Cumhuriyet Science Journal

Cumhuriyet Sci. J., 43(3) (2022) 454-459
DOI: https://doi.org/10.17776/csj.1063743

| csj.cumhuriyet.edu.tr | Founded: 2002 ISSN: 2587-2680 e-ISSN: 2587-246X Publisher: Sivas Cumhuriyet University

Constant Mean Curvature Surfaces Along a Spacelike Curve

Ergin Bayram [1,a,*]

[1] *Department of Mathematics, Faculty of Science, Ondokuz Mayıs University, Samsun, Türkiye.*
**Corresponding author*

Research Article

History
Received: 27/01/2022
Accepted: 01/08/2022

ABSTRACT

We construct constant mean curvature surfaces along a given spacelike curve in 3 dimensional Minkowski space. We parametrically present these surfaces using the famous Frenet frame of the curve in question. We give the sufficient conditions for the so called marching scale functions, which are the coefficients of the Frenet frame fields. We show that it is possible to obtain such surfaces for any given spacelike curve. Finally, the validity of the presented method is supported with illustrative examples.

Keywords: Spacelike curve, Constant mean curvature surfaces, Minkowski 3-space

[a] *erginbayram@yahoo.com* *https://orcid.org/ 0000-0003-2633-0991*

Introduction

The Gaussian and the mean curvature are a measure of how a surface curves. As an intrinsic quantity, the Gaussian curvature is one of the foundations of Riemannian geometry. In contrast, the mean curvature is an extrinsic quantity which measures how the surface lies in space. As the mean curvature is closely related to the character of the surface of a solid material, it is deeply connected to other sciences.

The mean curvature of a surface is half of the sum of principal curvatures at every point of the surface. A minimal surface is a surface which has zero mean curvature at every point. A surface with non-vanishing constant mean curvature is obtained by minimizing the area of the surface while preserving its volume. It can be physically modeled by a soap bubble.

We see surfaces almost in every differential geometry book [1-3]. There are several techniques to characterize surfaces. However, the construction of a surface is also an important issue. Current studies on surfaces have focused on finding surfaces with a common special curve [4 - 14]. Recently, Coşanoğlu and Bayram [15] obtained sufficient conditions for constant mean curvature surfaces through a prescribed curve in 3 dimensional Euclidean space. Mert and Karlığa [16] investigated timelike surfaces with constant angle in de-Sitter space. Mert and Atçeken [17] studied normal and binormal surfaces in hyperbolic 3–space.

In the present paper, analogous to Coşanoğlu and Bayram [15], we obtain parametric constant mean curvature surfaces through a given spacelike curve in 3 dimensional Minkowski space. We present constraints for these types of surfaces. The method is validated with several examples.

.

Materials and Methods

Apparatus

The real vector space $\mathbb{R}^3$ endowed with the metric tensor

$$\langle X, Y\rangle = -x_1y_1 + x_2y_2 + x_3y_3$$

is called the Minkowski 3-space and denoted by $\mathbb{R}^3_1$, where $X = (x_1, x_2, x_3)$, $Y = (y_1, y_2, y_3) \in \mathbb{R}^3$ [1] The Lorentzian vectorial product is defined by

$$X \times Y = (x_2y_3 - x_3y_2, x_1y_3 - x_3y_1, x_2y_1 - x_1y_2).$$

A vector $X \in \mathbb{R}^3_1$ is called timelike, spacelike or lightlike (null) if

$$\begin{cases} \langle X, X\rangle < 0, \\ \langle X, X\rangle > 0 \text{ or } X = \vec{0}, \\ \langle X, X\rangle = 0, \end{cases}$$

respectively. Similarly, a curve in $\mathbb{R}^3_1$ is called a timelike, spacelike or lightlike curve if its tangent vector field is always timelike, spacelike or lightlike, respectively.

The set $\{T, N, B\}$ denotes the moving Frenet-Serret frame through a curve α, where T, N and B are the tangent vector field, the principal normal vector field and the binormal vector field of the curve α, respectively.

