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# Robust Stability and Stable Member Problems for Multilinear Systems

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**ABSTRACT**
In this paper, we consider robust stability and stable member problems for linear systems whose characteristic polynomials are nonmonic polynomials with multilinear uncertainty. For both problems, the results are given by using the reflection (box) coefficients and the extreme point property of multilinear functions defined on the box. Finding stable member in a polynomial family is one of the hard problems of linear control theory. This issue is considered by visualizing the cases $n-l=2$ and $n-l=3$. Necessary and sufficient conditions for robust stability and the existence of a stable member of the multilinear polynomial family using the reflection coefficients are obtained. Several examples are provided.

**Keywords:** *Schur stability, Multilinear function, Stable member, Box coefficients.*

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## Introduction

Consider the following polynomial with real coefficients

$$p(z) = a_1 + a_2 z + \cdots + a_n z^{n-1} + a_{n+1} z^n \qquad (1)$$

where $a_{n+1} \neq 0$. If $a_{n+1} = 1$ the obtained polynomial

$$p(z) = a_1 + a_2 z + \cdots + a_n z^{n-1} + z^n \qquad (2)$$

is called a monic polynomial which corresponds to $n$ −dimensional vector $a = (a_1, a_2, \ldots, a_n)^T \in \mathbb{R}^n$. The polynomial (1) is called Schur stable polynomial if all roots lie in the open unit disc of the complex plane. The set of all monic Schur stable polynomials defines the set

$$\mathcal{D}_n = \{a \in \mathbb{R}^n : p(z) \text{ is Schur stable polynomial}\}. \qquad (3)$$

$\mathcal{D}_n$ is open, bounded and nonconvex subset in $\mathbb{R}^n$. The closure of $\mathcal{D}_n$ is

$$\overline{\mathcal{D}_n} = \{a \in \mathbb{R}^n : p(z) \text{ has all roots in the closed unit disc}\}.$$

Given a non-monic polynomial family $\mathcal{P}$ with multilinear uncertainity

$$\mathcal{P} = \{p(z,q) = a_1(q) + a_2(q)z + \cdots + a_n(q)z^{n-1} + a_{n+1}(q)z^n : q \in Q \subset \mathbb{R}^l\}. \qquad (4)$$

Here $Q$ is a box and $a_i\,(q): Q \to \mathbb{R}$ $(i = 1,2, \ldots, n+1)$ are multilinear functions, that are affine linear with respect to each component. Without loss of generality assume that $a_{n+1}(q) > 0$ for all $q \in Q$. If for $q \in Q$ the polynomial (4) is Schur stable the family (4) is said to be robust Schur stable. From now on the term stable will mean Schur stable.

It is well known that a multilinear polynomial family appears quite frequently in practical applications [1]. In [2], some conditions for the Schur stability of this family are given. In [3,4], sufficient conditions are given for ensuring Schur stability by using the Edge Theorem. In [5], it is suggested a simple algorithm for testing Schur stability of a multilinear family and given a result on Schur stability of a compact matrix family.

In Section 2 we discussed the robust stability of non-monic multilinear polynomial families using the reflection coefficients in [6,7] and give a necessary and sufficient condition for robust stability of the multilinear polynomial family.

The existence of a stable member in a matrix polytope and other related problems has been considered in many works (see [1,8,9] and references therein). Finding stable member in a polynomial family is one of the hard problems of linear control theory (see [10]). In Section 3, the necessary and sufficient condition for the existence of a stable member in the multilinear non-monic polynomial family is given. An application of this condition is shown in the example. In Section 4, a method is given for the presence of the stable element when the difference between degree and the number of uncertain parameters is 2 and 3.

