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Obtaining The Finite Difference Approximation of The Lame System By Using Barycentric Coordinates

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ABSTRACT

The elasto-plastic contact problem with an unknown contact domain (UCD) has attracted mathematicians, mechanics and engineers for decades. So, the problem of determining the stresses in the UCD is very important nowadays in terms of engineering and applied mathematics. To improve the finite element model, the remeshing algorithm is used for the considered indentation problem. The algorithm allows the determination of the UCD at each step of the indentation with high accuracy. This paper presents the analysis and numerical solution of the boundary value problem for the Lame system, and the modeling of the contact problem for rigid materials. By using barycentric coordinates, the finite difference approximation of the mathematical model of the deformation problem with undetermined bounded is obtained and the relations between the finite elements and finite differences are investigated.

Keywords: Barycentric coordinates, Finite Difference, Finite Element Method, Lame Equations.

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Introduction

The mathematical model of many engineering problems is expressed by undetermined bounded elliptic equations [1-3]. The most important feature of such problems is that some of the boundary conditions are given in the form of inequalities. Therefore, the general solution of the boundary problem satisfies not the integral identity but the integral inequality called the variational inequality [3]. Since the boundary conditions are in the form of inequalities, the generalized solution of the problem is sought in a closed convex subset of this space, not in any subspace of the Sobolev space. The solution of the equilibrium problem of an elastic body in any closed convex set (displacement set) was studied by the Italian mathematician Antonio Signorini in 1933 by bringing the functional to the minimization problem [4]. His study contributed greatly to the analysis of the boundary value problems of the elasticity theory in terms of variational inequalities. The variational inequalities of the elasticity theory were examined mathematically in G. Fichera's monograph [5], and then, the theory of variational inequalities was investigated by G. Duvaut, J.L. Lions [2], D. Kinderlehrerand, G. Stampacchia [3], J.L. Lions [6] and other authors. The extensive analysis of the numerical solutions of variational inequalities with undetermined bounded was given in various studies in the literature [1,7-9]. However, in all studies, when a numerical solution was found, a static mesh was used. That is, the indeterminate part of the boundary (contact domain) would be considered between the points of the static mesh. One of the most important difficulties of the problem is to determine the boundaries of the contact domain. The behavior of the solution at the boundary nodes for undetermined bounded elliptic equations was investigated previously [10]. The obtained results revealed the need to use adaptive (quasi/local-static) meshes in the solution of indeterminate bounded problems [11]. Weng P. et al. considered the elastic deformation of the indenter [12] in the contact process to establish a more accurate elastic-plastic transition model.

This study presents the processes of obtaining the numerical approximation of the variational inequality related to an elasto-plastic plane contact problem and its numerical solution. The main aims of the study were to obtain the finite difference approximation of the mathematical model of the deformation problem with undetermined bounded by using barycentric coordinates and investigate the relation between the finite elements and finite differences. The boundary value problem for the Lame system is detailed, and its variational formulation is given in Section 2. The local stiffness matrix (LSM) obtained by using barycentric coordinates and the finite difference equation (FDE) at any point obtained using the LSM are presented in Sections 3 and 4, respectively. Finally, the numerical solutions are presented, and the relations between the finite elements and finite differences are given in Section 5.

Materials and Methods

Formulation

In the case of plane deformation, the equilibrium problem of the object deformed by the effect of a rigid punch is modeled mathematically with the boundary value problem for the Lame equation:

$$-(\lambda+\mu)graddivu-\mu\Delta u=F,\ (x,y)\in\Omega\subset\mathbb{R}^2 \quad (1)$$

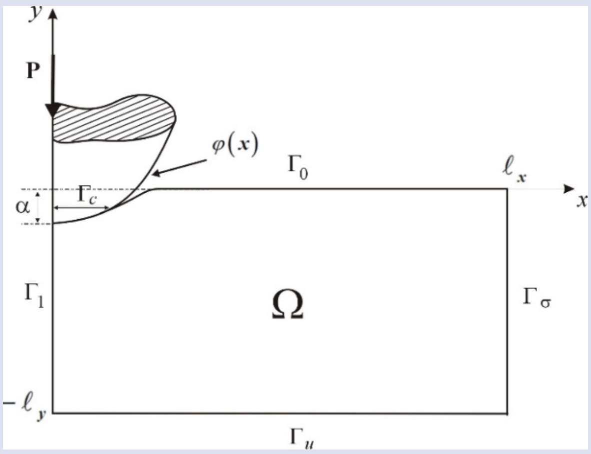


Figure 1. Geometry of the spherical indentation.

