generalized fractional integrals. These significant special cases of the integral operators (3) and (4) are presented as follows:
Let us consider $\varphi(\eta)=\eta$. Then, the operators (3) and (4) become to the Riemann integral.
If we select $\varphi(\eta)=\frac{\eta^\alpha}{\Gamma(\alpha)}$ and $\alpha>0$, the operators (3) and (4) reduce to the Riemann-Liouville fractional integrals $J_{a+}^\alpha F(x)$ and $J_{b-}^\alpha F(x)$, respectively. Here, $\Gamma$is Gamma function.

Let us note that $\varphi(\eta)=\frac{1}{k\Gamma_k(\alpha)}\eta^{\frac{\alpha}{k}}$ and $\alpha,k>0$. Then, the operators (3) and (4) become to the k-Riemann-Liouville fractional integrals $J_{a+,k}^{\alpha}F(x)$ and $J_{b-,k}^{\alpha}F(x)$, respectively. Here, Γ_k is k-Gamma function.

There are several papers on inequalities for generalized fractional integrals in the literature. In [8], Sarikaya and Ertuğral also established Hermite-Hadamard inequalities in the case of generalized fractional integrals. Moreover, Budak et al. proved midpoint type inequalities and extensions of Hermite-Hadamard inequalities in the papers [15] and [16], respectively. There have been a several number of research papers written on these subjects, (see, [17-19] and the references therein.

In [18], the authors are presented the generalized double fractional integrals as follows:

Definition 5. The Generalized double fractional integrals ${}_{a+,c+}I_{\varphi,\psi}$, ${}_{a+,d-}I_{\varphi,\psi}$, ${}_{b-,c+}I_{\varphi,\psi}$, ${}_{b-,d-}I_{\varphi,\psi}$, are described by

$$ {}_{a+,c+}I_{\varphi,\psi}\,F(x,y)=\int_a^x\int_c^y\frac{\varphi(x-\eta)}{x-\eta}\frac{\psi(y-\tau)}{y-\tau}F(\eta,\tau)\,d\tau d\eta,\ \ x>a,\ y>c, \tag{5}$$

$$ {}_{a+,d-}I_{\varphi,\psi}\,F(x,y)=\int_a^x\int_y^d\frac{\varphi(x-\eta)}{x-\eta}\frac{\psi(\tau-y)}{\tau-y}F(\eta,\tau)\,d\tau d\eta,\ \ x>a,\ y<d, \tag{6}$$

$$ {}_{b-,c+}I_{\varphi,\psi}\,F(x,y)=\int_x^b\int_c^y\frac{\varphi(\eta-x)}{\eta-x}\frac{\psi(y-\tau)}{y-\tau}F(\eta,\tau)\,d\tau d\eta,\ \ x<b,\ y>c, \tag{7}$$

and

$$ {}_{b-,d-}I_{\varphi,\psi}\,F(x,y)=\int_x^b\int_y^d\frac{\varphi(\eta-x)}{\eta-x}\frac{\psi(\tau-y)}{\tau-y}F(\eta,\tau)\,d\tau d\eta,\ \ x<b,\ y<d, \tag{8}$$

respectively. Here, $F\in L_1([a,b]\times[c,d])$ and the functions $\varphi,\psi:[0,\infty)\to[0,\infty)$ satisfy $\int_0^1\frac{\varphi(\eta)}{\eta}d\eta<\infty$ and $\int_0^1\frac{\psi(\tau)}{\tau}d\tau<\infty$, respectively.

By using Definition 5, well-known fractional integrals can be obtained by some special choices. For instance;

If we assign $\varphi(\eta)=\eta$ and $\psi(\tau)=\tau$, then the operators (5), (6), (7) and (8) become to the double Riemann integral.

For $\varphi(\eta)=\frac{\eta^{\alpha}}{\Gamma(\alpha)}$, $\psi(\tau)=\frac{\tau^{\beta}}{\Gamma(\beta)}$, $\alpha,\beta>0$, the operators (5), (6), (7) and (8) reduce to the Riemann-Liouville fractional integrals $J_{a+,c+}^{\alpha,\beta}F(x,y)$, $J_{a+,d-}^{\alpha,\beta}F(x,y)$, $J_{b-,c+}^{\alpha,\beta}F(x,y)$ and $J_{b-,d-}^{\alpha,\beta}F(x,y)$, respectively.

Let us consider $\varphi(\eta)=\frac{\eta^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ and $\psi(\tau)=\frac{\tau^{\frac{\beta}{k}}}{k\Gamma_k(\beta)}$, for $\alpha,\beta,k>0$. Then, the operators (5), (6), (7) and (8) reduce to the k-Riemann-Liouville fractional integrals $J_{a+,c+}^{\alpha,\beta,k}F(x,y)$, $J_{a+,d-}^{\alpha,\beta,k}F(x,y)$, $J_{b-,c+}^{\alpha,\beta,k}F(x,y)$ and $J_{b-,d-}^{\alpha,\beta,k}F(x,y)$, respectively. For more information and unexplained subjects, the reader is referred to [20-31], and the references therein.

In the paper [32], an identity for twice partially differentiable mappings involving the double generalized fractional integral is established as follows:

Lemma 1. [32] Let $F:\Delta\to\mathbb{R}$ be an absolutely continuous function on Δ such that the partial derivative of order $\frac{\partial^2F(\eta,\tau)}{\partial\eta\partial\tau}$ exist for all $(\eta,\tau)\in\Delta$. Then, the following equality for generalized fractional integrals holds:

$$\begin{aligned}
&\Phi(a,b,x;\,c,d,y)\\
&=\frac{(x-a)\,(y-c)}{\Upsilon(x,y)}\int_0^1\int_0^1\Lambda_1(x,\eta)\Lambda_2(y,\tau)\frac{\partial^2}{\partial\eta\partial\tau}F\big(\eta b+(1-\eta)\,(a+b-x),\tau d+(1-\tau)\,(c+d-y)\big)\,d\tau d\eta\\
&-\frac{(x-a)\,(d-y)}{\Upsilon(x,y)}\int_0^1\int_0^1\Lambda_1(x,\eta)\Delta_2(y,\tau)\frac{\partial^2}{\partial\eta\partial\tau}F\big(\eta b+(1-\eta)\,(a+b-x),\tau c+(1-\tau)\,(c+d-y)\big)\,d\tau d\eta\\
&-\frac{(b-x)\,(y-c)}{\Upsilon(x,y)}\int_0^1\int_0^1\Delta_1(x,\eta)\Lambda_2(y,\tau)\frac{\partial^2}{\partial\eta\partial\tau}F\big(\eta a+(1-\eta)\,(a+b-x),\tau d+(1-\tau)\,(c+d-y)\big)\,d\tau d\eta\\
&+\frac{(b-x)\,(d-y)}{\Upsilon(x,y)}\int_0^1\int_0^1\Delta_1(x,\eta)\Delta_2(y,\tau)\frac{\partial^2}{\partial\eta\partial\tau}F\big(\eta a+(1-\eta)\,(a+b-x),\tau c+(1-\tau)\,(c+d-y)\big)\,d\tau d\eta,
\end{aligned}$$