The unit speed spacelike curve α with timelike normal has spacelike vector fields T and B and timelike vector field N. For this setting we have,

$$T = -N \times B, \; N = B \times T, \; B = -T \times N.$$

$B(s)$ is the unique spacelike unit vector field orthogonal to the timelike plane $\{T(s), N(s)\}$ at every point $\alpha(s)$, such that the orientation of $\mathbb{R}_1^3$ and $\{T, N, B\}$ are the same. Then, we have the following Frenet formulas [18]

$$T' = \kappa N,\ N' = \kappa T + \tau B,\ B' = \tau N.$$

The arc-length spacelike curve α with spacelike normal has spacelike vector fields T and N and timelike vector field B. In this case,

$$T = -N \times B, \quad N = -B \times T, \quad B = T \times N.$$

The vector field $B(s)$ is the unique timelike unit vector field orthogonal to the spacelike plane $\{T(s), N(s)\}$ at each point $\alpha(s)$ so that the orientation of $\mathbb{R}_1^3$ and $\{T, N, B\}$ are the same. Then, we obtain the following Frenet formulas [19]

$$T' = \kappa N,\ N' = -\kappa T + \tau B,\ B' = \tau N.$$

The mean curvature of the surface $P(s,t)$ is given as

$$H(s,t) = -\left(\frac{\det(P_s, P_t, P_{ss})G - 2\det(P_s, P_t, P_{st})F + \det(P_s, P_t, P_{tt})E}{2(-\varepsilon W)^{\frac{3}{2}}}\right)(s,t),$$

where $E,\ F,\ G$ are the coefficients of the first fundamental form of the surface $P(s,t)$, $W = EG - F^2$ and

$$\varepsilon = \begin{cases} 1, \text{ if } P(s,t) \text{ is timelike}, \\ -1, \text{ if } P(s,t) \text{ is spacelike}, \end{cases}$$

[19].

Results and Discussion

Constant Mean Curvature Surfaces Along A Spacelike Curve

Let $\alpha(s)$ be a spacelike curve with timelike normal arc-length regular curve with curvature $\kappa(s)$ and torsion $\tau(s)$. Also, assume that $\alpha''(s) \neq \vec{0},\ \forall s.$ Parametric surfaces possessing $\alpha(s)$ can be written as

$$P(s,t) = \alpha(s) + u(s,t)T(s) + v(s,t)N(s) + w(s,t)B(s), \tag{1}$$

$L_1 \leq s \leq L_2,\ T_1 \leq t \leq T_2,$ where $\{T(s), N(s), B(s)\}$ is the Frenet-Serret frame of $\alpha(s)$. C^2 functions $u(s,t),\ v(s,t),\ w(s,t)$ are known as marching-scale functions. Note that, choosing distinct marching-scale functions corresponds to distinct surfaces along the curve $\alpha(s)$.

To simplify the calculations, we suppose that the curve $\alpha(s)$ is a t-parameter curve on the surface in Eqn. (1). So, we have

$$u(s,t_0) \equiv 0,\ v(s,t_0) \equiv 0,\ w(s,t_0) \equiv 0$$

for some $t_0 \in [T_1, T_2]$.

We make the following calculations required for the mean curvature.

$$\begin{aligned} P_s(s,t) = &\left(1 + u_s(s,t) + \kappa(s)v(s,t)\right)T(s) \\ &+ \left(\kappa(s)u(s,t) + v_s(s,t) + \tau(s)w(s,t)\right)N(s) \\ &+ \left(\tau(s)v(s,t) + w_s(s,t)\right)B(s), \end{aligned}$$