$$\begin{cases} u_2(s)\le\varphi(s),\quad \sigma_{22}(u)\le 0,\quad \sigma_{12}(u)=0, \\ \left[u_2(s)-\varphi(s)\right]\sigma_{22}(u)=0 \quad , \quad s\in\Gamma_0; \end{cases} \quad (2)$$

$$\begin{cases} \sigma_{ij}(u)n_j=f_i,\quad i,j=1,2, \\ u(s)\equiv\left(u_1(s),u_2(s)\right)=0,\quad s\in\Gamma_u; \end{cases} \quad (3)$$

$$\sigma_{12}(u)=0,\quad u_1(s)=0,\quad s\in\Gamma_l. \quad (4)$$

Here, let $\Gamma_0=\left\{(x,y)\middle|\, 0\le x<\ell_x,\, y=0\right\}$,
$\Gamma_u=\left\{(x,y)\middle|\, 0\le x<\ell_x,\, y=-\ell_y\right\}$,
$\Gamma_\sigma=\left\{(x,y)\middle|\, x=\ell_x,-\ell_y<y<0\right\}$ and
$\Gamma_1=\left\{(x,y)\middle|\, x=0,-\ell_y<y<0\right\}$ be the boundaries of the rectangular region
$\Omega=\left\{(x,y)\middle|\, 0<x<\ell_x,-\ell_y<y<0,\quad \ell_x,\ell_y>0\right\}$
filled by the rigid body in the Oxy plane (Figure 1). The contact domain of the rigid body with the punch is at the unknown boundary $\Gamma_c=\left\{(x,y)\middle|\, 0\le x\le a,\quad y=0\right\}$, where a is an uncertain constant, and the boundary of the contact domain is denoted by Γ_c.

Supposing that the material deforms as much as $\alpha>0$ through the Oy axis when it is compressed by the punch under the effect of any force P, that is, the maximal displacement of the apex of the punch is α, in case the cross-section of the punch is $y=\varphi(x)$, the contact domain of the material with the punch will be $\tilde{\Gamma}_c=\left\{(x,y)\middle|\, 0\le x\le a,\quad y=-\alpha+\varphi(x)\right\}$.

The components of the stress tensors are

$$\sigma_{ii}=\lambda div(u)+2\mu\frac{\partial u_i}{\partial x_i},\qquad \sigma_{ij}=\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right),$$

$i,j=1,2$.

while the components of the deformation tensors are

$$\varepsilon_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right).$$

The Lame constants λ and μ are defined as follows:

$$\lambda=\frac{E\nu}{(1+\nu)(1-2\nu)},\ \mu=\frac{E}{2(1+\nu)}.$$

Here, E is the modulus of elasticity, and ν is the Poisson's constant. The Lame constants λ and μ are non-linear in the case of plastic deformation, and they are defined depending on displacements as follows:

$$\tilde{\lambda}=\lambda+\frac{2}{3}\mu\omega(e_u(u)),\ \tilde{\mu}=\mu\left[1-\omega(e_u(u))\right]$$

where $\omega(e_u)$ is the function that characterizes the plasticity case [13]:

$$\begin{cases} 0<\omega(e_u)<\dfrac{d\left[e_u\omega(e_u)\right]}{de_u}<1, & e_u>e_0 \\ \omega(e_u)=0, & e_u\le e_0. \end{cases}$$

Here, e_0 is the elastic limit, and the intensity of deformation is

$$e_u(u)=\frac{\sqrt{2}}{3}\left\{(\varepsilon_{11}(u)-\varepsilon_{22}(u))^2+(\varepsilon_{22}(u)-\varepsilon_{33}(u))^2+(\varepsilon_{33}(u)-\varepsilon_{11}(u))^2+6\left[\varepsilon_{12}^2(u)+\varepsilon_{13}^2(u)+\varepsilon_{23}^2(u)\right]\right\}^{1/2}.$$

When $\omega(e_u)=0$ corresponding to the elastic case, even though Equation (1) is linear, the fact that the contact domain is not certain causes the problem to be non-linear.

Variational Formulation

It is known from the variational principle that the solution of problem (1) is minimized by the following functional:

$$J(u)=\frac{1}{2}a(u,u)-b(u) \quad (5)$$

where the bilinear and the linear parts of functional (5) are as follows, respectively:

$$a(u,v)=\iint_{\Omega}\left\{\lambda divu.divv+2\mu\left(\frac{\partial u_1}{\partial x}\frac{\partial v_1}{\partial x}+\frac{\partial u_2}{\partial y}\frac{\partial v_2}{\partial y}\right)+\mu\left(\frac{\partial u_2}{\partial x}+\frac{\partial u_1}{\partial y}\right)\left(\frac{\partial v_2}{\partial x}+\frac{\partial v_1}{\partial y}\right)\right\}dxdy,$$

$$b(v)=\iint_{\Omega}F_i v_i ds+\int_{\Gamma_\sigma}f_i v_i ds,\quad i=1,2. \tag{6}$$

So, the problem turns into a minimization problem $\exists u\in V \quad J(u)=\min_{v\in V}J(u)$ in the set $V=\left\{v\in H^1(\Omega)\middle| v(s)=0, s\in\Gamma_u; v_N(s)\le\varphi(s), s\in\Gamma_0\right\}$. When the problem is being solved, the contact domain Γ_c is determined as in the study by A.A. Ilyushin [13]. Since the contact region is assumed to be certain at each step, the inequalities in the boundary conditions turn into equalities, and the problem becomes linear.