In [6] a multilinear map $f(k_1, k_2, \ldots, k_n) = \left(f_1(k_1, k_2, \ldots, k_n), \ldots, f_n(k_1, k_2, \ldots, k_n)\right)^T$ has been defined by the multiplication of second and first order factors;
If $n$ is even

$$f_1(k_1,k_2,\dots,k_n)+f_2(k_1,k_2,\dots,k_n)z+\cdots+f_n(k_1,k_2,\dots,k_n)z^{n-1}+z^n=[z^2+(k_1k_2+k_1)z+k_2]\cdot[z^2+(k_3k_4+k_3)z+k_4]\cdots[z^2+(k_{n-1}k_n+k_{n-1})z+k_n],$$

if $n$ is odd

$$f_1(k_1,k_2,\dots,k_n)+f_2(k_1,k_2,\dots,k_n)z+\cdots+f_n(k_1,k_2,\dots,k_n)z^{n-1}+z^n=[z^2+(k_1k_2+k_1)z+k_2]\cdots[z^2+(k_{n-2}k_{n-1}+k_{n-2})z+k_{n-1}]\cdot(z+k_n).$$

**Proposition 1 ([6]):** $p(z)$ is a Schur polynomial if and only if there exist numbers $k_j\in(-1,1)$ such that $a_i=f_i(k_1,k_2,\dots,k_n)$ $(i,j=1,2,\dots,n)$.

From Proposition 1 follows the following

**Proposition 2:** $a\in\overline{\mathfrak{D}_n}$ if and only if there exist numbers $k_j\in[-1,1]$ such that $a_i=f_i(k_1,k_2,\dots,k_n)$ $(i,j=1,2,\dots,n)$. The defined above map $f$ is a multilinear. The set $Q=\{(q_1,q_2,\dots,q_l)^T: q_i^-\le q_i\le q_i^+, i=1,2,\dots,l\}$ is called a box. The following theorem shows that a multilinear image of a box is a polytope, that is convex hull of a finite number of points.

**Theorem 1 (The mapping theorem [3], p.247):** Let $f: Q\to\mathbb{R}^n$ be a multilinear map, where $Q$ is a box with the set of extreme points $\{q^i\}$, then convex hull of the image $f(Q)$ equals $co\{f(q^i)\}$, where $co$ stands for the convex hull.

Let $K$ be the $n$-dimensional cube defined by

$$K=\{(k_1,k_2,\dots,k_n): -1\le k_1\le 1,\dots,-1\le k_n\le 1\}.$$

**Proposition 3:** $f(\partial K)=\partial\mathcal{D}_n$, where $\partial K$ is the boundary of the set $K$.

*Proof.* Take any $x\in f(\partial K)$. Then there exists $(k_1^0,k_2^0,\dots,k_n^0)\in\partial K$ such that $f(k_1^0,k_2^0,\dots,k_n^0)=x$. Without loss of generality assume that $x\in\{k\in K: k_1=1\}$, then $x=f(1,k_2^0,\dots,k_n^0)$. Three cases are possible.

$f(1,k_2^0,\dots,k_n^0)\in\mathcal{D}_n$. This case is impossible, since the second order factor $[s^2+(k_2^0+1)s+k_2^0]$ from the definition of $f$ has unstable factor $(s+1)$.

$f(1,k_2^0,\dots,k_n^0)$ is an exterior point of $\mathcal{D}_n$. This case is impossible as well, since any neighbourhood of $(1,k_2^0,\dots,k_n^0)$ contains element from $K^0$ (the set of interior point of $K$) and we obtain a contradiction to Proposition 1.

It remains the case $x\in\partial\mathcal{D}_n$ which proves $f(\partial K)\subset\partial\mathcal{D}_n$.

Conversely, assume that $x\in\partial\mathcal{D}_n$. Then there exists a sequence $x^m\in\mathcal{D}_n$ such that $x^m\to x$ as $m\to\infty$. By Proposition 1 there exists $k^m\in K^0$ such that $f(k^m)=x^m$. The set $K$ is compact and without loss of generality assume that $k^m\to k\in K$. Then $f(k)=x$. The inclusion $k\in K^0$ is impossible due to Proposition 1 and equality $f(k)=x$ and openness of $\mathcal{D}_n$ (Recall that any vector $y\in\partial\mathcal{D}_n$ is unstable.). Consequently $k\in\partial K$ and $x\in f(\partial K)$.