where $\Delta:=[a,b]\times[c,d]$,

$$\Phi(a,b,x;\ c,d,y)=F(a+b-x,c+d-y)$$

$$-\frac{1}{\chi_2(y)}\left[\ {}_{d-}I_{\psi}F(a+b-x,c+d-y)+\ {}_{c+}I_{\psi}F(a+b-x,c+d-y)\right]$$

$$-\frac{1}{\chi_1(x)}\left[\ {}_{b-}I_{\varphi}F(a+b-x,c+d-y)+\ {}_{a+}I_{\varphi}F(a+b-x,c+d-y)\right]$$

$$+\frac{1}{\Upsilon(x,y)}\Big[\ {}_{b-,d-}I_{\varphi,\psi}F(a+b-x,c+d-y)+\ {}_{b-,c+}I_{\varphi,\psi}F(a+b-x,c+d-y)$$

$$+\ {}_{a+,d-}I_{\varphi,\psi}F(a+b-x,c+d-y)+\ {}_{a+,c+}I_{\varphi,\psi}F(a+b-x,c+d-y)\Big],$$

and

$$\begin{cases}\chi_1(x)=\Lambda_1(x,0)+\Delta_1(x,0),\\ \chi_2(y)=\Lambda_2(y,0)+\Delta_2(y,0),\\ \Upsilon(x,y)=\chi_1(x)\chi_2(y).\end{cases}$$

Throughout this paper for brevity, let us consider

$$\Lambda_1(x,\eta)=\int_{\eta}^{1}\frac{\varphi\big((b-x)\,u\big)}{u}du,\quad \Delta_1(x,\eta)=\int_{\eta}^{1}\frac{\varphi\big((x-a)\,u\big)}{u}du,$$

and

$$\Lambda_2(y,\tau)=\int_{\tau}^{1}\frac{\psi\big((d-y)\,u\big)}{u}du,\quad \Delta_2(y,\tau)=\int_{\tau}^{1}\frac{\psi\big((y-c)\,u\big)}{u}du.$$

Midpoint Inequalities for Co-ordinated (s_1,s_2)-Convex Function Involving Generalized Fractional Integrals Discussion

Theorem 1. Suppose that the assumptions of Lemma 1 hold. Suppose also that the function $\left|\frac{\partial^2 F}{\partial\eta\partial\tau}\right|$ is co-ordinated (s_1,s_2)-convex on Δ. Then, we have the following inequality for generalized fractional integrals

$$\begin{aligned}
&|\Phi(a,b,x;\ c,d,y)|\\
&\le\frac{(x-a)\,(y-c)}{\Upsilon(x,y)}\Bigg[\lambda_1\mu_1\left|\frac{\partial^2}{\partial\eta\partial\tau}F(b,d)\right|+\lambda_1\mu_2\left|\frac{\partial^2}{\partial\eta\partial\tau}F(b,c+d-y)\right|\\
&+\lambda_2\mu_1\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a+b-x,d)\right|+\lambda_2\mu_2\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a+b-x,c+d-y)\right|\Bigg]\\
&+\frac{(x-a)\,(d-y)}{\Upsilon(x,y)}\Bigg[\lambda_1\mu_4\left|\frac{\partial^2}{\partial\eta\partial\tau}F(b,c)\right|+\lambda_1\mu_3\left|\frac{\partial^2}{\partial\eta\partial\tau}F(b,c+d-y)\right|\\
&+\lambda_2\mu_4\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a+b-x,c)\right|+\lambda_2\mu_3\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a+b-x,c+d-y)\right|\Bigg]\\
&+\frac{(b-x)\,(y-c)}{\Upsilon(x,y)}\Bigg[\lambda_4\mu_1\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a,d)\right|+\lambda_4\mu_2\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a,c+d-y)\right|\\
&+\lambda_3\mu_1\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a+b-x,d)\right|+\lambda_3\mu_2\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a+b-x,c+d-y)\right|\Bigg]\\
&+\frac{(b-x)\,(d-y)}{\Upsilon(x,y)}\Bigg[\lambda_4\mu_4\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a,c)\right|+\lambda_4\mu_3\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a,c+d-y)\right|\\
&+\lambda_3\mu_4\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a+b-x,c)\right|+\lambda_3\mu_3\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a+b-x,c+d-y)\right|\Bigg].
\end{aligned}$$

Here,

$$\begin{cases} \lambda_1 = \int_0^1 \eta^{s_1} \Lambda_1(x,\eta)d\eta, & \lambda_2 = \int_0^1 (1-\eta)^{s_1} \Lambda_1(x,\eta)d\eta, \\ \lambda_3 = \int_0^1 (1-\eta)^{s_1} \Delta_1(x,\eta)d\eta, & \lambda_4 = \int_0^1 \eta^{s_1} \Delta_1(x,\eta)d\eta, \end{cases} \tag{9}$$

and

$$\begin{cases} \mu_1 = \int_0^1 \tau^{s_2} \Lambda_2(y,\tau)d\tau, & \mu_2 = \int_0^1 (1-\tau)^{s_2} \Lambda_2(y,\tau)d\tau, \\ \mu_3 = \int_0^1 (1-\tau)^{s_2} \Delta_2(y,\tau)d\tau, & \mu_4 = \int_0^1 \tau^{s_2} \Delta_2(y,\tau)d\tau. \end{cases} \tag{10}$$

Proof. Let us take the modulus in Lemma 1. Then, it follows:

$$\begin{aligned} &|\Phi(a,b,x;\ c,d,y)| \\ &\leq \frac{(x-a)\,(y-c)}{\Upsilon(x,y)} \int_0^1 \int_0^1 \Lambda_1(x,\eta)\Lambda_2(y,\tau) \left| \frac{\partial^2}{\partial\eta\partial\tau} F\big(\eta b + (1-\eta)\,(a+b-x), \tau d + (1-\tau)\,(c+d-y)\big) \right| d\tau d\eta \\ &+ \frac{(x-a)\,(d-y)}{\Upsilon(x,y)} \int_0^1 \int_0^1 \Lambda_1(x,\eta)\Delta_2(y,\tau) \left| \frac{\partial^2}{\partial\eta\partial\tau} F\big(\eta b + (1-\eta)\,(a+b-x), \tau c + (1-\tau)\,(c+d-y)\big) \right| d\tau d\eta \\ &+ \frac{(b-x)\,(y-c)}{\Upsilon(x,y)} \int_0^1 \int_0^1 \Delta_1(x,\eta)\Lambda_2(y,\tau) \left| \frac{\partial^2}{\partial\eta\partial\tau} F\big(\eta a + (1-\eta)\,(a+b-x), \tau d + (1-\tau)\,(c+d-y)\big) \right| d\tau d\eta \\ &+ \frac{(b-x)\,(d-y)}{\Upsilon(x,y)} \int_0^1 \int_0^1 \Delta_1(x,\eta)\Delta_2(y,\tau) \left| \frac{\partial^2}{\partial\eta\partial\tau} F\big(\eta a + (1-\eta)\,(a+b-x), \tau c + (1-\tau)\,(c+d-y)\big) \right| d\tau d\eta. \end{aligned} \tag{11}$$