Obtaining Local Stiffness Matrix by Using Barycentric Coordinates

Suppose that the region Ω in which the problem is defined is divided into quadrilateral finite elements. In this case, we define the equal-step mesh as follows:

$$\omega_{h\tau}=\left\{\left\{(x_i,y_j)\middle| x_{i+1}=x_i+h,\, y_{j+1}=y_j+\tau,\,(x_1,y_1)=(0,-\ell_y),\,(x_N,y_M)=(\ell_x,0),\, i=\overline{1,N-1},\, j=\overline{1,M-1}\right\}\right\}.$$

In the quadrilateral finite elements, the barycentric coordinates are equal to the form function, which is the projection of the Lagrange basis function $\xi_{ij}(x,y)$ on the finite element $e_{mn}=\left\{(x,y)\middle| x_m\le x\le x_{m+1},\, y_n\le y\le y_{n+1}\right\}$, and it can be defined as

$$L_q(x,y)=\xi_{ij}(x,y)\Big|_{e_{mn}},\quad q=1,2,3,4. \tag{7}$$

Since this form function is the same as the barycentric coordinates in the quadrilateral finite element, it can be written in general with single indices as follows [14-15]:

$$L_q(x,y)=\frac{a_q+b_q x+c_q y+d_q xy}{S},\quad q=1,2,3,4. \tag{8}$$

Here, q is the local number of the vertices of the finite quadrilateral element e_{mn} (from down to up and from left to right), and S is the area of this finite element. It is ascertained that

$$\begin{aligned}
&a_1=x_{i+1}y_{j+1},\quad b_1=-y_{j+1},\quad c_1=-x_{i+1},\quad d_1=1;\\
&a_2=-x_{i+1}y_j,\quad b_2=-y_j,\quad c_2=x_{i+1},\quad d_2=-1;\\
&a_3=x_i y_{j+1},\quad b_3=y_{j+1},\quad c_3=x_i,\quad d_3=-1;\\
&a_4=x_i y_j,\quad b_4=-y_j,\quad c_4=-x_i,\quad d_4=1;
\end{aligned} \tag{9}$$

for the finite quadrilateral elements e_{ij}, since the components of the local stiffness matrix corresponding to the finite element e_{mn} are calculated with the help of the bilinear form $a(u,v)$,

$$a_{mn}^{1}\left(L_p,L_q\right)=\iint_{e_{mn}}\left\{(\lambda+2\mu)\frac{\partial L_p}{\partial x}\frac{\partial L_q}{\partial x}+\mu\frac{\partial L_p}{\partial y}\frac{\partial L_q}{\partial y}\right\}dxdy\ ;$$

$$a_{mn}^{2}\left(L_p,L_q\right)=\iint_{e_{mn}}\left\{\lambda\frac{\partial L_p}{\partial x}\frac{\partial L_q}{\partial y}+\mu\frac{\partial L_p}{\partial y}\frac{\partial L_q}{\partial x}\right\}dxdy\ ;$$

$$a_{mn}^{3}\left(L_p,L_q\right)=\iint_{e_{mn}}\left\{\lambda\frac{\partial L_p}{\partial y}\frac{\partial L_q}{\partial x}+\mu\frac{\partial L_p}{\partial x}\frac{\partial L_q}{\partial y}\right\}dxdy\ ; \qquad (10)$$

$$a_{mn}^{4}\left(L_p,L_q\right)=\iint_{e_{mn}}\left\{(\lambda+2\mu)\frac{\partial L_p}{\partial y}\frac{\partial L_q}{\partial y}+\mu\frac{\partial L_p}{\partial x}\frac{\partial L_q}{\partial x}\right\}dxdy\ .$$

Given that

$$\frac{\partial}{\partial x}L_i\left(x,y\right)=\frac{b_i+d_iy}{S},\quad \frac{\partial}{\partial y}L_i\left(x,y\right)=\frac{c_i+d_ix}{S},\quad i=1,2,3,4\,.$$

The expressions for the components of the local stiffness matrix are obtained as follows:

$$a_{m}^{1}\left(L_p,L_q\right)=\frac{h\tau}{6}\frac{1}{S^2}\Big\{\left(\lambda+2\mu\right)\Big[6b_pb_q+3\left(b_pd_q+b_qd_p\right)\left(y_{j+1}+y_j\right)+2d_pd_q\left(y_{j+1}^2+y_{j+1}y_j+y_j^2\right)\Big]$$
$$\mu\Big[6c_pc_q+3\left(c_pd_q+c_qd_p\right)\left(x_{i+1}+x_i\right)+2d_pd_q\left(x_{i+1}^2+x_{i+1}x_i+x_i^2\right)\Big]\Big\},$$

