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Analysis of Exact Solutions of a Mathematical Model by New Function Method

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ABSTRACT

In this article, the new function method is used to obtain the wave solutions of the nonlinear Klein-Gordon equation. Since the Klein-Gordon equation is a nonlinear partial differential equation containing exponential functions, it was decided to apply the new function method, which was defined on the assumption of a nonlinear auxiliary differential equation containing exponential functions. Thus, it aims to reach wave solutions not found in the literature. The considered method can be easily applied to this type of nonlinear problem that is difficult to solve and gives us solutions. Here, two new exact solutions are obtained. Then two and three-dimensional density and contour graphs are drawn by selecting the appropriate parameters to analyze the physical behavior of these solutions. The Mathematica package program was effectively used in all calculations and graphic drawings.

Keywords: New function method, Klein-Gordon Equation with an exponential nonlinearity, Mathematical model.

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Introduction

Each of the nonlinear partial differential equations is a mathematical model of problems in physics, chemistry, biology, mechanics, and health. Therefore, the solutions to such equations are essential to researchers. Based on this thought, various methods have been developed and applied to investigate the exact or approximate solutions of nonlinear mathematical models. Some of those methods are, respectively, the trial equation method [1], the Kudryashov method [2], the first integral method [3], the extended trial equation method [4], the new extended direct algebraic method [5], the modified exponential function method [6], the sine-cosine method [7], the extended tanh-function method [8], the variational iteration method [9], Adomian decomposition method [10], the Chebyshey-Tau method [11]. Apart from these, a new method called the “new function method” has been proposed in the paper [12]. Subsequently, some authors have applied this approach to various nonlinear partial differential equations [13-18]. This method makes it easy to find the exact solutions of the nonlinear partial differential equations that contain exponential, trigonometric, or hyperbolic functions and are difficult to solve. Because the nonlinear partial differential equations containing such functions can be easily solved by assuming an auxiliary ordinary differential equation containing functions of the same type and thus by simplifying the derivative term and hyperbolic, trigonometric, or exponential functions in these equations, for this reason, the new function method was used to determine the exact solutions of the nonlinear Klein-Gordon equation containing the exponential function in this study. All details about this method and its applications to nonlinear partial differential equations can be found in the articles published on this subject in the literature.

In this study, the new function method is used for the equation given below as the Klein-Gordon equation with exponential nonlinearity [19]

$$\phi_{tt}=\phi_{xx}+ae^{\beta\phi}+be^{2\beta\phi}. \tag{1}$$

The nonlinear Klein-Gordon equations are considered to model some nonlinear phenomena. Also, this nonlinear physical model has exponential nonlinearity. Detailed information on modeling these equations is given in the paper [20].

New Function Method

In this study, we analyze the traveling wave solutions of the Klein-Gordon equation using the new function method. First of all, the general form of the nonlinear mathematical model is given as

$$P\left(\phi_{xx},\phi_{tt},e^{\beta\phi},e^{2\beta\phi},\ldots\right)=0 \tag{2}$$

and then the wave transforms $\phi(x,t)=\phi(\eta)$, $\eta=k(x-ct)$ is substituted into Eq. (2), where $c\neq 0$ is the velocity of the wave and k is wave number. Thus, the nonlinear partial differential equation is obtained in the form of a nonlinear ordinary differential equation as follows

$$N(\phi', \phi'' \ldots) = 0 . \tag{3}$$

According to the method, suppose that the function ϕ satisfies the following equation

$$H(\phi'') = R(r(\phi)), \tag{4}$$

where H, R and r are continuous or differentiable functions on *R*. Also, the derivative concepts required in Eq. (4) are considered as

$$\phi' = h(r(\phi)), \quad \phi'' = h(r(\phi))r'(\phi)h'(r(\phi)) . \tag{5}$$

Substituting Eq. (5) into Eq. (4), we obtain

$$H\left(h(r(\phi))r'(\phi)h'(r(\phi))\right) = R(r(\phi)) . \tag{6}$$

If we take $\varpi = r(\phi)$, then we can write

$$H\left(\varpi' h(\varpi)h'(\varpi)\right) = R(\varpi) . \tag{7}$$

Thus, an ordinary differential equation containing the single variable function h and its derivatives up to the requested order is obtained from Equation (7). Then, integrating the obtained ordinary differential equation, it is easily converted into a form as follows:

$$\frac{d\phi}{h(r(\phi))} = d\eta \Rightarrow \int \frac{d\phi}{h(r(\phi))} = \int d\eta = \eta + S . \tag{8}$$