## Stability of a Non-monic Multilinear Family

Consider the set $\mathcal{D}_n$ (see equation (3)). The boundary set $\partial\mathcal{D}_n$ of $\mathcal{D}_n$ consists of three parts ([7])

$$\partial\mathcal{D}_n=B_1\cup B_{-1}\cup B_c$$

where

$B_1=\{a\in\mathbb{R}^n: p(z)$ has all roots in the closed disc $|z|\le 1$ and has at least one root $z=1\}$,

$B_{-1}=\{a\in\mathbb{R}^n: p(z)$ has all roots in the closed disc $|z|\le 1$ and has at least one root $z=-1\}$,

$B_c=\{a\in\mathbb{R}^n: p(z)$ has all roots in the closed disc $|z|\le 1$ and has at least one complex root $z=e^{j\theta}, 0<\theta<\pi\}$.

The following proposition has been proved in [6]. It gives parametric description of the boundary set $\partial\mathcal{D}_n$.

**Proposition 4 ([6]):** a) Let $n$ be even. Then the surface $B_1$ has the parametric equation

$$x_i=f_i(-1,k_2,\dots,k_n), i=1,2,\dots,n$$
$$-1\le k_2\le 1,\dots,-1\le k_n\le 1$$

and the surface $B_{-1}$ has the parametric equation

$$x_i=f_i(1,k_2,\dots,k_n), i=1,2,\dots,n,$$
$$-1\le k_2\le 1,\dots,-1\le k_n\le 1.$$

b) Let $n$ be odd. Then the surface $B_1$ has the parametric equation

$$x_i=f_i(k_1,k_2,\dots,k_{n-1},-1), i=1,2,\dots,n$$
$$-1\le k_1\le 1,\dots,-1\le k_{n-1}\le 1,$$

and the surface $B_{-1}$ has the parametric equation

$$x_i=f_i(k_1,k_2,\dots,k_{n-1},1), i=1,2,\dots,n$$
$$-1\le k_1\le 1,\dots,-1\le k_{n-1}\le 1.$$

c) The surface has the parametric equation

$$x_i=f_i(k_1,1,k_3,\dots,k_{n-1},k_n), i=1,2,\dots,n$$
$$-1\le k_1\le 1, -1\le k_3\le 1,\dots,-1\le k_n\le 1.$$

Now we give stability condition of the family (4).

Define the functions $(i=1,2,\dots,n)$

$$F_i(q_1,\dots,q_l,k_1,k_2,\dots,k_n)=F_i(q,k)=a_{n+1}(q)f_i(k)-a_i(q) \qquad (5)$$

**Theorem 2:** Assume that the family (4) is given, where $n$ is even, $a_i(q): Q\to\mathbb{R}$ $(i=1,2,\dots,n+1)$ are multilinear functions, $Q\subset\mathbb{R}^l$ is a box and $a_{n+1}(q)>0$ for all $q\in Q$. Assume also that this family has a stable member $p(z,q^*)$. Then (4) is robust stable if and only if the following conditions a), b), c) are satisfied simultaneously.

a) The system

$$F_i(q_1,q_2,\dots,q_l,-1,k_2,\dots,k_n)=0 \qquad (6)$$

has no solution on the box $Q\times[-1,1]^{n-1}$ $(i=1,2,\dots,n)$.

b) The system

$$F_i(q_1,q_2,\dots,q_l,1,k_2,\dots,k_n)=0 \qquad (7)$$

has no solution on the box $Q\times[-1,1]^{n-1}$ $(i=1,2,\dots,n)$.

c) The system

$$F_i(q_1,q_2,\dots,q_l,k_1,1,k_3,\dots,k_n)=0 \qquad (8)$$

has no solution on the box $Q \times [-1,1]^{n-1}$ $(i = 1,2,\dots,n)$.