$$a_{m}^{2}\left(L_p,L_q\right)=\frac{h\tau}{4}\frac{1}{S^2}\Big\{\lambda\Big[4b_pc_q+2b_pd_q\left(x_{i+1}+x_i\right)+2c_qd_p\left(y_{j+1}+y_j\right)+d_pd_q\left(x_{i+1}+x_i\right)\left(y_{j+1}+y_j\right)\Big]$$
$$\mu\Big[4b_qc_p+2c_pd_q\left(y_{j+1}+y_j\right)+2b_qd_p\left(x_{i+1}+x_i\right)+d_pd_q\left(x_{i+1}+x_i\right)\left(y_{j+1}+y_j\right)\Big]\Big\},$$

$$a_{m}^{3}\left(L_p,L_q\right)=\frac{h\tau}{4}\frac{1}{S^2}\Big\{\lambda\Big[4b_qc_p+2c_pd_q\left(y_{j+1}+y_j\right)+2b_qd_p\left(x_{i+1}+x_i\right)+d_pd_q\left(x_{i+1}+x_i\right)\left(y_{j+1}+y_j\right)\Big]$$
$$\mu\Big[4b_pc_q+2b_pd_q\left(x_{i+1}+x_i\right)+2c_qd_p\left(y_{j+1}+y_j\right)+d_pd_q\left(x_{i+1}+x_i\right)\left(y_{j+1}+y_j\right)\Big]\Big\},$$

$$a_{m}^{4}\left(L_p,L_q\right)=\frac{h\tau}{6}\frac{1}{S^2}\Big\{\left(\lambda+2\mu\right)\Big[6c_pc_q+3\left(c_pd_q+c_qd_p\right)\left(x_{i+1}+x_i\right)+2d_pd_q\left(x_{i+1}^2+x_{i+1}x_i+x_i^2\right)\Big]$$
$$\mu\Big[6b_pb_q+3\left(b_pd_q+b_qd_p\right)\left(y_{j+1}+y_j\right)+2d_pd_q\left(y_{j+1}^2+y_{j+1}y_j+y_j^2\right)\Big]\Big\},$$

where *p,q=1,2,3,4.*

Finally, considering (9) for any finite element e_{mn}, the components of the local stiffness matrix A defined as

$$A=\left[A_{ij}\right]_{8\times8}=\begin{bmatrix}\mathcal{A}_{11} & \mathcal{A}_{12}\\ \mathcal{A}_{21} & \mathcal{A}_{22}\end{bmatrix}$$

are obtained as follows:

$$\mathcal{A}_{11}=\begin{bmatrix} \mu\frac{h}{3\tau}+(\lambda+2\mu)\frac{\tau}{3h} & -\mu\frac{h}{3\tau}+(\lambda+2\mu)\frac{\tau}{6h} & \mu\frac{h}{6\tau}-(\lambda+2\mu)\frac{\tau}{3h} & -\mu\frac{h}{6\tau}-(\lambda+2\mu)\frac{\tau}{6h} \\ -\mu\frac{h}{3\tau}+(\lambda+2\mu)\frac{\tau}{6h} & \mu\frac{h}{3\tau}+(\lambda+2\mu)\frac{\tau}{3h} & -\mu\frac{h}{6\tau}-(\lambda+2\mu)\frac{\tau}{6h} & \mu\frac{h}{6\tau}-(\lambda+2\mu)\frac{\tau}{3h} \\ \mu\frac{h}{6\tau}-(\lambda+2\mu)\frac{\tau}{3h} & -\mu\frac{h}{6\tau}-(\lambda+2\mu)\frac{\tau}{6h} & \mu\frac{h}{3\tau}+(\lambda+2\mu)\frac{\tau}{3h} & -\mu\frac{h}{3\tau}+(\lambda+2\mu)\frac{\tau}{6h} \\ -\mu\frac{h}{6\tau}-(\lambda+2\mu)\frac{\tau}{6h} & \mu\frac{h}{6\tau}-(\lambda+2\mu)\frac{\tau}{3h} & -\mu\frac{h}{3\tau}+(\lambda+2\mu)\frac{\tau}{6h} & \mu\frac{h}{3\tau}+(\lambda+2\mu)\frac{\tau}{3h} \end{bmatrix}$$

$$\mathcal{A}_{12}=\left(\mathcal{A}_{21}\right)^{T}=\begin{bmatrix} \frac{\lambda+\mu}{4} & \frac{-\lambda+\mu}{4} & \frac{\lambda-\mu}{4} & -\frac{\lambda+\mu}{4} \\ \frac{\lambda-\mu}{4} & -\frac{\lambda+\mu}{4} & \frac{\lambda+\mu}{4} & \frac{-\lambda+\mu}{4} \\ \frac{-\lambda+\mu}{4} & \frac{\lambda+\mu}{4} & -\frac{\lambda+\mu}{4} & \frac{\lambda-\mu}{4} \\ -\frac{\lambda+\mu}{4} & \frac{\lambda-\mu}{4} & \frac{-\lambda+\mu}{4} & \frac{\lambda+\mu}{4} \end{bmatrix}$$