$$P_t(s,t)=u_t(s,t)T(s)+v_t(s,t)N(s)+w_t(s,t)B(s),$$

$$P_s(s,t_0)=T(s),$$

$$P_t(s,t_0)=u_t(s,t_0)T(s)+v_t(s,t_0)N(s)+w_t(s,t_0)B(s),$$

$$P_{ss}(s,t_0)=\kappa(s)N(s),$$

$$\begin{aligned}P_{st}(s,t_0)=P_{ts}(s,t_0)=&\left(u_{ts}(s,t_0)+\kappa(s)v_t(s,t_0)\right)T(s)\\&+\left(\kappa(s)u_t(s,t_0)+v_{ts}(s,t_0)+\tau(s)w_t(s,t_0)\right)N(s)\\&+\left(\tau(s)v_t(s,t_0)+w_{ts}(s,t_0)\right)B(s)\end{aligned}$$

$$P_{tt}(s,t_0)=u_{tt}(s,t_0)T(s)+v_{tt}(s,t_0)N(s)+w_{tt}(s,t_0)B(s),$$

$$\det\left(P_s(s,t_0),P_t(s,t_0),P_{ss}(s,t_0)\right)=-\kappa(s)w_t(s,t_0),$$

$$\begin{aligned}\det\left(P_s(s,t_0),P_t(s,t_0),P_{st}(s,t_0)\right)=&\,v_t(s,t_0)\left(v_t(s,t_0)\tau(s)+w_{ts}(s,t_0)\right)\\&-w_t(s,t_0)\left(u_t(s,t_0)\kappa(s)+v_{ts}(s,t_0)\right.\\&\left.+\tau(s)w_t(s,t_0)\right),\end{aligned}$$

$$\det\left(P_s(s,t_0),P_t(s,t_0),P_{tt}(s,t_0)\right)=v_t(s,t_0)w_{tt}(s,t_0)-w_t(s,t_0)v_{tt}(s,t_0),$$

where subscript denotes the partial derivative with respect to the parameter in question. Hence, the surface $P(s,t)$ in Eqn. (1) has the following mean curvature along the curve $\alpha(s)$

$$H(s,t_0)=\frac{\kappa w_t\left(u_t^2-v_t^2+w_t^2\right)+2u_t\left[v_t\left(v_t\tau+w_{ts}\right)-w_t\left(\kappa u_t+v_{ts}+\tau w_t\right)\right]+w_tv_{tt}-v_tw_{tt}}{2\left(\varepsilon\left(v_t^2-w_t^2\right)\right)^{\frac{3}{2}}}(s,t_0).$$

Theorem 1 : The surface $P(s,t)$ in Eqn. (1) has constant mean curvature along the spacelike curve $\alpha(s)$ with spacelike binormal if one of the following conditions is satisfied:

i) $u(s,t_0)=v(s,t_0)=w(s,t_0)=w_t(s,t_0)=w_{tt}(s,t_0)\equiv 0\neq u_t(s,t_0)=\ v_t(s,t_0),\ \tau(s)=$ constant,
ii) $u(s,t_0)=v(s,t_0)=w(s,t_0)=v_t(s,t_0)=v_{tt}(s,t_0)\equiv 0\neq u_t(s,t_0)=\ w_t(s,t_0),\ \tau(s)=$ constant,
iii) $v_t(s,t_0)\neq 0\equiv u(s,t_0)=v(s,t_0)=w(s,t_0)=u_t(s,t_0)=w_t(s,t_0)=w_{tt}(s,t_0),$
iv) $w_t(s,t_0)\neq 0\equiv u(s,t_0)=v(s,t_0)=w(s,t_0)=u_t(s,t_0)=v_t(s,t_0)=v_{tt}(s,t_0),\ \kappa(s)=$ constant,

If $\alpha(s),\ L_1\leq s\leq L_2$ is a spacelike curve with timelike binormal arc-length regular curve having curvature $\kappa(s)$ and torsion $\tau(s)$, then we have the theorem below.

