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Singularities of the Ruled Surfaces According to RM Frame and Natural Lift Curves

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ABSTRACT

In this study, the ruled surface generated by the natural lift curve in IR^3 is obtained by using the isomorphism between unit dual sphere, DS^2 and the subset of the tangent bundle of unit 2-sphere, $T\bar{M}$. Then, exploiting E. Study mapping and the isomorphism mentioned below, each natural lift curve on $T\bar{M}$ corresponds the ruled surface in IR^3. Moreover, the singularities of this ruled surface are examined according to RM vectors and these ruled surfaces have been classified. Some examples are given to support the main results.

Keywords: Ruled surface, Dual space, Natural lift curve, RM vectors.

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Introduction

In geometry, there are many orthonormal frames such as Frenet, Bishop, RM frames, etc. for investigating the geometric structures of curves and surfaces, [1-4]. These frames describe the kinematic properties of a particle moving along a continuous, differentiable curve in Euclidean space. Furthermore, they have several advantages and application areas in differential geometry, engineering, surface modeling, etc. For example, due to minimal twist, the rotation minimizing frame (RMF) is extensively used in computer graphics, surface modeling, motion design and control in computer animation and robotics, streamline visualization, and tool path planning in CAD/CAM, [5-7].

The theory of curves is a significiant subject in differential geometry. Considering the relation among Frenet vectors, some special curves such as natural lift curve, some curve couples like Bertrand, Mannheim, Involute-evolute pairs, etc. have attracted many mathematicians in literature. Among these special curves, the well-known curve, which is called the natural lift curve, has firstly been encountered in J. A. Thorpe's book, [8]. In this book, the natural lift curve is defined as a curve obtained by the endpoints of the unit tangent vectors of the given curve. Taking into consideration the definition of the natural lift curve, the fundamental properties of this curve is examined in [9]. Then, the Frenet vector fields of the natural lift curve are calculated in [10]. Moreover, Frenet operators of the natural lift curve are calculated by using the Frenet operators of the main curve in detail.

In addition to the theory of curves, the surface theory is an important subject in differential geometry. There are several special surfaces such as minimal surfaces, ruled surfaces, parallel surfaces, circular surfaces, etc. One of the well-known surfaces is the ruled surface which consists the union of one parametric family of lines. The lines of this family are called the generators of the ruled surface. In literature, the properties of these surfaces have been studied in different spaces, [11-15]. Moreover, by using the theorems and definitions of dual space in [16], the isomorphism among unit dual sphere, the tangent bundle of unit 2-sphere, and non-cylindirical ruled surfaces are investigated in [17]. In the light of this study, a one-to-one correspondence between and is mentioned in [18]. In that study, according to E. Study mapping, to each curve on corresponds a ruled surface in Euclidean 3-space, Furthermore, exploiting this relation, each curve on corresponds a ruled surface in Then, inspired by [18], the isomorphism among , the subset of the tangent bundle of unit 2-sphere, and the ruled surface generated by natural lift curves in are examined in [19]. Furthermore, the developability condition of this ruled surface is given in the same study.

The problem of singularities for surfaces has attrached many mathematicians such as [20-22] in literature. In [21], basic notations and properties of space curves have been described. Also, the classification of singularities of the rectifying developable and the Darboux developable of a space curve are reviewed in detail. In [22], Legendre curves in unit tangent bundle by using rotation minimizing vector fields have been investigated. Then, ruled surfaces corresponding to these curves are specified. Additionally, the singularities of these ruled surfaces have been analyzed and classified.

In literature, there has not been any research about the singularities of the ruled surface generated by the natural lift curve according to RM frame. In order to fill this gap, this study is organized as follows: In Section 2, the properties of natural lift curves and RM vector fields are denoted. In Section 3, the isomorphism among , the subset of the tangent bundle of unit 2-sphere, and the ruled surface generated by natural lift curves in is examined in detail. Furthermore, the singularities of the ruled surfaces generated by the natural lift curves according to RM vectors are mentioned. Some examples are given to illustrate the main theorems. In Section 4, the obtained results are discussed.