Since $\left|\frac{\partial^2 F}{\partial\eta\partial\tau}\right|$ is co-ordinated (s_1, s_2)-convex, we have

$$\begin{aligned} &\int_0^1 \int_0^1 \Lambda_1(x,\eta)\Lambda_2(y,\tau) \left| \frac{\partial^2}{\partial\eta\partial\tau} F\big(\eta b + (1-\eta)\,(a+b-x), \tau d + (1-\tau)\,(c+d-y)\big) \right| d\tau d\eta \\ &\leq \int_0^1 \int_0^1 \Lambda_1(x,\eta)\Lambda_2(y,\tau) \Bigg(\eta^{s_1}\tau^{s_2} \left| \frac{\partial^2}{\partial\eta\partial\tau} F(b,d) \right| + \eta^{s_1}(1-\tau)^{s_2} \left| \frac{\partial^2}{\partial\eta\partial\tau} F(b,c+d-y) \right| \\ &\quad + (1-\eta)^{s_1}\tau^{s_2} \left| \frac{\partial^2}{\partial\eta\partial\tau} F(a+b-x,d) \right| + (1-\eta)^{s_1}(1-\tau)^{s_2} \left| \frac{\partial^2}{\partial\eta\partial\tau} F(a+b-x,c+d-y) \right| \Bigg) d\tau d\eta \\ &= \lambda_1\mu_1 \left| \frac{\partial^2}{\partial\eta\partial\tau} F(b,d) \right| + \lambda_1\mu_2 \left| \frac{\partial^2}{\partial\eta\partial\tau} F(b,c+d-y) \right| \\ &\quad + \lambda_2\mu_1 \left| \frac{\partial^2}{\partial\eta\partial\tau} F(a+b-x,d) \right| + \lambda_2\mu_2 \left| \frac{\partial^2}{\partial\eta\partial\tau} F(a+b-x,c+d-y) \right|. \end{aligned} \tag{12}$$

Similarly, it follows

$$\begin{aligned} &\int_0^1 \int_0^1 \Lambda_1(x,\eta)\Delta_2(y,\tau) \left| \frac{\partial^2}{\partial\eta\partial\tau} F\big(\eta b + (1-\eta)\,(a+b-x), \tau c + (1-\tau)\,(c+d-y)\big) \right| d\tau d\eta \\ &\leq \lambda_1\mu_4 \left| \frac{\partial^2}{\partial\eta\partial\tau} F(b,c) \right| + \lambda_1\mu_3 \left| \frac{\partial^2}{\partial\eta\partial\tau} F(b,c+d-y) \right| \\ &\quad + \lambda_2\mu_4 \left| \frac{\partial^2}{\partial\eta\partial\tau} F(a+b-x,c) \right| + \lambda_2\mu_3 \left| \frac{\partial^2}{\partial\eta\partial\tau} F(a+b-x,c+d-y) \right|, \end{aligned} \tag{13}$$

$$\begin{aligned} &\int_0^1 \int_0^1 \Delta_1(x,\eta)\Lambda_2(y,\tau) \left| \frac{\partial^2}{\partial\eta\partial\tau} F\big(\eta a + (1-\eta)\,(a+b-x), \tau d + (1-\tau)\,(c+d-y)\big) \right| d\tau d\eta \\ &\leq \lambda_4\mu_1 \left| \frac{\partial^2}{\partial\eta\partial\tau} F(a,d) \right| + \lambda_4\mu_2 \left| \frac{\partial^2}{\partial\eta\partial\tau} F(a,c+d-y) \right| \\ &\quad + \lambda_3\mu_1 \left| \frac{\partial^2}{\partial\eta\partial\tau} F(a+b-x,d) \right| + \lambda_3\mu_2 \left| \frac{\partial^2}{\partial\eta\partial\tau} F(a+b-x,c+d-y) \right|, \end{aligned} \tag{14}$$

and

$$\int_0^1 \int_0^1 \Delta_1(x,\eta)\Delta_2(y,\tau)\left|\frac{\partial^2}{\partial\eta\partial\tau}F\big(\eta a+(1-\eta)\,(a+b-x),\tau c+(1-\tau)\,(c+d-y)\big)\right| d\tau d\eta$$
$$\leq \lambda_4\mu_4\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a,c)\right| + \lambda_4\mu_3\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a,c+d-y)\right| \tag{15}$$
$$+\lambda_3\mu_4\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a+b-x,c)\right| + \lambda_3\mu_3\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a+b-x,c+d-y)\right|.$$

If it is substituted the inequalities (12)-(15) in (11), then the desired results are obtained. This finishes the proof of Theorem 1.

Corollary 1. Let us consider $\varphi(\eta)=\eta$ and $\psi(\tau)=\tau$ for all $(\eta,\tau)\in\Delta$ in Theorem 1. Let us also consider $s_1=s_2=s$, $x=\frac{a+b}{2}$ and $y=\frac{c+d}{2}$. Then, we have the following midpoint type inequality for Riemann-Liouville fractional integrals

$$\left|F\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{1}{d-c}\left[\,{}_{d-}I_{\psi}F\left(\frac{a+b}{2},\frac{c+d}{2}\right)+\,{}_{c+}I_{\psi}F\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]\right.$$
$$-\frac{1}{b-a}\left[\,{}_{b-}I_{\varphi}F\left(\frac{a+b}{2},\frac{c+d}{2}\right)+\,{}_{a+}I_{\varphi}F\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]$$
$$+\frac{1}{(b-a)\,(d-c)}\left[\,{}_{b-,d-}I_{\varphi,\psi}F\left(\frac{a+b}{2},\frac{c+d}{2}\right)+\,{}_{b-,c+}I_{\varphi,\psi}F\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right.$$
$$\left.\left.+\,{}_{a+,d-}I_{\varphi,\psi}F\left(\frac{a+b}{2},\frac{c+d}{2}\right)+\,{}_{a+,c+}I_{\varphi,\psi}F\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]\right|$$
$$\leq\frac{(b-a)\,(d-c)}{16}\left[\frac{1}{(s+1)^2(s+2)^2}\left(\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a,c)\right|+\left|\frac{\partial^2}{\partial\eta\partial\tau}F(b,d)\right|+\left|\frac{\partial^2}{\partial\eta\partial\tau}F(b,c)\right|+\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a,d)\right|\right)\right.$$
$$+\frac{2}{(s+1)\,(s+2)^2}\left(\left|\frac{\partial^2}{\partial\eta\partial\tau}F\left(a,\frac{c+d}{2}\right)\right|+\left|\frac{\partial^2}{\partial\eta\partial\tau}F\left(b,\frac{c+d}{2}\right)\right|\right.$$
$$\left.\left.+\left|\frac{\partial^2}{\partial\eta\partial\tau}F\left(\frac{a+b}{2},c\right)\right|+\left|\frac{\partial^2}{\partial\eta\partial\tau}F\left(\frac{a+b}{2},d\right)\right|\right)+\frac{4}{(s+2)^2}\left(\left|\frac{\partial^2}{\partial\eta\partial\tau}F\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\right)\right]$$
$$\leq\frac{(b-a)\,(d-c)}{16}\left(\frac{2^{2s}+2^{s+2}(s+1)+4(s+1)^2}{2^{2s}(s+1)^2(s+2)^2}\right)$$
$$\times\left[\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a,c)\right|+\left|\frac{\partial^2}{\partial\eta\partial\tau}F(b,d)\right|+\left|\frac{\partial^2}{\partial\eta\partial\tau}F(b,c)\right|+\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a,d)\right|\right].$$

Remark 1. Let us consider $s_1=s_2=1$ in Corollary 1. Then, Corollary 1 reduces to [5,Theorem 2].