*Proof.* Assume that the family (4) is robust stable. From the condition $a_{n+1}(q) > 0$ it follows that the family

$$\tilde{\mathcal{P}} = \left\{ \frac{a_1(q)}{a_{n+1}(q)} + \frac{a_2(q)}{a_{n+1}(q)} z + \cdots + \frac{a_n(q)}{a_{n+1}(q)} z^{n-1} + z^n : q \in Q \right\}$$

is robust stable as well. Consequently for all $q \in Q$ the vector $v(q) = \left(\frac{a_1(q)}{a_{n+1}(q)}, \dots, \frac{a_n(q)}{a_{n+1}(q)}\right)$ is not contained in the boundary $\partial\mathcal{D}_n = B_1 \cup B_{-1} \cup B_c$ of the open set $\mathcal{D}_n$. By Proposition 1 and 4

$v(q) \notin B_1 \Rightarrow$ The system (6) has no solution on $Q \times [-1,1]^{n-1}$,

$v(q) \notin B_{-1} \Rightarrow$ The system (7) has no solution on $Q \times [-1,1]^{n-1}$,

$v(q) \notin B_c \Rightarrow$ The system (8) has no solution on $Q \times [-1,1]^{n-1}$.

Conversely, if the systems (6), (7) and (8) have no solutions then $\tilde{P} \subset \mathcal{D}_n$ or $\tilde{P} \subset \mathcal{D}_n^c$, where $\mathcal{D}_n^c$ is the complementary of $\mathcal{D}_n$. Since the family $\mathcal{P}$ and consequently the family $\tilde{P}$ has a stable member then $\tilde{P} \subset \mathcal{D}_n$, from this it follows that the family (4) is robust stable.

The systems (6), (7), (8) can be investigated by using The Mapping Theorem (Theorem 1) and splitting evaluation algorithm (see [5]). Divide the box $Q \times [-1,1]^{n-1}$ into small subboxes and if the convex hull of the images of vertices does not include the zero then eliminate this small subbox.

**Example 1:** Consider robust stability problem for the following multilinear family

$$\begin{aligned} p(z,q) = {} & (7 - q_1q_2 - 2q_1)z^6 + (2 + q_1 + 0.5q_2)z^5 \\ & + (2.5 + q_1 + 0.1q_2 - q_1q_2)z^4 \\ & + (1.5 + q_1q_2)z^3 \\ & + (0.5 + q_1 - q_1q_2)z^2 \\ & + (-0.7 + q_1 + 0.5q_2)z + 0.4 - q_1 \\ & + q_2 - 0.5q_1q_2, \end{aligned}$$

$q_1 \in [6,10]$, $q_2 \in [3,5]$. For $q_1 = 6$, $q_2 = 3$ the polynomial is stable. Using the equations of the boundary $\partial\mathcal{D}_n$ in the parametric forms ((6), (7), (8)) write three multilinear systems of equations

$F_i(q,k) = a_{n+1}(q) f_i(k) - a_i(q)$

$$a_7(q_1,q_2) f_i(1,k_2,k_3,\dots,k_6) - a_i(q_1,q_2) = 0 \quad (9)$$
$(\mathrm{i} = 1,\dots,6)$

$$a_7(q_1,q_2) f_i(-1,k_2,k_3,\dots,k_6) - a_i(q_1,q_2) = 0 \quad (10)$$
$(\mathrm{i} = 1,\dots,6)$

$$a_7(q_1,q_2) f_i(k_1,1,k_3,\dots,k_6) - a_i(q_1,q_2) = 0 \quad (11)$$
$(\mathrm{i} = 1,\dots,6)$

where $a_i(q_1,q_2)$ are the coefficients of $p(z,q)$ and

$$(q_1,q_2,k_1,\dots,k_6) \in B = [6,10] \times [3,5] \times [-1,1] \times \cdots \times [-1,1].$$

We have to show that all three systems (9)-(11) have no solutions. Here we use splitting-elimination algorithm (see [5]) with the use of The Mapping Theorem (divide the box $B$ into small subboxes, if for a subbox the zero is not contained in the convex hull of the images of vertices, then eliminate this subbox).

For the systems (9), (10) and (11) all subboxes are eliminated after 2, 2 and 418 steps totally 16 sec, respectively. Therefore the given family does not intersect the boundary of $\mathcal{D}_n$ and has a stable member. Consequently the family is robust stable.