Theorem 2 : The surface $P(s,t)$ in Eqn. (1) has constant mean curvature along the spacelike curve $\alpha(s)$ with timelike binormal if one of the following conditions is satisfied :

i) $u(s,t_0)=v(s,t_0)=w(s,t_0)=w_t(s,t_0)=w_{tt}(s,t_0)\equiv 0\neq u_t(s,t_0)=\ v_t(s,t_0),\ \tau(s)=$ constant,
ii) $u(s,t_0)=v(s,t_0)=w(s,t_0)=v_t(s,t_0)=v_{tt}(s,t_0)\equiv 0\neq u_t(s,t_0)=\ w_t(s,t_0),\ \kappa(s)+\tau(s)=$ constant,
iii) $v_t(s,t_0)\neq 0\equiv u(s,t_0)=v(s,t_0)=w(s,t_0)=u_t(s,t_0)=w_t(s,t_0)=w_{tt}(s,t_0),$
iv) $w_t(s,t_0)\neq 0\equiv u(s,t_0)=v(s,t_0)=w(s,t_0)=u_t(s,t_0)=v_t(s,t_0)=v_{tt}(s,t_0),\ \kappa(s)=$ constant,

Example 1 : In this example, we construct surfaces with constant mean curvature along a given spacelike curve with timelike normal vector field. The unit speed spacelike curve with timelike normal $\alpha(s)=\left(\frac{1}{2}\cosh\left(\sqrt{2}s\right),\frac{\sqrt{2}s}{2},\frac{1}{2}\sinh\left(\sqrt{2}s\right)\right)$ has the following Frenet apparatus :

$$T(s)=\left(\frac{\sqrt{2}}{2}\sinh\left(\sqrt{2}s\right),\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\cosh\left(\sqrt{2}s\right)\right),$$

$$N(s)=\left(\cosh\left(\sqrt{2}s\right),0,\sinh\left(\sqrt{2}s\right)\right),$$

$$B(s)=\left(-\frac{\sqrt{2}}{2}\sinh\left(\sqrt{2}s\right),\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\cosh\left(\sqrt{2}s\right)\right),$$

$$\kappa(s)=1,\quad \tau(s)=-1.$$

Marching-scale functions $u(s,t)=v(s,t)=t,\ w(s,t)\equiv 0$ and $t_0=0,$ satisfies Theorem 1 (i) and the surface

$$P_1(s,t)=\left(\left(t+\frac{1}{2}\right)\cosh\left(\sqrt{2}s\right)+\frac{t\sqrt{2}}{2}\sinh\left(\sqrt{2}s\right),\frac{\sqrt{2}}{2}(s+t),\right.$$
$$\left.\left(t+\frac{1}{2}\right)\sinh\left(\sqrt{2}s\right)+\frac{t\sqrt{2}}{2}\cosh\left(\sqrt{2}s\right)\right),$$

$-1\le s\le 1,\ \ 0\le t\le 1$ with constant mean curvature $H(s,0)=-1$ along the spacelike curve $\alpha(s)$ is obtained (Figure 1).

Choosing marching-scale functions $u(s,t)=w(s,t)=t,\ \ v(s,t)\equiv 0$ and $t_0=0,$ satisfies Theorem 1 (ii) and we immediately get the surface

$$P_2(s,t)=\left(\frac{1}{2}\cosh\left(\sqrt{2}s\right),\frac{\sqrt{2}}{2}(s+2t),\frac{1}{2}\sinh\left(\sqrt{2}s\right)\right),$$

with constant mean curvature $H(s,0)=1$ along the curve $\alpha(s)$ (Figure 2).

Example 2 The spacelike curve with timelike binormal

$$\alpha(s)=\left(\frac{4}{9}\sinh(3s),\frac{4}{9}\cosh(3s),\frac{5}{3}s\right)$$

has the following Frenet apparatus :

$$T(s)=\left(\frac{4}{3}\cosh(3s),\frac{4}{3}\sinh(3s),\frac{5}{3}s\right),$$

$$N(s)=\left(\sinh(3s),\cosh(3s),0\right),$$

$$B(s)=\left(-\frac{5}{3}\cosh(3s),-\frac{5}{3}\sinh(3s),-\frac{4}{3}\right),$$

$$\kappa(s)=4,\quad \tau(s)=-5.$$

Choosing marching-scale functions $v(s,t)=t,\ u(s,t)=w(s,t)\equiv 0$ and $t_0=0,$ satisfies Theorem 2 (iii) and we obtain the surface

$$P_3(s,t)=\left(\left(\frac{4}{9}+t\right)\sinh(3s),\left(\frac{4}{9}+t\right)\cosh(3s),\frac{5}{3}s\right),$$

$-1\le s\le 1,\ \ 0\le t\le 1$ with constant mean curvature $H(s,0)=0$ along the curve $\alpha(s)$ (Figure 3) .