$$\mathcal{A}_{22}=\begin{bmatrix} \mu\frac{\tau}{3h}+(\lambda+2\mu)\frac{h}{3\tau} & \mu\frac{\tau}{6h}-(\lambda+2\mu)\frac{h}{3\tau} & -\mu\frac{\tau}{3h}+(\lambda+2\mu)\frac{h}{6\tau} & -\mu\frac{\tau}{6h}-(\lambda+2\mu)\frac{h}{6\tau} \\ \mu\frac{\tau}{6h}-(\lambda+2\mu)\frac{h}{3\tau} & \mu\frac{\tau}{3h}+(\lambda+2\mu)\frac{h}{3\tau} & -\mu\frac{\tau}{6h}-(\lambda+2\mu)\frac{h}{6\tau} & -\mu\frac{\tau}{3h}+(\lambda+2\mu)\frac{h}{6\tau} \\ -\mu\frac{\tau}{3h}+(\lambda+2\mu)\frac{h}{6\tau} & -\mu\frac{\tau}{6h}-(\lambda+2\mu)\frac{h}{6\tau} & \mu\frac{\tau}{3h}+(\lambda+2\mu)\frac{h}{3\tau} & \mu\frac{\tau}{6h}-(\lambda+2\mu)\frac{h}{3\tau} \\ -\mu\frac{\tau}{6h}-(\lambda+2\mu)\frac{h}{6\tau} & -\mu\frac{\tau}{3h}+(\lambda+2\mu)\frac{h}{6\tau} & \mu\frac{\tau}{6h}-(\lambda+2\mu)\frac{h}{3\tau} & \mu\frac{\tau}{3h}+(\lambda+2\mu)\frac{h}{3\tau} \end{bmatrix}$$

Obtaining the Finite Difference Equation (FDE) at Any Point Using the Local Stiffness Matrix

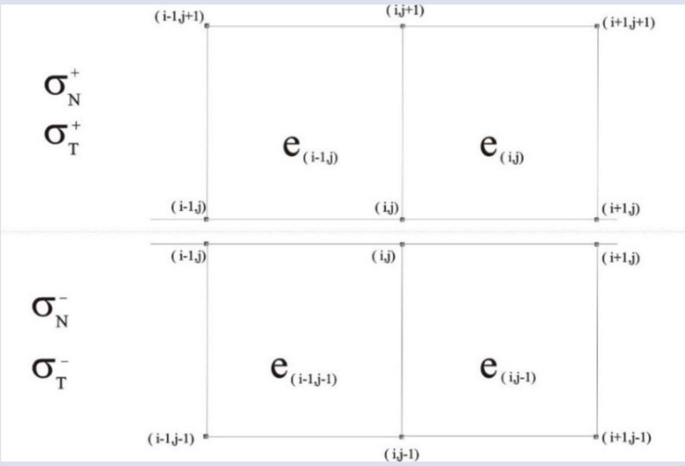


Figure 2. The interlayer finite elements.

In order to obtain the approximating expression of both normal σ_N and tangential σ_T components of the stresses, A_{ij} needs to be multiplied by u_{ij} and v_{ij}, where u_{ij} and v_{ij} are the displacement vectors on the x-axis and y-axis directions, respectively. Then, the results of these multiplications need to be summed up.

Firstly, the normal component of the stresses $\sigma_N=\sigma_{22}=(\lambda+2\mu)\frac{\partial v}{\partial y}+\lambda\frac{\partial u}{\partial x}$ has to be obtained. For the sake of simplicity, the top and bottom parts are considered separately in Figure 2. For this purpose, the FDE for the stress tensors $\sigma_N = \sigma_N^- + \sigma_N^+$ is obtained in the normal direction by processing the relevant lines of the local stiffness matrix A and grouping them according to the displacement vectors u and v. So, to obtain the FDE of σ_N^-, the 6th and 8th lines of the local stiffness matrix A have to be multiplied by unknown vectors belonging to each finite element $e_{i,j-1}$ and $e_{i-1,j-1}$, respectively (Figure 3).

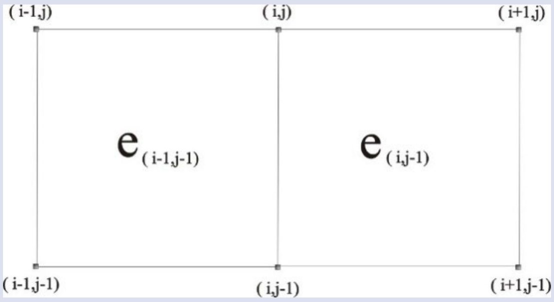


Figure 3. The bottom parts of interlayer finite elements.