**Example 2:** Consider the given multilinear family

$$\begin{aligned} p(z,q) = {} & (1 - q_1 + 3q_2 + 3q_1q_2)z^7 + (q_1q_2 + q_1 - q_2 - 6)z^6 \\ & + (-5q_1q_2 - 6q_1 + 5q_2 + 12)z^5 \\ & + (8q_1q_2 + 13q_1 - 7q_2 - 8)z^4 \\ & + (-4q_1q_2 - 11q_1 - q_2)z^3 \\ & + (q_1q_2 + 8q_2 - 1)z^2 \\ & + (4q_1 - 4q_2 + 4)z - 4 + q_1 - q_2, \end{aligned}$$

$q_1 \in [-1,4]$, $q_2 \in [-2,5]$. Using the equations of the boundary $\partial\mathcal{D}_\mathrm{n}$ we obtain three multilinear systems of equations that correspond to the multilinear family.

For the equation systems, all subboxes are eliminated after 78, 122 and 256 steps totally 63 sec, respectively. There are no solutions. Therefore the given family does not intersect the boundary of $\mathcal{D}_n$. For $q_1 = 0$, $q_2 = 0$ the polynomial is not stable. As a result the family has no stable member.

## Existence of a Stable Member

Consider the family (4), we are interested in the existence of $q^* \in Q$ such that $p(z,q^*)$ becomes Schur stable. This problems of such types are important in the control theory ([11]).

**Theorem 3:** There exists a stable member in $\mathcal{P}$ if and only if the following multilinear system

$$F_i(q_1,\dots,q_l,k_1,\dots,k_n) = a_{n+1}(q) f_i(k) - a_i(q) = 0 \quad (12)$$
$(i = 1,2,\dots,n)$

has a solution in $Q \times (-1,1)^n$.

*Proof.* There exists a stable member in $\mathcal{P}$ if and only if there exists a stable member in $\tilde{\mathcal{P}}$. By Proposition 1 for a given $q^* \in Q$ the polynomial $\tilde{p}\,(z,q^*) \in \tilde{\mathcal{P}}$ is stable if and only if there exists $k^* \in (-1,1)^n$ such that

$$\frac{a_i(q^*)}{a_{n+1}(q^*)} = f_i(k^*) \qquad (i = 1,2,\dots,n)$$

By the definition of $F_i$ this means that the system (12) has solution $(q^*,k^*) \in Q \times (-1,1)^n$. System (12) is a multilinear system defined on a box. Its solution can be searched by splitting-elimination algorithm:

Divide $Q \times (-1,1)^n$ into small subboxes and for a small subbox the convex hull of vertices does not include the zero then the eliminate this subbox by the Mapping Theorem. By this way we eliminate a great number of subboxes. If a remaining subbox has small volume then check its center for stability, i.e. if $(q^c,k^c)$ is a center then check the polynomial $p(z,q^c)$ for stability.

**Example 3:** Consider the given multilinear family

$$p(z,q) = (q_1 - 0.2q_2 + 3.6)z^5 + (q_1 + 0.3q_2 + 1.1)z^4 + (0.5q_1q_2 - 1.5q_1 + q_2 - 3)z^3 + (q_1q_2 - 3q_1 + 1.25q_2 - 3.75)z^2 + (q_2 - q_1 - 5)z + q_1 + q_2 - 1,$$

$q_1 \in [-4,-1]$, $q_2 \in [2,5]$. The splitting-elimination algorithm gives 388 remaining subboxes of

$$B = [-1,1]^5 \times [-4,-1] \times [2,5].$$

When the centers of the remaining 388 subboxes are examined, for the center $q_1^c = -\frac{17}{8}$, $q_2^c = \frac{25}{8}$ of the box

$$[0.5,1] \times [-0.5,0] \times [-1,-0.5] \times [0,1] \times [0,1] \times \left[-\frac{5}{2},-\frac{7}{4}\right] \times \left[\frac{11}{4},\frac{7}{2}\right]$$

the polynomial is stable.