If we choose $u(s,t)=v(s,t)\equiv 0, w(s,t)=t$ and $t_0=0,$ Theorem 2 (iv) is satisfied and we obtain the surface

$$P_4(s,t)=\left(\frac{4}{9}\sinh(3s)-\frac{5}{3}t\cosh(3s),\frac{4}{9}\cosh(3s)-\frac{5}{3}t\sinh(3s),\frac{5s-4t}{3}\right),$$

$-1\le s\le 1,\ \ 0\le t\le 1$ with constant mean curvature $H(s,0)=-2$ along the curve $\alpha(s)$ (Figure 4).

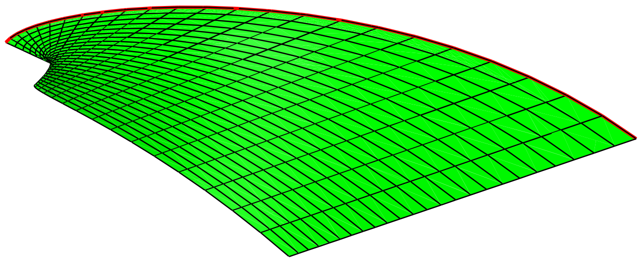

Figure 1. Constant mean curvature surface $P_1(s,t)$ along the spacelike curve $\alpha(s)$

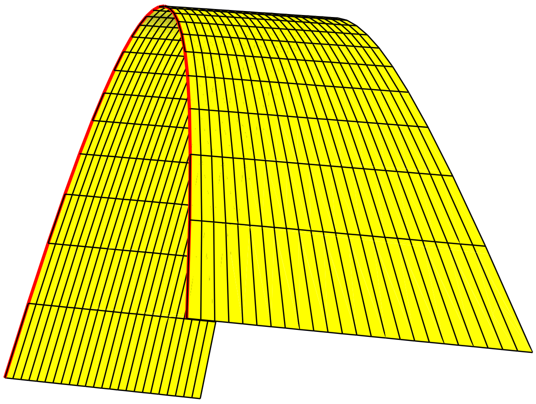

Figure 2. Constant mean curvature surface $P_2(s,t)$ along the spacelike curve $\alpha(s)$

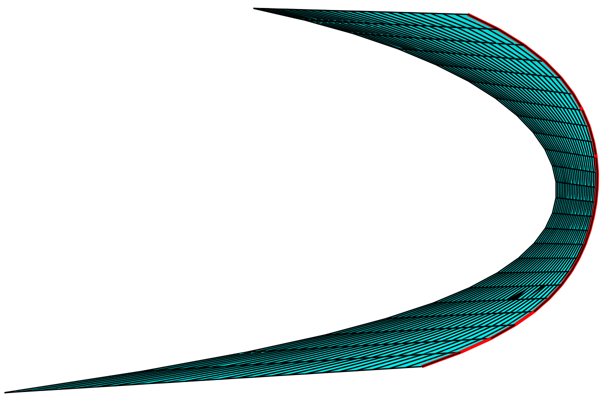

Figure 3. Constant mean curvature surface $P_3(s,t)$ along the spacelike curve $\alpha(s)$