Here, the unknown vectors u_{ij}, v_{ij} belonging to the finite elements $e_{i,j-1}$ and $e_{i-1,j-1}$ on the nodal points (x_i, y_j) are as follows:

$$e_{i-1,j-1} \quad \Rightarrow \quad (u_{i-1,j-1}, u_{i-1,j}, u_{i,j-1}, u_{i,j}, v_{i-1,j-1}, v_{i-1,j}, v_{i,j-1}, v_{i,j}),$$

$$e_{i,j-1} \quad \Rightarrow \quad (u_{i,j-1}, u_{i,j}, u_{i+1,j-1}, u_{i+1,j}, v_{i,j-1}, v_{i,j}, v_{i+1,j-1}, v_{i+1,j}).$$

Then, the results of these multiplications have to be summed up, and σ_N^- is obtained as follows:

$$\begin{aligned}
\sigma_N^- &= A_{61}u_{i,j-1} + A_{62}u_{i,j} + A_{63}u_{i+1,j-1} + A_{64}u_{i+1,j} + A_{65}v_{i,j-1} + A_{66}v_{i,j} + A_{67}v_{i+1,j-1} + A_{68}v_{i+1,j} \\
&+ A_{81}u_{i-1,j-1} + A_{82}u_{i-1,j} + A_{83}u_{i,j-1} + A_{84}u_{i,j} + A_{85}v_{i-1,j-1} + A_{86}v_{i-1,j} + A_{87}v_{i,j-1} + A_{88}v_{i,j} \\
&= h\left[(\lambda + 2\mu) v_{\bar{y}}^{(i,j)} + \lambda u_{\mathring{x}}^{(i,j)} - \tau \frac{1}{2}(\lambda + \mu) u_{\mathring{x}\bar{y}}^{(i,j)} - \tau \frac{\mu}{6}\left(v_{\bar{x}x}^{(i,j-1)} + 2v_{\bar{x}x}^{(i,j)} \right) + h^2 \frac{(\lambda + 2\mu)}{6} v_{\bar{y}\bar{x}x}^{(i,j)} \right].
\end{aligned}$$

Similarly, in order to obtain the FDE of σ_N^+, the relevant lines (5th and 7th) of the local stiffness matrix A have to be multiplied by unknown vectors for the finite elements $e_{i,j}$ and $e_{i-1,j}$, respectively (Figure 4).

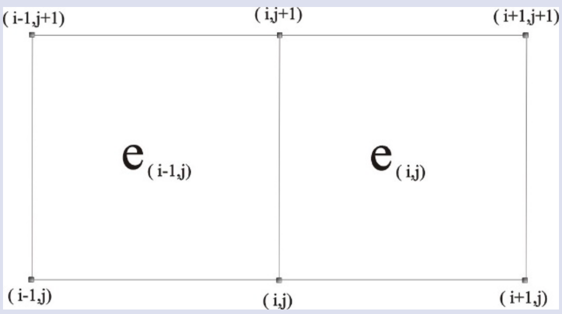


Figure 4. The top parts of interlayer finite elements.

$$e_{i-1,j} \quad \Rightarrow \quad (u_{i-1,j}, u_{i-1,j+1}, u_{i,j}, u_{i,j+1}, v_{i-1,j}, v_{i-1,j+1}, v_{i,j}, v_{i,j+1})$$

$$e_{i,j} \quad \Rightarrow \quad (u_{i,j}, u_{i,j+1}, u_{i+1,j}, u_{i+1,j+1}, v_{i,j}, v_{i,j+1}, v_{i+1,j}, v_{i+1,j+1})$$

Then, the results of these multiplications have to be summed up as above, and σ_N^+ is obtained as follows:

$$\sigma_N^+ = A_{51}u_{i,j} + A_{52}u_{i,j+1} + A_{53}u_{i+1,j} + A_{54}u_{i+1,j+1} + A_{55}v_{i,j} + A_{56}v_{i,j+1} + A_{57}v_{i+1,j} + A_{58}v_{i+1,j+1}$$
$$+ A_{71}u_{i-1,j} + A_{72}u_{i-1,j+1} + A_{73}u_{i,j} + A_{74}u_{i,j+1} + A_{75}v_{i-1,j} + A_{76}v_{i-1,j+1} + A_{77}v_{i,j} + A_{78}v_{i,j+1}$$
$$= -h\left[\left((\lambda+2\mu)v_y^{(i,j)} + \lambda u_{\dot{x}}^{(i,j)}\right) + \frac{1}{2}\tau(\lambda+\mu)u_{\dot{x}y}^{(i,j)} + \tau\frac{\mu}{6}\left(v_{x\bar{x}}^{(i,j+1)} + 2v_{x\bar{x}}^{(i,j)}\right) + h^2\frac{(\lambda+2\mu)}{6}v_{yx\bar{x}}^{(i,j)}\right].$$

Hence, $\sigma_N = \sigma_N^- + \sigma_N^+ = 0$ is satisfied for the normal component of the stresses, and

$$\sigma_N^- + \sigma_N^+ = h\left[(\lambda+2\mu)v_{\bar{y}}^{(i,j)} + \lambda u_{\dot{x}}^{(i,j)} - \tau\frac{1}{2}(\lambda+\mu)u_{\dot{x}\bar{y}}^{(i,j)} - \tau\frac{\mu}{6}\left(v_{x\bar{x}}^{(i,j-1)} + 2v_{x\bar{x}}^{(i,j)}\right) + h^2\frac{(\lambda+2\mu)}{6}v_{\bar{y}x\bar{x}}^{(i,j)}\right]$$
$$-h\left[\left((\lambda+2\mu)v_y^{(i,j)} + \lambda u_{\dot{x}}^{(i,j)}\right) + \frac{1}{2}\tau(\lambda+\mu)u_{\dot{x}y}^{(i,j)} + \tau\frac{\mu}{6}\left(v_{x\bar{x}}^{(i,j+1)} + 2v_{x\bar{x}}^{(i,j)}\right) + h^2\frac{(\lambda+2\mu)}{6}v_{yx\bar{x}}^{(i,j)}\right] = 0.$$