## Stable Member for the Cases $n - l = 2$ and $n - l = 3$

As pointed out in [5] stabilization problem for unstable plant can be reduced to the following:

Let $A$ be $n \times l$ matrix with full rank, $U^0 \in \mathbb{R}^n$, $c = (c_1, c_2, \dots, c_l)^T$ and $k = (k_1, k_2, \dots, k_n)^T$. Is there exists $(c,k) \in \mathbb{R}^l \times \mathbb{R}^n$ such that

$$Ac + U^0 = f(k) \qquad (13)$$

where $f: \mathbb{R}^n \to \mathbb{R}^n$ is defined in Introduction and $l < n$? By solving the first $l$ equations in (13) with respect to $c_1, c_2, \dots c_l$ and inserting into the lost $n - l$ equations the obtained $c_1 = b_1(k), \dots, c_l = b_l(k)$ a multilinear system consisting of $(n - l)$ equations

$$\begin{aligned} &g_1(k_1,k_2,\dots,k_n)=0 \\ &\vdots \\ &g_{n-l}(k_1,k_2,\dots,k_n)=0 \end{aligned} \qquad (14)$$

is obtained. Consequently there exists a stabilizing vector $c$ if and only if the system (14) has a solution in $[-1,1]^n$.
Here we consider the cases $n - l = 2$ and $n - l = 3$.
In this case, it is possible to display rough image of $(-1,1)^n$ under map $g$.

From the equation (14) it follows that there exists a stabilizing vector if and only if the zero is contained in the image
$g((-1,1)^n) = \{g(k): k \in (-1,1)^n\}$
where $g = (g_1, \dots, g_{n-1})^T$. Gridding and displaying this image for the cases $n - l = 2$ and $n - l = 3$ may give positive results.

The following procedure can be suggested.

By gridding, display the "rough" image $g((-1,1)^n)$. This "rough" image gives a hint of the existence (or nonexistence) of a solution of (14).

Choose sufficient small $\varepsilon > 0$. Consider for a point $g(k^*)$ for which the distance between $g(k^*)$ and the origin is less than ε.

Calculate $c^* = b(k^*)$ and check $c^*$ for a stabilizing parameter.

**Example 4:** Let the family (2) be as

$$p(z,c) = z^5 + (c_3 - 0.4)z^4 + (c_2 - 0.1c_3 - 1.19)z^3 + (c_1 - 0.1c_2 - 0.06c_3 + 0.876)z^2 + (-0.1c_1 - 0.06c_2)z - 0.06c_1.$$

Here

$$A = \begin{bmatrix} -0.06 & 0 & 0 \\ -0.1 & -0.06 & 0 \\ 1 & -0.1 & -0.06 \\ 0 & 1 & -0.1 \\ 0 & 0 & 1 \end{bmatrix}, U^0 = \begin{bmatrix} 0 \\ 0 \\ 0.876 \\ -1.19 \\ -0.4 \end{bmatrix}.$$

Therefore $n = 5, l = 3, n - l = 2$. After corresponding calculations we conclude that $g_1(k_1, \dots, k_5)$ has 27 terms whereas $g_2(k_1, \dots, k_5)$ has 22 terms.

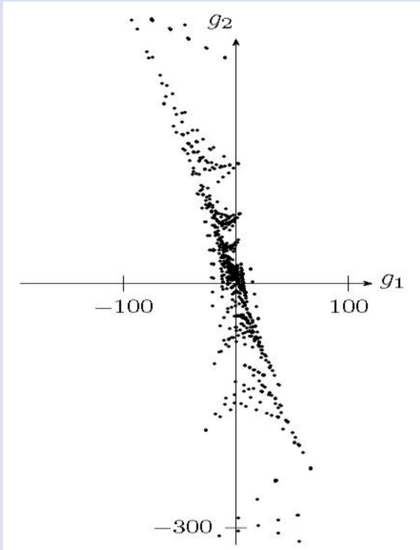


Figure 1. The "rough" images of g([-1,1]^5 ).