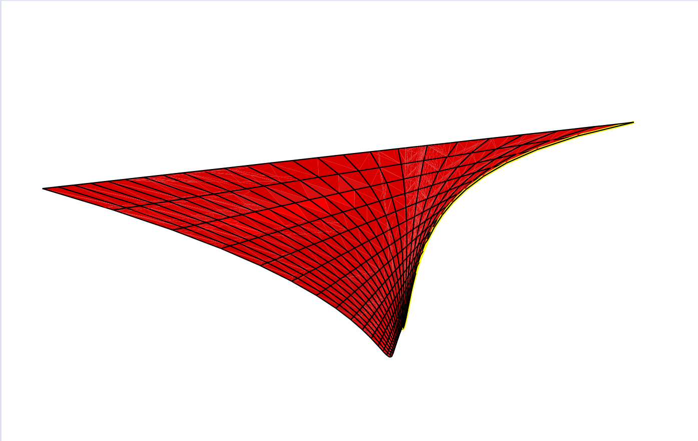

Figure 4. Constant mean curvature surface P_4 (s,t) along the spacelike curve α(s)

Acknowledgment

The author would like to present his sincere appreciation to the Editor and referees for their time and efforts.

Conflicts of interest

The authors state that there is no conflict of interests.

References

[1] O'Neill B., Semi-Riemannian geometry with applications to relativity, Academic Press Lim., London, (1983).

[2] Willmore T.J., An introduction to differential geometry, Oxford University Press, Delhi, (1959).

[3] Carmo M.P., Differential geometry of curves and surfaces, Englewood Cliffs, Prentice Hall, (1976).

[4] Wang G.J., Tang K., Tai C.L., Parametric representation of a surface pencil with a common spatial geodesic, *Comput. Aided Design,* 36 (2004) 447-459.

[5] Kasap E., Akyıldız F.T., Surfaces with common geodesic in Minkowski 3-space, *Appl. Math. Comput*, 177 (2006) 260-270.

[6] Li C.Y., Wang R.H., Zhu C.G., Parametric representation of a surface pencil with a common line of curvature, *Comput. Aided Design*, 43 (2011) 1110-1117.

[7] Bayram E., Güler F., Kasap E., Parametric representation of a surface pencil with a common asymptotic curve, *Comput. Aided Design*, 44 (2012) 637-643.

[8] Bayram E., Bilici M., Surface family with a common involute asymptotic curve, *Int. J. Geom. Methods Modern Phys*., 13 (2016) 1650062.

[9] Güler F., Bayram E., Kasap E., Offset surface pencil with a common asymptotic curve, *Int. J. Geom. Methods Modern Phys.,* 15 (2018) 1850195.

[10] Şaffak Atalay G., Kasap E., Surfaces family with common null asymptotic, *Appl. Math. Comput*., 260 (2015) 135-139.

[11] Yüzbaşı Z.K., On a family of surfaces with common asymptotic curve in the Galilean space G_3, *J. Nonlinear Sci. Appl.,* 9 (2016), 518-523.

[12] Şaffak Atalay G.,Bayram E., Kasap E., Surface family with a common asymptotic curve in Minkowski 3-space, *Journal of Science and Arts*, 2 (43) (2018) 357-368.

[13] Yoon D.W., Yüzbaşı Z.K., Bektaş M., An approach for surfaces using an asymptotic curve in Lie Group, *Journal of Advanced Physics*, 6 (4) (2017) 586-590.

[14] Bayram E., Surface pencil with a common adjoint curve, *Turkish Journal of Mathematics*, 44 (2020) 1649 -1659.

[15] Coşanoğlu H., Bayram E., Surfaces with constant mean curvature along a curve in 3-dimensional Euclidean space, Süleyman Demirel University, *Journal of Natural and Applied Sciences*, 24 (3) (2020) 533-538.

[16] Mert T., Karlığa B., Timelike surfaces with constant angle in de-Sitter space, *Boletim da Sociedade Paranaense de Matemática*, 35 (2017), 79-93.

[17] Mert T., Atçeken M., Some special ruled surfaces in hyperbolic 3-space, *Konuralp Journal of Mathematics*, 9 (2), (2021), 298-309.

[18] Walrave J., Curves and surfaces in Minkowski space, PhD. Thesis, K. U. Leuven Faculteit Der Wetenschappen, 1995.

[19] Lopez R., Differential geometry of curves and surfaces in Lorentz-Minkowski space, *Int. El. J. of Geometry*, 7 (1) (2014) 44-107.