Secondly, the FDE of the tangential component of the stresses $\sigma_T = \sigma_{12} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$ is obtained. σ_T is calculated so that $\sigma_T = \sigma_T^- + \sigma_T^+$, by the same way as σ_N. In order to obtain the FDE of σ_T^-, the 2nd and 4th lines of the local stiffness matrix A have to be multiplied by unknown vectors belonging to each finite element $e_{i,j-1}$ and $e_{i-1,j-1}$, respectively. Then, the results of these multiplications have to be summed up, and σ_T^- is obtained as follows:

$$\sigma_T^- = A_{21}u_{i,j-1} + A_{22}u_{i,j} + A_{23}u_{i+1,j-1} + A_{24}u_{i+1,j} + A_{25}v_{i,j-1} + A_{26}v_{i,j} + A_{27}v_{i+1,j-1} + A_{28}v_{i+1,j}$$
$$+ A_{41}u_{i-1,j-1} + A_{42}u_{i-1,j} + A_{43}u_{i,j-1} + A_{44}u_{i,j} + A_{45}v_{i-1,j-1} + A_{46}v_{i-1,j} + A_{47}v_{i,j-1} + A_{48}v_{i,j}$$
$$= h\left\{\mu\left(u_{\bar{y}}^{(i,j)} + v_{\dot{x}}^{(i,j)}\right) + \frac{1}{6}\left[h^2\mu u_{\bar{y}x\bar{x}}^{(i,j)} - \tau(\lambda+2\mu)\left(u_{x\bar{x}}^{(i,j-1)} + 2u_{x\bar{x}}^{(i,j)}\right)\right] - \frac{1}{2}\tau(\lambda+\mu)v_{\bar{y}\dot{x}}^{(i,j)}\right\}.$$

Likewise, to obtain the FDE of σ_T^+, the relevant lines (1st and 3rd) of the local stiffness matrix A have to be multiplied by unknown vectors for $e_{i,j}$ and $e_{i-1,j}$, respectively, followed by summing up the results of these multiplications. Accordingly, σ_T^+ is obtained as follows:

$$\sigma_T^+ = A_{11}u_{i,j} + A_{12}u_{i,j+1} + A_{13}u_{i+1,j} + A_{14}u_{i+1,j+1} + A_{15}v_{i,j} + A_{16}v_{i,j+1} + A_{17}v_{i+1,j} + A_{18}v_{i+1,j+1}$$
$$+ A_{31}u_{i-1,j} + A_{32}u_{i-1,j+1} + A_{33}u_{i,j} + A_{34}u_{i,j+1} + A_{35}v_{i-1,j} + A_{36}v_{i-1,j+1} + A_{37}v_{i,j} + A_{38}v_{i,j+1}$$
$$= -h\left\{\mu\left(u_y^{(i,j)} + v_{\dot{x}}^{(i,j)}\right) + \frac{1}{6}\left[h^2\mu u_{yx\bar{x}}^{(i,j)} + \tau(\lambda+2\mu)\left(2u_{x\bar{x}}^{(i,j)} + u_{x\bar{x}}^{(i,j+1)}\right)\right] + \frac{1}{2}\tau(\lambda+\mu)v_{y\dot{x}}^{(i,j)}\right\}.$$

Hence, the tangential component of the stresses $\sigma_T = \sigma_T^- + \sigma_T^+ = 0$ is satisfied, and