Natural Lift Curves and RM Vector Fields

In this section, we recall some basic definitions and theorems about the tangent bundle of unit 2-sphere and properties of the natural lift curve of the given curve. Moreover, the difference between the Frenet frame and RM frame of natural lift curve has been mentioned.
Assume that S^2 is the unit 2-sphere in IR^3. The tangent bundle of S^2 is denoted as

$$TS^2 = \left\{(\gamma, v) \in IR^3 \times IR^3 : |\gamma| = 1, \langle \gamma, v \rangle = 0\right\}, \tag{1}$$

where " $\langle , \rangle$ " is the inner product and " $|,|$ " is the norm in IR^3, respectively, [18]. Also, the unit tangent bundle of S^2 is

$$UTS^2 = \left\{(\gamma, v) \in IR^3 \times IR^3 : |\gamma| = |v| = 1, \langle \gamma, v \rangle = 0\right\}. \tag{2}$$

Let $T\bar{M}$ be a subset of TS^2, defined by

$$T\bar{M} = \left\{(\bar{\gamma}, \bar{v}) \in IR^3 \times IR^3 : |\bar{\gamma}| = 1, \langle \bar{\gamma}, \bar{v} \rangle = 0\right\}, \tag{3}$$

where $\bar{\gamma}$ and $\bar{v}$ represent the derivatives of γ and v, respectively, [19]. Moreover,

$$UT\bar{M} = \left\{(\bar{\gamma}, \bar{v}) \in IR^3 \times IR^3 : |\bar{\gamma}| = |\bar{v}| = 1, \langle \bar{\gamma}, \bar{v} \rangle = 0\right\}. \tag{4}$$

Definition 1 For the curve γ, $\bar{\gamma}$ is called the natural lift of γ on $T\bar{M}$, which produces in the following equation, [19]:

$$\bar{\Gamma}(s) = (\bar{\gamma}(s), \bar{v}(s)) = (\gamma'(s)\big|_{\gamma(s)}, v'(s)\big|_{v(s)}). \tag{5}$$

Accordingly, we can write

$$\frac{d\bar{\Gamma}(s)}{ds} = \frac{d}{ds}(\Gamma'(s)\big|_{\Gamma(s)}) = D_{\Gamma'(s)}\Gamma'(s).$$

Here, D refers the Levi-Civita connection in IR^3. We have

$$T\bar{M} = \bigcup_{p \in \bar{M}} T_p\bar{M},$$

where $T_p\bar{M}$ is the tangent space of $\bar{M}$ at p, [9]. Additionally, the Frenet formulas along the natural lift curve $\bar{\Gamma}(s)$ are

$$\begin{pmatrix} \nabla_{\bar{T}}\bar{T}(s) \\ \nabla_{\bar{N}}\bar{N}(s) \\ \nabla_{\bar{B}}\bar{B}(s) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \bar{\kappa}(s)/|\bar{W}(s)| & 0 & \bar{\tau}(s)/|\bar{W}(s)| \\ \bar{\tau}(s)/|\bar{W}(s)| & 0 & \bar{\kappa}(s)/|\bar{W}(s)| \end{pmatrix} \begin{pmatrix} \bar{T}(s) \\ \bar{N}(s) \\ \bar{B}(s) \end{pmatrix},$$

where $\bar{W}(s)$ is Darboux vector field of the mean curve $\Gamma(s)$ in [10].

Assume that $\bar{\gamma}$ is a smooth curve in (M, g). A normal vector field $\bar{N}$ over $\bar{\gamma}$ is called a rotation minimizing vector field (in short, RM vector field) if it is parallel with respect to the normal connection of $\bar{\gamma}$. That is, $\nabla_{\bar{\gamma}'}\bar{N}$ and $\bar{\gamma}'$ are proportional.

The orthonormal frame which consists of the tangent vectors $\bar{T}$ and two normal vector fields $\bar{N}_1$ and $\bar{N}_2$ is called RM frame along a curve $\bar{\gamma} = \bar{\gamma}(s)$. The Frenet type equations are defined as

$$\begin{pmatrix} \nabla_{\bar{T}}\bar{T}(s) \\ \nabla_{\bar{T}}\bar{N}_1(s) \\ \nabla_{\bar{T}}\bar{N}_2(s) \end{pmatrix} = \begin{pmatrix} 0 & \bar{\kappa}_1(s) & \bar{\kappa}_2(s) \\ -\bar{\kappa}_1(s) & 0 & 0 \\ -\bar{\kappa}_2(s) & 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{T}(s) \\ \bar{N}_1(s) \\ \bar{N}_2(s) \end{pmatrix}.$$