Corollary 2. In Theorem 1, if we assign $\varphi(\eta)=\frac{\eta^{\alpha}}{\Gamma(\alpha)}$ and $\psi(\tau)=\frac{\tau^{\beta}}{\Gamma(\beta)}$ for all $(\eta,\tau)\in\Delta$ and if we choose $s_1=s_2=s$, and $y=\frac{c+d}{2}$, then we have the following midpoint type inequality for Riemann-Liouville fractional integrals

$$\left|F\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{2^{\beta-1}\Gamma(\beta+1)}{(d-c)^{\beta}}\left[J_{d-}^{\beta}F\left(\frac{a+b}{2},\frac{c+d}{2}\right)+J_{c+}^{\beta}F\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]\right.$$

$$-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{b-}^{\alpha}F\left(\frac{a+b}{2},\frac{c+d}{2}\right)+J_{a+}^{\alpha}F\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]$$

$$+\frac{2^{\alpha+\beta-2}\Gamma(\alpha+1)\,\Gamma(\beta+1)}{(b-a)^{\alpha}(d-c)^{\beta}}\left[J_{b-,\,d-}^{\alpha,\,\beta}F\left(\frac{a+b}{2},\frac{c+d}{2}\right)+J_{b-,\,c+}^{\alpha,\,\beta}F\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right.$$

$$\left.\left.+J_{a+,\,d-}^{\alpha,\,\beta}F\left(\frac{a+b}{2},\frac{c+d}{2}\right)+J_{a+,\,c+}^{\alpha,\,\beta}F\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]\right|$$

$$\leq\frac{(b-a)\,(d-c)}{16}\left[\frac{\alpha\beta}{(s+1)^{2}(s+\alpha+1)\,(s+\beta+1)}\right.$$

$$\times\left(\left|\frac{\partial^{2}}{\partial\eta\partial\tau}F(a,c)\right|+\left|\frac{\partial^{2}}{\partial\eta\partial\tau}F(a,d)\right|+\left|\frac{\partial^{2}}{\partial\eta\partial\tau}F(b,c)\right|+\left|\frac{\partial^{2}}{\partial\eta\partial\tau}F(b,d)\right|\right)$$

$$+\frac{2\alpha\left(\frac{1}{s+1}-\mathfrak{P}(\beta+1,s+1)\right)}{(s+1)\,(s+\alpha+1)}\left(\left|\frac{\partial^{2}}{\partial\eta\partial\tau}F\left(a,\frac{c+d}{2}\right)\right|+\left|\frac{\partial^{2}}{\partial\eta\partial\tau}F\left(b,\frac{c+d}{2}\right)\right|\right)$$

$$+\frac{2\beta\left(\frac{1}{s+1}-\mathfrak{P}(\alpha+1,s+1)\right)}{(s+1)\,(s+\beta+1)}\left(\left|\frac{\partial^{2}}{\partial\eta\partial\tau}F\left(\frac{a+b}{2},c\right)\right|+\left|\frac{\partial^{2}}{\partial\eta\partial\tau}F\left(\frac{a+b}{2},d\right)\right|\right)$$

$$\left.+4\left(\frac{1}{s+1}-\mathfrak{P}(\alpha+1,s+1)\right)\left(\frac{1}{s+1}-\mathfrak{P}(\beta+1,s+1)\right)\left|\frac{\partial^{2}}{\partial\eta\partial\tau}F\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\right]$$

$$\leq\frac{(b-a)\,(d-c)}{16}\left[\frac{\alpha\beta}{(s+1)^{2}(s+\alpha+1)\,(s+\beta+1)}+\frac{2\alpha\left(\frac{1}{s+1}-\mathfrak{P}(\beta+1,s+1)\right)}{2^{s}(s+1)\,(s+\alpha+1)}\right.$$

$$\left.+\frac{2\beta\left(\frac{1}{s+1}-\mathfrak{P}(\alpha+1,s+1)\right)}{2^{s}(s+1)\,(s+\beta+1)}+\frac{4}{2^{2s}}\left(\frac{1}{s+1}-\mathfrak{P}(\alpha+1,s+1)\right)\left(\frac{1}{s+1}-\mathfrak{P}(\beta+1,s+1)\right)\right]$$

$$\times\left(\left|\frac{\partial^{2}}{\partial\eta\partial\tau}F(a,c)\right|+\left|\frac{\partial^{2}}{\partial\eta\partial\tau}F(a,d)\right|+\left|\frac{\partial^{2}}{\partial\eta\partial\tau}F(b,c)\right|+\left|\frac{\partial^{2}}{\partial\eta\partial\tau}F(b,d)\right|\right)$$

Remark 2. If it is chosen $s_1 = s_2 = 1$ in Corollary 2, then Corollary 2 becomes to [32, Corollary 1].

Corollary 3. In Theorem 1, if we assign $\varphi(\eta)=\frac{\eta^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ and $\psi(\tau)=\frac{\tau^{\frac{\beta}{k}}}{k\Gamma_k(\beta)}$ for all $(\eta,\tau)\in\Delta$ and if we choose $s_1=s_2=s$, $x=\frac{a+b}{2}$ and $y=\frac{c+d}{2}$, then we have the following midpoint type inequality for k-Riemann-Liouville fractional integrals

$$\left|F\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{2^{\frac{\beta}{k}-1}\Gamma_k(\beta+k)}{(d-c)^{\frac{\beta}{k}}}\left[J_{d-,k}^{\beta}F\left(\frac{a+b}{2},\frac{c+d}{2}\right)+J_{c+,k}^{\beta}F\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]\right.$$

$$-\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}\left[J_{b-,k}^{\alpha}F\left(\frac{a+b}{2},\frac{c+d}{2}\right)+J_{a+,k}^{\alpha}F\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]$$

$$+\frac{2^{\frac{\alpha+\beta}{k}-2}\Gamma_k(\alpha+k)\,\Gamma_k(\beta+k)}{(b-a)^{\frac{\alpha}{k}}(d-c)^{\frac{\beta}{k}}}\left[J_{b-,d-}^{\alpha,\beta,k}F\left(\frac{a+b}{2},\frac{c+d}{2}\right)+J_{b-,c+}^{\alpha,\beta,k}F\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right.$$

$$\left.\left.+J_{a+,d-}^{\alpha,\beta,k}F\left(\frac{a+b}{2},\frac{c+d}{2}\right)+J_{a+,c+}^{\alpha,\beta,k}F\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]\right|$$

$$\leq\frac{(b-a)\,(d-c)}{16}\left[\frac{\alpha\beta}{(s+1)^2(sk+\alpha+k)\,(sk+\beta+k)}\right.$$

$$\times\left(\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a,c)\right|+\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a,d)\right|+\left|\frac{\partial^2}{\partial\eta\partial\tau}F(b,c)\right|+\left|\frac{\partial^2}{\partial\eta\partial\tau}F(b,d)\right|\right)$$