Gridding the cube $[-1,1]^5$ with step size $h = 0.25$ and displaying the image $g([-1,1]^5)$ gives the Fig. 1.
For $\varepsilon = 0.5$, $k^*$ is calculated as
$k^* = (-0.75, -0.5, 0.75, -0.25, 0.5)^T$ which gives stabilizing vector $c^* = (-1.041, 1.215, 0.877)^T$.

**Example 5:** Consider the following "famous" example from [9,10];
$p(z,c) = p_0(z) + c_1p_1(z) + c_2p_2(z)$,
where $p_0(z) = z^5 - 0.1z^4 - 1.9825z^3 + 0.1772z^2 + 0.8211z$, $p_1(z) = z^2 - 0.5z + 0.8$, $p_2(z) = z^3 - 0.5z^2 + 0.8z$.

Here

$$A = \begin{bmatrix} 0.8 & 0 \\ -0.5 & 0.8 \\ 1 & -0.5 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, U^0 = \begin{bmatrix} 0 \\ 0.8211 \\ 0.1772 \\ -1.9825 \\ -0.1 \end{bmatrix}$$

and

$$\begin{aligned} g_1(k_1,k_2,k_3,k_4,k_5) &= 0.625(k_1k_2k_4k_5 + k_2k_3k_4k_5 - k_1k_4k_5 - k_2k_3k_5 - k_2k_4) \\ &+ 0.859375k_2k_4k_5 - k_1k_2k_3k_4k_5 + k_1k_2k_3k_5 + k_1k_3k_4k_5 + k_1k_2k_4 \\ &- k_1k_3k_5 + k_2k_3k_4 - k_1k_4 - k_2k_3 + k_2k_5 + k_4k_5 + 0.6903875, \end{aligned}$$

$$g_2(k_1,k_2,k_3,k_4,k_5) = 1.25(-k_1k_2k_4k_5 - k_2k_3k_4k_5 + k_1k_4k_5 + k_2k_3k_5 + k_2k_4) - k_1k_2k_3k_4 + 0.78125k_2k_4k_5 - 3.008875 + k_1k_2k_3 - k_1k_2k_5 + k_1k_3k_4 - k_3k_4k_5 - k_1k_3 + k_1k_5 + k_3k_5 + k_2 + k_4$$

$$g_3(k_1,k_2,k_3,k_4,k_5) = -0.1 - k_1k_2 - k_3k_4 + k_1 + k_3 - k_5.$$

Gridding $[-1,1]^5$ with step size $h = 0.03$ and displaying the image $g([-1,1]^5)$ gives the Fig. 2.

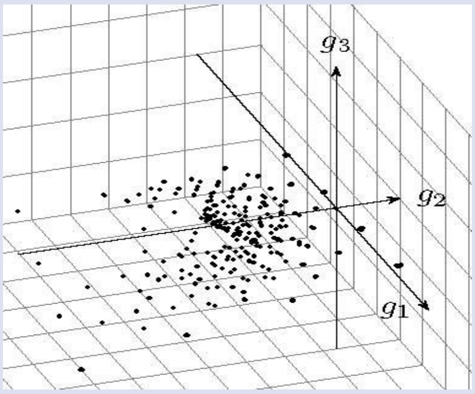


Figure 2. The "rough" images of g([-1,1]^5 ).

For $\varepsilon = 0.1$, $k^*$ is calculated as $k^* = (-0.99,-0.96,0.99,-0.96,-0.03)^T$ and the corresponding $c^* = (-0.03456,0.104025)^T$ is stabilizing vector.

## Conclusion

In this study, determining the robust stability of families of nonmonic multilinear polynomials and searching for stable member in these families are discussed. Reflection (box) coefficients have been used to analyze these problems. The set of $n$-th order Schur stable polynomials can be characterized by the reflection coefficients. For these problems, the results are given by using the reflection coefficients and the multilinear functions' extreme point property. Hence a multilinear equations system is obtained. Solution of this system of equations can be investigated with the division-elimination algorithm. One of the hard problems in linear control theory is considered by visualizing for the cases $n - l = 2$ and $n - l = 3$. A number of examples are given to illustrate the results.

## Conflicts of interest

The author declared there is no conflict of interest associated with this work.

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