Here, S is an integral constant. The traveling wave solutions of Eq. (1) can be constructed by calculating these integrals.

Applications for The Klein-Gordon Equation with An Exponential Nonlinearity

By using the wave transform, the Klein-Gordon equation with exponential nonlinearity is reduced to the following nonlinear ordinary differential equation form:

$$k^2(c^2-1)\phi'' = ae^{\beta\phi} + be^{2\beta\phi} . \tag{9}$$

We suppose that the auxiliary equation based on the exponential function is as follows

$$\phi' = h\left(e^{\beta\phi}\right) . \tag{10}$$

Both sides of Eq. (10) are derived with respect to η, we get

$$\phi'' = \beta e^{\beta\phi} h\left(e^{\beta\phi}\right)h'\left(e^{\beta\phi}\right) . \tag{11}$$

Substituting Eq. (11) into Eq. (9), we have

$$k^2\left(c^2-1\right)\beta h\left(e^{\beta\phi}\right)h'\left(e^{\beta\phi}\right) = a + be^{\beta\phi} . \tag{12}$$

If λ is written instead of $e^{\beta\phi}$ in Eq. (12), then the new form of the ordinary differential equation is given as

$$k^2\left(c^2-1\right)\beta h(\lambda)h'(\lambda) = a + b\lambda . \tag{13}$$

When Eq. (13) is integrated with respect to λ, we obtain

$$h^2(\lambda) = \frac{1}{\beta k^2\left(c^2-1\right)}\left(b\lambda^2 + 2a\lambda + S\right), \tag{14}$$

where S is an integration constant. From Eq. (14), we can easily write the following function

$$h(\lambda) = \pm\sqrt{\frac{1}{\beta k^2\left(c^2-1\right)}\left(b\lambda^2 + 2a\lambda + S\right)} . \tag{15}$$

Letting $e^{\beta\phi} = \lambda$, we have $\phi = \frac{1}{\beta}\ln\lambda$. Differentiating both sides of this equality with respect to η, we get

$$\phi' = \frac{1}{\beta}\frac{\lambda'}{\lambda} = h\left(e^{\beta\phi}\right) = h(\lambda) . \tag{16}$$

From Eq. (16), the following differential equation is obtained

$$\lambda' = \frac{d\lambda}{d\eta} = \beta\lambda h(\lambda) = \sqrt{\frac{\beta}{k^2\left(c^2-1\right)}\lambda^2\left(b\lambda^2 + 2a\lambda + S\right)} \tag{17}$$

When Eq. (17) is solved by using the Mathematica program, the equality is obtained as

$$\eta + Q = \frac{\sqrt{bk^2\left(c^2-1\right)}\arctan\left[\frac{S + a\lambda}{\sqrt{S^2 + \lambda S(2a + b\lambda)}}\right]}{\sqrt{\beta S}}, \tag{18}$$

where Q is an integration constant. $\lambda_i \ (i = 1, \cdots, 4)$ are roots of equation $\left(b\lambda^4 + 2a\lambda^3 + S\lambda^2\right) = 0$

$$\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = -\frac{a + \sqrt{a^2 - bS}}{b}, \lambda_4 = \frac{-a + \sqrt{a^2 - bS}}{b} \tag{19}$$

Then, we obtain the following cases.