$$+\frac{2\alpha\left(\frac{1}{s+1}-\mathfrak{B}\left(\frac{\beta}{k}+1,s+1\right)\right)}{(s+1)\,(sk+\alpha+k)}\left(\left|\frac{\partial^2}{\partial\eta\partial\tau}F\left(a,\frac{c+d}{2}\right)\right|+\left|\frac{\partial^2}{\partial\eta\partial\tau}F\left(b,\frac{c+d}{2}\right)\right|\right)$$

$$+\frac{2\beta\left(\frac{1}{s+1}-\mathfrak{B}\left(\frac{\alpha}{k}+1,s+1\right)\right)}{(s+1)\,(sk+\beta+k)}\left(\left|\frac{\partial^2}{\partial\eta\partial\tau}F\left(\frac{a+b}{2},c\right)\right|+\left|\frac{\partial^2}{\partial\eta\partial\tau}F\left(\frac{a+b}{2},d\right)\right|\right)$$

$$\left.+4\left(\frac{1}{s+1}-\mathfrak{B}\left(\frac{\alpha}{k}+1,s+1\right)\right)\left(\frac{1}{s+1}-\mathfrak{B}\left(\frac{\beta}{k}+1,s+1\right)\right)\left|\frac{\partial^2}{\partial\eta\partial\tau}F\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\right]$$

$$\leq\frac{(b-a)\,(d-c)}{16}\left[\frac{\alpha\beta}{(s+1)^2(sk+\alpha+k)\,(sk+\beta+k)}+\frac{2\alpha\left(\frac{1}{s+1}-\mathfrak{B}\left(\frac{\beta}{k}+1,s+1\right)\right)}{2^s(s+1)\,(sk+\alpha+k)}\right.$$

$$\left.+\frac{2\beta\left(\frac{1}{s+1}-\mathfrak{B}\left(\frac{\alpha}{k}+1,s+1\right)\right)}{2^s(s+1)\,(sk+\beta+k)}+\frac{4}{2^{2s}}\left(\frac{1}{s+1}-\mathfrak{B}\left(\frac{\alpha}{k}+1,s+1\right)\right)\left(\frac{1}{s+1}-\mathfrak{B}\left(\frac{\beta}{k}+1,s+1\right)\right)\right]$$

$$\times\left(\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a,c)\right|+\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a,d)\right|+\left|\frac{\partial^2}{\partial\eta\partial\tau}F(b,c)\right|+\left|\frac{\partial^2}{\partial\eta\partial\tau}F(b,d)\right|\right).$$

Remark 3. Let us consider $s_1=s_2=1$ in Corollary 3. Then, Corollary 3 reduces to [32, Corollary 2].

Theorem 2. Assume that the assumptions of Lemma 1 are valid. Assume also that the mapping $\left|\frac{\partial^2 F}{\partial\eta\partial\tau}\right|^q$, $q>1$ is co-ordinated (s_1,s_2)-convex on Δ. Then, the following inequality for generalized fractional integrals holds:

$$|\Phi(a,b,x;\ c,d,y)| \leq \frac{(x-a)\,(y-c)}{\Upsilon(x,y)}\left(\int_0^1\int_0^1 [\Lambda_1(x,\eta)\Lambda_2(y,\tau)]^p d\tau d\eta\right)^{\frac{1}{p}}$$

$$\times\left(\frac{\left|\frac{\partial^2}{\partial\eta\partial\tau}F(b,d)\right|^q+\left|\frac{\partial^2}{\partial\eta\partial\tau}F(b,c+d-y)\right|^q+\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a+b-x,d)\right|^q+\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a+b-x,c+d-y)\right|^q}{(s_1+1)\,(s_2+1)}\right)^{\frac{1}{q}}$$

$$+\frac{(x-a)\,(d-y)}{\Upsilon(x,y)}\left(\int_0^1\int_0^1 [\Lambda_1(x,\eta)\Delta_2(y,\tau)]^p d\tau d\eta\right)^{\frac{1}{p}}$$

$$\times\left(\frac{\left|\frac{\partial^2}{\partial\eta\partial\tau}F(b,c)\right|^q+\left|\frac{\partial^2}{\partial\eta\partial\tau}F(b,c+d-y)\right|^q+\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a+b-x,c)\right|^q+\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a+b-x,c+d-y)\right|^q}{(s_1+1)\,(s_2+1)}\right)^{\frac{1}{q}}$$

$$+\frac{(b-x)\,(y-c)}{\Upsilon(x,y)}\left(\int_0^1\int_0^1 [\Delta_1(x,\eta)\Lambda_2(y,\tau)]^p d\tau d\eta\right)^{\frac{1}{p}}$$

$$\times\left(\frac{\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a,d)\right|^q+\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a,c+d-y)\right|^q+\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a+b-x,d)\right|^q+\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a+b-x,c+d-y)\right|^q}{(s_1+1)\,(s_2+1)}\right)^{\frac{1}{q}}$$

$$+\frac{(b-x)\,(d-y)}{\Upsilon(x,y)}\left(\int_0^1\int_0^1 [\Delta_1(x,\eta)\Delta_2(y,\tau)]^p d\tau d\eta\right)^{\frac{1}{p}}$$

$$\times\left(\frac{\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a,c)\right|^q+\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a,c+d-y)\right|^q+\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a+b-x,c)\right|^q+\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a+b-x,c+d-y)\right|^q}{(s_1+1)\,(s_2+1)}\right)^{\frac{1}{q}}.$$

Here, $\frac{1}{p}+\frac{1}{q}=1$and $\Phi(a,b,x;\ c,d,y)$ are defined as in Lemma 1.

Proof. With the help of Hölder inequality and co-ordinated (s_1,s_2)-convexity of $\left|\frac{\partial^2 F}{\partial\eta\partial\tau}\right|^q$, it follows

$$\int_0^1\int_0^1 \Lambda_1(x,\eta)\Lambda_2(y,\tau)\left|\frac{\partial^2}{\partial\eta\partial\tau}F\big(\eta b+(1-\eta)\,(a+b-x),\tau d+(1-\tau)\,(c+d-y)\big)\right| d\tau d\eta$$

$$\leq\left(\int_0^1\int_0^1 [\Lambda_1(x,\eta)\Lambda_2(y,\tau)]^p d\tau d\eta\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1\int_0^1 \left|\frac{\partial^2}{\partial\eta\partial\tau}F\big(\eta b+(1-\eta)\,(a+b-x),\tau d+(1-\tau)\,(c+d-y)\big)\right|^q d\tau d\eta\right)^{\frac{1}{q}} \tag{16}$$

$$\leq\left(\int_0^1\int_0^1 [\Lambda_1(x,\eta)\Lambda_2(y,\tau)]^p d\tau d\eta\right)^{\frac{1}{p}}$$

$$\times\left(\frac{\left|\frac{\partial^2}{\partial\eta\partial\tau}F(b,d)\right|^q+\left|\frac{\partial^2}{\partial\eta\partial\tau}F(b,c+d-y)\right|^q+\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a+b-x,d)\right|^q+\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a+b-x,c+d-y)\right|^q}{(s_1+1)\,(s_2+1)}\right)^{\frac{1}{q}}.$$