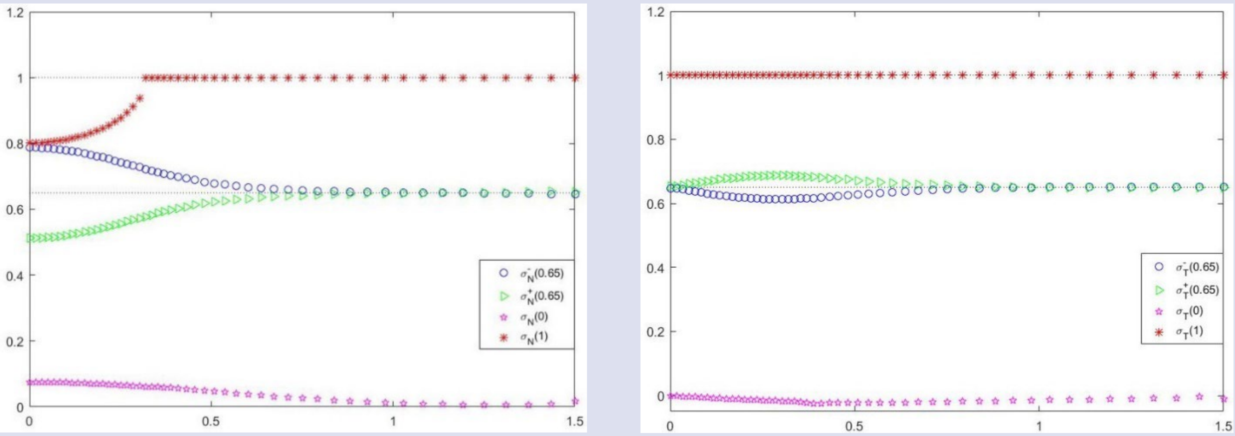
$$\sigma_T^- + \sigma_T^+ = h\left\{\mu\left(u_{\bar{y}}^{(i,j)} + v_{\dot{x}}^{(i,j)}\right) + \frac{1}{6}\left[h^2\mu u_{\bar{y}x\bar{x}}^{(i,j)} - \tau(\lambda+2\mu)\left(u_{x\bar{x}}^{(i,j-1)} + 2u_{x\bar{x}}^{(i,j)}\right)\right] - \frac{1}{2}\tau(\lambda+\mu)v_{\bar{y}\dot{x}}^{(i,j)}\right\} +$$
$$-h\left\{\mu\left(u_y^{(i,j)} + v_{\dot{x}}^{(i,j)}\right) + \frac{1}{6}\left[h^2\mu u_{yx\bar{x}}^{(i,j)} + \tau(\lambda+2\mu)\left(2u_{x\bar{x}}^{(i,j)} + u_{x\bar{x}}^{(i,j+1)}\right)\right] + \frac{1}{2}\tau(\lambda+\mu)v_{y\dot{x}}^{(i,j)}\right\} = 0.$$

Numerical Results

The numerical solution of the problem is obtained using barycentric coordinates for the quadrilateral finite elements. The size mesh $N_x \times N_y$ as $N_x = 50$ and $N_y = 21$ is considered in the rectangular region Ω to obtain the numerical solution. A local adaptive mesh is used to find the contact area Γ_c with less error, and therefore, the mesh steps are considered smaller in the area close to the contact area (number of points in the contact area is $N_{a_c} = 19$).

For the numerical experiments, the modules of elasticity, Poisson's constant and elasticity limit of these materials are $E = 210\,GPa$, $\nu = 0.3$ and $e_0 = 0.027$, respectively. The geometric parameters of the region and the punch are then defined as $R = 0.2\times10^{-2}m$, $l_x = 1.5\times10^{-2}m$, $l_y = 1\times10^{-2}m$ such that $\varphi(x) = \sqrt{R^2 - x^2}$. In the case where the punch deforms the rigid body by $\alpha = 0.5\times10^{-4}m$, the initial value of the contact zone is taken as $\tilde{a}_c = 0.2\times10^{-3}m$, and the contact zone is found as $\tilde{a}_c = 3.1935\times10^{-3}m$ in seven iterations. Since the force acting on the punch to deform the rigid body by as much as $\alpha = 0.5 \times 10^{-4} m$ is defined as $P[\alpha] = \int_{\Gamma_c(\alpha)} \sigma_N(u)dx$, it is found to be $P(\alpha) = 4.953 \times 10^{-2} GPa$. The values of the stress tensors σ_N and σ_T are calculated in each layer of the mesh defined in the region, and it is determined that the equilibrium conditions are satisfied ($P^+(\alpha) = 4.9532 \times 10^{-2} GPa$ and $P^-(\alpha) = 4.9584 \times 10^{-2} GPa$).

The plots of the functions σ_N and σ_T at the top of the deforming body and the thickness $y = 0.65$ are given in Figure 5. (a) and (b), respectively ($N_y^{0.65} = 23$).



(a) (b)

Figure 5. The plots of the functions (a) σ_N and (b) σ_T at the top of the deforming body and *y=0.65*.

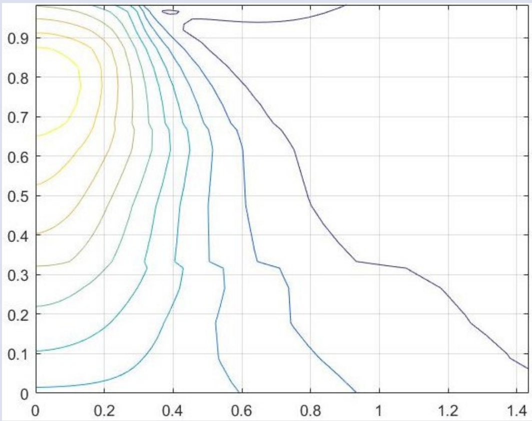


Figure 6. The distribution of plasticity in the rigid body.

Conclusion

Finite difference equations of the mathematical model of the elasto-plastic plane contact problem with undetermined bounded corresponding to the Lame equation system were obtained by using barycentric coordinates. Then, the geometric interpretations of the numerical solution obtained with the help of the prepared computer program were presented.

Conflict of interests

The authors state that did not have conflict of interests.

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