CASE-1:

$$\lambda(x,t) = \frac{S\left(-a\,\mathrm{sech}[\sigma]^2 + \sqrt{\left(-a^2 + bS\right)\mathrm{sech}[\sigma]^2\tanh[\sigma]^2}\right)}{a^2 - bS\tanh[\sigma]^2}, \tag{20}$$

where $\sigma = \left(\frac{\sqrt{S\beta}\left(Q + k\left(x - ct\right)\right)}{bk^2\left(c^2 - 1\right)} \right)$, $k \neq 0$ and $c \neq \pm 1$. Substituting the function λ in Eq. (20) into the relation $\phi = \frac{1}{\beta}\ln\lambda$, we get

$$\phi(\eta) = \frac{1}{\beta}\ln\left[\frac{S\left(-a\,\mathrm{sec}h[\sigma]^2 + \sqrt{\left(-a^2 + bS\right)\mathrm{sec}h[\sigma]^2 \tanh[\sigma]^2}\right)}{a^2 - bS\tanh[\sigma]^2} \right] \quad (21)$$

By determining the suitable values of the parameters in the exact solution of the nonlinear mathematical model, two-dimensional, three-dimensional, density, and contour graphs of this function were drawn. The Mathematica program is used to obtain these graphics.

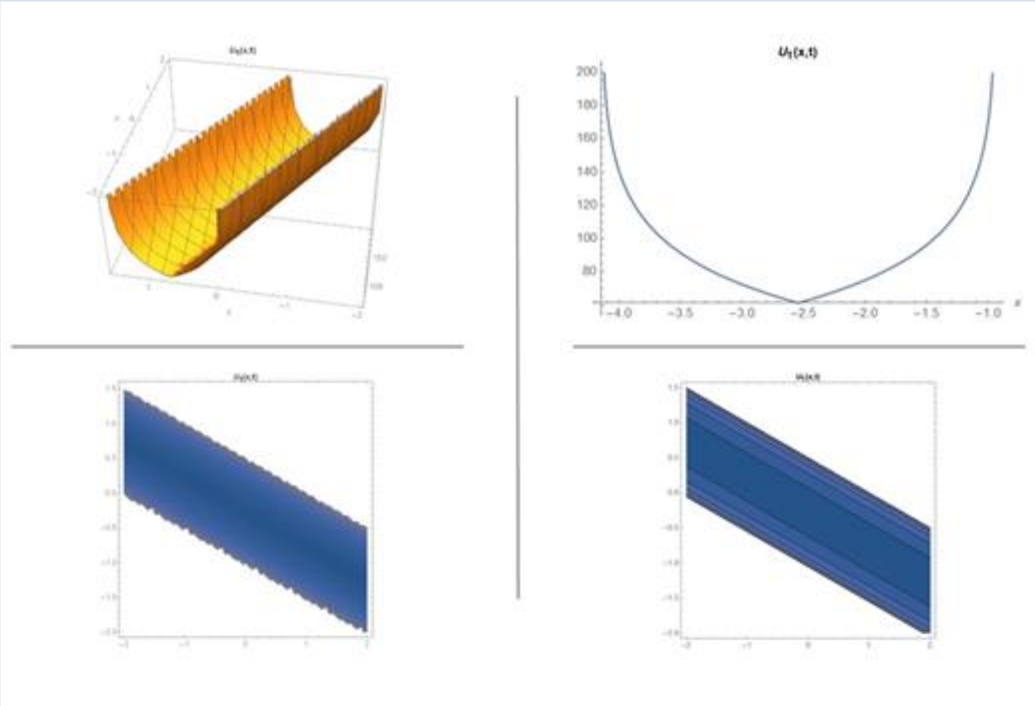

Figure 1. Three-dimensional, density and contour graphs of solution (21) at c=-2, S=1.26, a=-0.2, b=1, Q=1.2, k=2.2, β=0.03 and two-dimensional graph for t=1.

CASE-2:

$$\lambda(x,t) = \frac{S\left(a\,\mathrm{sec}h[\sigma]^2 + \sqrt{\left(-a^2 + bS\right)\mathrm{sec}h[\sigma]^2 \tanh[\sigma]^2}\right)}{-a^2 + bS\tanh[\sigma]^2} \quad (22)$$

where $\sigma = \left(\frac{\sqrt{S\beta}\left(Q + k\left(x - ct\right)\right)}{bk^2\left(c^2 - 1\right)} \right)$, $k \neq 0$ and $c \neq \pm 1$.