Similarly, we obtain

$$\int_0^1\int_0^1 \Lambda_1(x,\eta)\Delta_2(y,\tau)\left|\frac{\partial^2}{\partial\eta\partial\tau}F\big(\eta b+(1-\eta)(a+b-x),\tau c+(1-\tau)(c+d-y)\big)\right|d\tau d\eta$$
$$\leq\left(\int_0^1\int_0^1[\Lambda_1(x,\eta)\Delta_2(y,\tau)]^p d\tau d\eta\right)^{\frac{1}{p}} \tag{17}$$
$$\times\left(\frac{\left|\frac{\partial^2}{\partial\eta\partial\tau}F(b,c)\right|^q+\left|\frac{\partial^2}{\partial\eta\partial\tau}F(b,c+d-y)\right|^q+\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a+b-x,c)\right|^q+\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a+b-x,c+d-y)\right|^q}{(s_1+1)(s_2+1)}\right)^{\frac{1}{q}},$$

$$\int_0^1\int_0^1 \Delta_1(x,\eta)\Lambda_2(y,\tau)\left|\frac{\partial^2}{\partial\eta\partial\tau}F\big(\eta a+(1-\eta)(a+b-x),\tau d+(1-\tau)(c+d-y)\big)\right|d\tau d\eta$$
$$\leq\left(\int_0^1\int_0^1[\Delta_1(x,\eta)\Lambda_2(y,\tau)]^p d\tau d\eta\right)^{\frac{1}{p}} \tag{18}$$
$$\times\left(\frac{\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a,d)\right|^q+\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a,c+d-y)\right|^q+\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a+b-x,d)\right|^q+\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a+b-x,c+d-y)\right|^q}{(s_1+1)(s_2+1)}\right)^{\frac{1}{q}},$$

and

$$\int_0^1\int_0^1 \Delta_1(x,\eta)\Delta_2(y,\tau)\left|\frac{\partial^2}{\partial\eta\partial\tau}F\big(\eta a+(1-\eta)(a+b-x),\tau c+(1-\tau)(c+d-y)\big)\right|d\tau d\eta$$
$$\leq\left(\int_0^1\int_0^1[\Delta_1(x,\eta)\Delta_2(y,\tau)]^p d\tau d\eta\right)^{\frac{1}{p}} \tag{19}$$
$$\times\left(\frac{\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a,c)\right|^q+\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a,c+d-y)\right|^q+\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a+b-x,c)\right|^q+\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a+b-x,c+d-y)\right|^q}{(s_1+1)(s_2+1)}\right)^{\frac{1}{q}}.$$

If it is substituted the inequalities (16)-(19) in (11), then the required results are obtained. This ends of the proof of Theorem 2.

Corollary 4. Let us consider $\varphi(\eta)=\eta$ and $\psi(\tau)=\tau$ for all $(\eta,\tau)\in\Delta$ in Theorem 2. Let us also consider $s_1=s_2=s$, $x=\frac{a+b}{2}$ and $y=\frac{c+d}{2}$. Then, we have the following midpoint type inequality for Riemann-Liouville fractional integrals

$$\Bigg|F\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{1}{d-c}\left[{}_{d-}I_{\psi}F\left(\frac{a+b}{2},\frac{c+d}{2}\right)+{}_{c+}I_{\psi}F\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]$$
$$-\frac{1}{b-a}\left[{}_{b-}I_{\varphi}F\left(\frac{a+b}{2},\frac{c+d}{2}\right)+{}_{a+}I_{\varphi}F\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]$$
$$+\frac{1}{(b-a)(d-c)}\Bigg[{}_{b-,d-}I_{\varphi,\psi}F\left(\frac{a+b}{2},\frac{c+d}{2}\right)+{}_{b-,c+}I_{\varphi,\psi}F\left(\frac{a+b}{2},\frac{c+d}{2}\right)$$
$$+{}_{a+,d-}I_{\varphi,\psi}F\left(\frac{a+b}{2},\frac{c+d}{2}\right)+{}_{a+,c+}I_{\varphi,\psi}F\left(\frac{a+b}{2},\frac{c+d}{2}\right)\Bigg]\Bigg|$$

$$\leq \frac{(b-a)(d-c)}{16(p+1)^{\frac{2}{p}}(s+1)^{\frac{2}{q}}}$$

$$\times\left[\left(\left|\frac{\partial^2}{\partial\eta\partial\tau}F(b,d)\right|^q+\left|\frac{\partial^2}{\partial\eta\partial\tau}F\left(b,\frac{c+d}{2}\right)\right|^q+\left|\frac{\partial^2}{\partial\eta\partial\tau}F\left(\frac{a+b}{2},d\right)\right|^q+\left|\frac{\partial^2}{\partial\eta\partial\tau}F\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q\right)^{\frac{1}{q}}\right.$$

$$+\left(\left|\frac{\partial^2}{\partial\eta\partial\tau}F(b,c)\right|^q+\left|\frac{\partial^2}{\partial\eta\partial\tau}F\left(b,\frac{c+d}{2}\right)\right|^q+\left|\frac{\partial^2}{\partial\eta\partial\tau}F\left(\frac{a+b}{2},c\right)\right|^q+\left|\frac{\partial^2}{\partial\eta\partial\tau}F\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q\right)^{\frac{1}{q}}$$

$$+\left(\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a,d)\right|^q+\left|\frac{\partial^2}{\partial\eta\partial\tau}F\left(a,\frac{c+d}{2}\right)\right|^q+\left|\frac{\partial^2}{\partial\eta\partial\tau}F\left(\frac{a+b}{2},d\right)\right|^q+\left|\frac{\partial^2}{\partial\eta\partial\tau}F\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q\right)^{\frac{1}{q}}$$

$$\left.+\left(\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a,c)\right|^q+\left|\frac{\partial^2}{\partial\eta\partial\tau}F\left(a,\frac{c+d}{2}\right)\right|^q+\left|\frac{\partial^2}{\partial\eta\partial\tau}F\left(\frac{a+b}{2},c\right)\right|^q+\left|\frac{\partial^2}{\partial\eta\partial\tau}F\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{(b-a)(d-c)}{16(p+1)^{\frac{2}{p}}(s+1)^{\frac{2}{q}}}$$

$$\times\left[\left(\frac{1+2^{s+1}+2^{2s}}{2^{2s}}\left|\frac{\partial^2}{\partial\eta\partial\tau}F(b,d)\right|^q+\frac{1+2^s}{2^{2s}}\left(\left|\frac{\partial^2}{\partial\eta\partial\tau}F(b,c)\right|^q+\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a,d)\right|^q\right)+\frac{1}{2^s}\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a,c)\right|^q\right)^{\frac{1}{q}}\right.$$

$$+\left(\frac{1+2^{s+1}+2^{2s}}{2^{2s}}\left|\frac{\partial^2}{\partial\eta\partial\tau}F(b,c)\right|^q+\frac{1+2^s}{2^{2s}}\left(\left|\frac{\partial^2}{\partial\eta\partial\tau}F(b,d)\right|^q+\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a,c)\right|^q\right)+\frac{1}{2^s}\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a,d)\right|^q\right)^{\frac{1}{q}}$$

$$+\left(\frac{1+2^{s+1}+2^{2s}}{2^{2s}}\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a,d)\right|^q+\frac{1+2^s}{2^{2s}}\left(\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a,c)\right|^q+\left|\frac{\partial^2}{\partial\eta\partial\tau}F(b,d)\right|^q\right)+\frac{1}{2^s}\left|\frac{\partial^2}{\partial\eta\partial\tau}F(b,c)\right|^q\right)^{\frac{1}{q}}$$

$$\left.+\left(\frac{1+2^{s+1}+2^{2s}}{2^{2s}}\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a,c)\right|^q+\frac{1+2^s}{2^{2s}}\left(\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a,d)\right|^q+\left|\frac{\partial^2}{\partial\eta\partial\tau}F(b,c)\right|^q\right)+\frac{1}{2^s}\left|\frac{\partial^2}{\partial\eta\partial\tau}F(b,d)\right|^q\right)^{\frac{1}{q}}\right].$$

Theorem 3. Let us note that the assumptions of Lemma 1 hold. If the function $\left|\frac{\partial^2 F}{\partial\eta\partial\tau}\right|^q$, $q \geq 1$ is co-ordinated convex on Δ, then the following inequality for generalized fractional integral holds:

$$
\begin{aligned}
&|\Phi(a,b,x;\ c,d,y)| \\
&\leq \frac{(x-a)\,(y-c)}{\Upsilon(x,y)}\left(\int_0^1\int_0^1 \Lambda_1(x,\eta)\Lambda_2(y,\tau)d\tau d\eta\right)^{1-\frac{1}{q}} \\
&\times\left[\lambda_1\mu_1\left|\frac{\partial^2}{\partial\eta\partial\tau}F(b,d)\right|^q+\lambda_1\mu_2\left|\frac{\partial^2}{\partial\eta\partial\tau}F(b,c+d-y)\right|^q\right. \\
&\left.+\lambda_2\mu_1\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a+b-x,d)\right|^q+\lambda_2\mu_2\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a+b-x,c+d-y)\right|^q\right]^{\frac{1}{q}} \\
&+\frac{(x-a)\,(d-y)}{\Upsilon(x,y)}\left(\int_0^1\int_0^1 \Lambda_1(x,\eta)\Delta_2(y,\tau)d\tau d\eta\right)^{1-\frac{1}{q}} \\
&\times\left[\lambda_1\mu_4\left|\frac{\partial^2}{\partial\eta\partial\tau}F(b,c)\right|^q+\lambda_1\mu_3\left|\frac{\partial^2}{\partial\eta\partial\tau}F(b,c+d-y)\right|^q\right. \\
&\left.+\lambda_2\mu_4\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a+b-x,c)\right|^q+\lambda_2\mu_3\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a+b-x,c+d-y)\right|^q\right]^{\frac{1}{q}} \\
&+\frac{(b-x)\,(y-c)}{\Upsilon(x,y)}\left(\int_0^1\int_0^1 \Delta_1(x,\eta)\Lambda_2(y,\tau)d\tau d\eta\right)^{1-\frac{1}{q}} \\
&\times\left[\lambda_4\mu_1\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a,d)\right|^q+\lambda_4\mu_2\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a,c+d-y)\right|^q\right. \\
&\left.+\lambda_3\mu_1\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a+b-x,d)\right|^q+\lambda_3\mu_2\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a+b-x,c+d-y)\right|^q\right]^{\frac{1}{q}} \\
&+\frac{(b-x)\,(d-y)}{\Upsilon(x,y)}\left(\int_0^1\int_0^1 \Delta_1(x,\eta)\Delta_2(y,\tau)d\tau d\eta\right)^{1-\frac{1}{q}} \\
&\times\left[\lambda_4\mu_4\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a,c)\right|^q+\lambda_4\mu_3\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a,c+d-y)\right|^q\right. \\
&\left.+\lambda_3\mu_4\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a+b-x,c)\right|^q+\lambda_3\mu_3\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a+b-x,c+d-y)\right|^q\right]^{\frac{1}{q}}.
\end{aligned}
$$

Here, $\Phi(a,b,x;\ c,d,y)$ is defined as in Lemma 1, A_i, $i=1,2,3,4$ are described as in (9) and B_i, $i=1,2,3,4$ are defined as in (10).

Proof. Power mean inequality and co-ordinated (s_1,s_2)-convexity of $\left|\frac{\partial^2 F}{\partial\eta\partial\tau}\right|^q$ yield

$$\int_0^1\int_0^1 \Lambda_1(x,\eta)\Lambda_2(y,\tau)\left|\frac{\partial^2}{\partial\eta\partial\tau}F\big(\eta b+(1-\eta)(a+b-x),\tau d+(1-\tau)(c+d-y)\big)\right| d\tau d\eta$$

$$\leq \left(\int_0^1\int_0^1 \Lambda_1(x,\eta)\Lambda_2(y,\tau)d\tau d\eta\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1\int_0^1 \Lambda_1(x,\eta)\Lambda_2(y,\tau)\left|\frac{\partial^2}{\partial\eta\partial\tau}F\big(\eta b+(1-\eta)(a+b-x),\tau d+(1-\tau)(c+d-y)\big)\right|^q d\tau d\eta\right)^{\frac{1}{q}}$$

$$\leq \left(\int_0^1\int_0^1 \Lambda_1(x,\eta)\Lambda_2(y,\tau)d\tau d\eta\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1\int_0^1 \Lambda_1(x,\eta)\Lambda_2(y,\tau)\left[\eta\tau\left|\frac{\partial^2}{\partial\eta\partial\tau}F(b,d)\right|^q+\eta(1-\tau)\left|\frac{\partial^2}{\partial\eta\partial\tau}F(b,c+d-y)\right|^q\right.\right. \tag{20}$$

$$\left.\left.+(1-\eta)\tau\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a+b-x,d)\right|^q+(1-\eta)(1-\tau)\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a+b-x,c+d-y)\right|^q\right]d\tau d\eta\right)^{\frac{1}{q}}$$

$$= \left(\int_0^1\int_0^1 \Lambda_1(x,\eta)\Lambda_2(y,\tau)d\tau d\eta\right)^{1-\frac{1}{q}}$$

$$\times\left(\lambda_1\mu_1\left|\frac{\partial^2}{\partial\eta\partial\tau}F(b,d)\right|^q+\lambda_1\mu_2\left|\frac{\partial^2}{\partial\eta\partial\tau}F(b,c+d-y)\right|^q\right.$$

$$\left.+\lambda_2\mu_1\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a+b-x,d)\right|^q+\lambda_2\mu_2\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a+b-x,c+d-y)\right|^q\right)^{\frac{1}{q}}.$$