Substituting the function λ in Eq. (22) into the relation $\phi = \frac{1}{\beta}\ln\lambda$, we get

$$\phi(\eta) = \frac{1}{\beta}\ln\left[\frac{S\left(a\,\mathrm{sec}h[\sigma]^2 + \sqrt{\left(-a^2 + bS\right)\mathrm{sec}h[\sigma]^2 \tanh[\sigma]^2}\right)}{-a^2 + bS\tanh[\sigma]^2} \right] \quad (23)$$

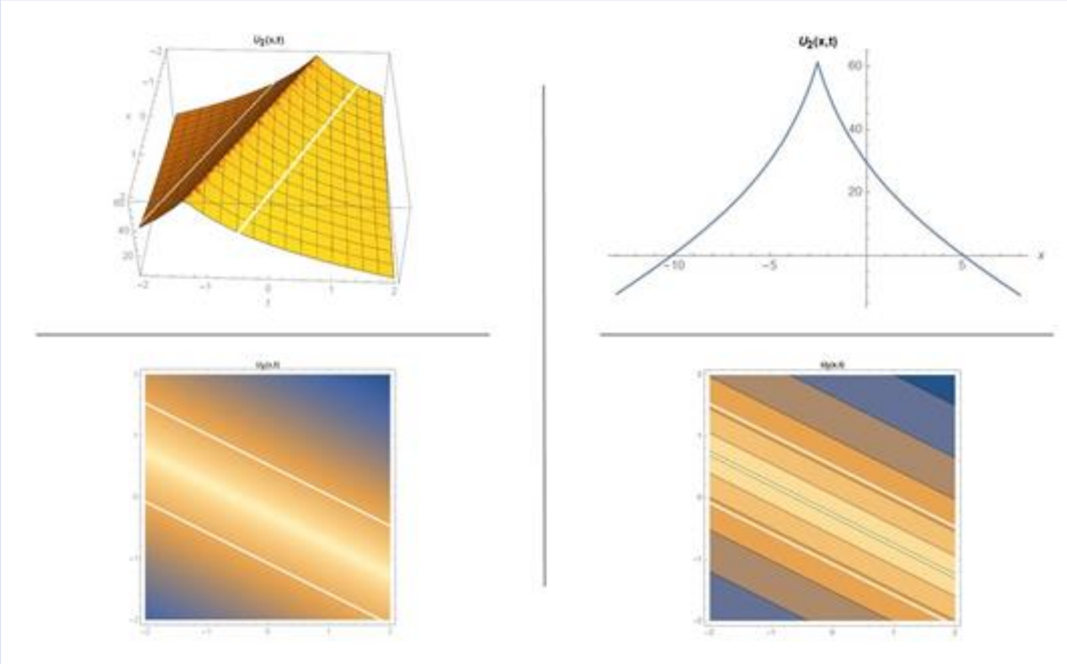

Figure 2. Three-dimensional, density and contour graphs of solution (23) at c=-2, S=1.26, a=-0.2, b=1, Q=1.2, k=2.2, β=0.03 and two-dimensional graph for t=1.

Conclusions

A new function method is used to obtain the soliton solutions of the Klein-Gordon equation. Thus, functions with periodic characteristics were obtained with the help of this method. Finding functions with this type of property provides an advantage in interpreting and making assumptions about the behavior of the mathematical model. Since periodic functions exhibit similar motion behaviors in all intervals, it allows interpretation of the behavior pattern for any interval beforehand. The solutions of trigonometric functions can be written in a logarithmic function. In addition, when the two-three dimensional density and contour graphs of the obtained solution function were examined, the periodic functions obtained as the solution function of the mathematical model represented the behavioral models. The results show that the new function technique is a very effective and convenient mathematical tool for solving nonlinear partial differential equations.

Conflicts of interest

The authors declared no conflict of interests.

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