Similarly, we obtain

$$\int_0^1\int_0^1 \Lambda_1(x,\eta)\Delta_2(y,\tau)\left|\frac{\partial^2}{\partial\eta\partial\tau}F\big(\eta b+(1-\eta)(a+b-x),\tau c+(1-\tau)(c+d-y)\big)\right| d\tau d\eta$$

$$\leq \left(\int_0^1\int_0^1 \Lambda_1(x,\eta)\Delta_2(y,\tau)d\tau d\eta\right)^{1-\frac{1}{q}} \tag{21}$$

$$\times\left(\lambda_1\mu_4\left|\frac{\partial^2}{\partial\eta\partial\tau}F(b,c)\right|^q+\lambda_1\mu_3\left|\frac{\partial^2}{\partial\eta\partial\tau}F(b,c+d-y)\right|^q\right.$$

$$\left.+\lambda_2\mu_4\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a+b-x,c)\right|^q+\lambda_2\mu_3\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a+b-x,c+d-y)\right|^q\right)^{\frac{1}{q}},$$

$$\int_0^1\int_0^1 \Delta_1(x,\eta)\Lambda_2(y,\tau)\left|\frac{\partial^2}{\partial\eta\partial\tau}F\big(\eta a+(1-\eta)(a+b-x),\tau d+(1-\tau)(c+d-y)\big)\right| d\tau d\eta$$

$$\leq \left(\int_0^1\int_0^1 \Delta_1(x,\eta)\Lambda_2(y,\tau)d\tau d\eta\right)^{1-\frac{1}{q}} \tag{22}$$

$$\left(\lambda_4\mu_1\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a,d)\right|^q+\lambda_4\mu_2\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a,c+d-y)\right|^q\right.$$

$$\left.+\lambda_3\mu_1\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a+b-x,d)\right|^q+\lambda_3\mu_2\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a+b-x,c+d-y)\right|^q\right)^{\frac{1}{q}},$$

and

$$\int_0^1\int_0^1 \Delta_1(x,\eta)\Delta_2(y,\tau)\left|\frac{\partial^2}{\partial\eta\partial\tau}F\big(\eta a+(1-\eta)(a+b-x),\tau c+(1-\tau)(c+d-y)\big)\right| d\tau d\eta$$

$$\leq \left(\int_0^1\int_0^1 \Delta_1(x,\eta)\Delta_2(y,\tau)d\tau d\eta\right)^{1-\frac{1}{q}} \tag{23}$$

$$\left(\lambda_4\mu_4\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a,c)\right|^q+\lambda_4\mu_3\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a,c+d-y)\right|^q\right.$$

$$\left.+\lambda_3\mu_4\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a+b-x,c)\right|^q+\lambda_3\mu_3\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a+b-x,c+d-y)\right|^q\right)^{\frac{1}{q}}.$$

If we substitute the inequalities (20)-(23) in (11), then we establish desired result. This completes the proof of Theorem 3.

Corollary 5. Let us consider $\varphi(\eta)=\eta$ and $\psi(\tau)=\tau$ for all $(\eta,\tau)\in\Delta$ in Theorem 3. Let us also consider $s_1=s_2=s$, $x=\frac{a+b}{2}$ and $y=\frac{c+d}{2}$. Then, we have the following midpoint type inequality for Riemann-Liouville fractional integrals

$$\begin{aligned}
&\left|\Phi\left(a,b,\frac{a+b}{2};c,d,\frac{c+d}{2}\right)\right| = F\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
&-\frac{1}{d-c}\left[{}_{d-}I_{\psi}F\left(\frac{a+b}{2},\frac{c+d}{2}\right)+{}_{c+}I_{\psi}F\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]\\
&-\frac{1}{b-a}\left[{}_{b-}I_{\phi}F\left(\frac{a+b}{2},\frac{c+d}{2}\right)+{}_{a+}I_{\phi}F\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]\\
&+\frac{1}{(b-a)(d-c)}\left[{}_{b-,d-}I_{\phi,\psi}F\left(\frac{a+b}{2},\frac{c+d}{2}\right)+{}_{b-,c+}I_{\phi,\psi}F\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right.\\
&\left.+{}_{a+,d-}I_{\phi,\psi}F\left(\frac{a+b}{2},\frac{c+d}{2}\right)+{}_{a+,c+}I_{\phi,\psi}F\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]\\
\leq{}&\frac{4^{\frac{1}{q}}(b-a)(d-c)}{64}\\
\times{}&\left[\left(\frac{\left|\frac{\partial^2}{\partial\eta\partial\tau}F(b,d)\right|^q}{(s+1)^2(s+2)^2}+\frac{\left|\frac{\partial^2}{\partial\eta\partial\tau}F\left(b,\frac{c+d}{2}\right)\right|^q}{(s+1)(s+2)^2}+\frac{\left|\frac{\partial^2}{\partial\eta\partial\tau}F\left(\frac{a+b}{2},d\right)\right|^q}{(s+1)(s+2)^2}+\frac{\left|\frac{\partial^2}{\partial\eta\partial\tau}F\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q}{(s+2)^2}\right)^{\frac{1}{q}}\right.\\
&+\left(\frac{\left|\frac{\partial^2}{\partial\eta\partial\tau}F(b,c)\right|^q}{(s+1)^2(s+2)^2}+\frac{\left|\frac{\partial^2}{\partial\eta\partial\tau}F\left(b,\frac{c+d}{2}\right)\right|^q}{(s+1)(s+2)^2}+\frac{\left|\frac{\partial^2}{\partial\eta\partial\tau}F\left(\frac{a+b}{2},c\right)\right|^q}{(s+1)(s+2)^2}+\frac{\left|\frac{\partial^2}{\partial\eta\partial\tau}F\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q}{(s+2)^2}\right)^{\frac{1}{q}}\\
&+\left(\frac{\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a,d)\right|^q}{(s+1)^2(s+2)^2}+\frac{\left|\frac{\partial^2}{\partial\eta\partial\tau}F\left(a,\frac{c+d}{2}\right)\right|^q}{(s+1)(s+2)^2}+\frac{\left|\frac{\partial^2}{\partial\eta\partial\tau}F\left(\frac{a+b}{2},d\right)\right|^q}{(s+1)(s+2)^2}+\frac{\left|\frac{\partial^2}{\partial\eta\partial\tau}F\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q}{(s+2)^2}\right)^{\frac{1}{q}}\\
&\left.+\left(\frac{\left|\frac{\partial^2}{\partial\eta\partial\tau}F(a,c)\right|^q}{(s+1)^2(s+2)^2}+\frac{\left|\frac{\partial^2}{\partial\eta\partial\tau}F\left(a,\frac{c+d}{2}\right)\right|^q}{(s+1)(s+2)^2}+\frac{\left|\frac{\partial^2}{\partial\eta\partial\tau}F\left(\frac{a+b}{2},c\right)\right|^q}{(s+1)(s+2)^2}+\frac{\left|\frac{\partial^2}{\partial\eta\partial\tau}F\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q}{(s+2)^2}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Remark 4. If we consider suitable choices of $\varphi(\eta)$ and $\psi(\tau)$ then the new inequalities can be obtained for some forms of fractional integrals, namely, Riemann-Liouville fractional integral, k-Riemann-Liouville fractional integral, conformable fractional integral, Hadamard fractional integrals, Katugampola fractional integrals, etc.

Conclusion

In this paper, some Hermite-Hadamard type inequalities are established for the case of differentiable co-ordinated (s_1," " s_2)-convex functions. In other words, the generalizations of the midpoint type inequalities are proved for the case of differentiable co-ordinated (s_1," " s_2)-convex functions in the second sense on the rectangle from the plain. Moreover, several inequalities are given for the case of Riemann-Liouville fractional integrals and k-Riemann-Liouville fractional integrals by choosing the special cases of our obtained main results. In future studies, improvements or generalizations of our results can be investigated by using different kinds of convex function classes or other types of fractional integral operators. Furthermore, the authors can extend the results by choosing bounded functions and also try to give discrete versions of the findings for the future studies.

Author contributions

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Conflicts of interest

The author declare that they have no competing interests.

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