Here, $\bar{\kappa}_1(s)$ and $\bar{\kappa}_2(s)$ are called natural curvatures of RM frame which provide the following equations given below, [22]:

$$\bar{\kappa}(s) = \sqrt{\bar{\kappa}_1^2(s) + \bar{\kappa}_2^2(s)}$$

and

$$\bar{\tau}(s) = \theta'(s) = \frac{\bar{\kappa}_1(s)\bar{\kappa}_2'(s) - \bar{\kappa}_1'(s)\bar{\kappa}_2(s)}{\bar{\kappa}_1^2(s) + \bar{\kappa}_2^2(s)}.$$

Here, $\theta(s) = \arg(\bar{\kappa}_1(s), \bar{\kappa}_2(s)) = \arctan(\frac{\bar{\kappa}_2(s)}{\bar{\kappa}_1(s)})$. θ' denotes the derivative of θ according to arc-length parameter s, [22]. Furthermore, the following theorem is denoted:

Theorem 1 Let $\bar{\gamma} : I \subseteq \mathbb{R} \to S^2$ be a regular smooth curve with the Frenet apparatus $\{\bar{T}, \bar{N}, \bar{B}, \bar{\kappa}, \bar{\tau}\}$. The following assumptions are satisfied:

(i) If $\bar{T}(s)$ and $\bar{N}_1(s)$ are RM vector fields of the curve $\bar{\gamma}$, the curve $\bar{\Gamma}(s) = (\bar{T}(s), \bar{N}_1(s))$ is natural lift curve on $T\bar{M}$

(ii) If $\bar{T}(s)$ and $\bar{N}_2(s)$ are RM vector fields of the curve $\bar{\gamma}$, the curve $\bar{\Gamma}(s) = (\bar{T}(s), \bar{N}_2(s))$ is natural lift curve on $T\bar{M}$

.**Proof.** Using the definition of natural lift curve, the proof can be easily seen.

From the definition of $UT\bar{M}$, we may introduce the new frame satisfying $\bar{\mu}(s) = \bar{\gamma}(s) \times \bar{v}(s)$. It can be simply seen that $\langle \bar{\gamma}(s), \bar{v}(s) \rangle = \langle \bar{\gamma}(s), \bar{\mu}(s) \rangle = 0$. Moreover, the Frenet frame along $\bar{\gamma}(s)$ can be expressed as

$$\begin{pmatrix} \bar{\gamma}'(s) \\ \bar{v}'(s) \\ \bar{\mu}'(s) \end{pmatrix} = \begin{pmatrix} 0 & \bar{l}(s) & \bar{m}(s) \\ -\bar{l}(s) & 0 & \bar{n}(s) \\ -\bar{m}(s) & -\bar{n}(s) & 0 \end{pmatrix} \begin{pmatrix} \bar{\gamma}(s) \\ \bar{v}(s) \\ \bar{\mu}(s) \end{pmatrix},$$

where

$\bar{l}'(s) = \langle \bar{\gamma}'(s), \bar{v}(s) \rangle$, $\bar{m}'(s) = \langle \bar{\gamma}'(s), \bar{\mu}(s) \rangle$ and

$\bar{n}'(s) = \langle \bar{v}'(s), \bar{\mu}(s) \rangle$ are curvature functions of $\bar{\Gamma}(s)$.

Theorem 2 Let $\bar{\Gamma}(s) = (\bar{\gamma}(s), \bar{v}(s))$ be the natural lift curve on $T\bar{M}$. If $\bar{l}(s) = 0$, the vectors $(\bar{\gamma}(s), \bar{v}(s))$ are the RM vector fields of the $\bar{\mu}$-direction curve $\bar{\beta}$ (i.e., $\bar{\beta}(s) = \int \bar{\mu}(s)ds$), the set $\{\bar{\gamma}, \bar{v}, \bar{\mu}\}$ is RM frame.

Proof. Let $\bar{\Gamma}(s) = (\bar{\gamma}(s), \bar{v}(s))$ be the natural lift curve on $T\bar{M}$. If $\bar{l}(s) = 0$, we write

$$\begin{pmatrix} \bar{\mu}'(s) \\ \bar{\gamma}'(s) \\ \bar{v}'(s) \end{pmatrix} = \begin{pmatrix} 0 & -\bar{m}(s) & -\bar{n}(s) \\ \bar{m}(s) & 0 & 0 \\ \bar{n}(s) & 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{\mu}(s) \\ \bar{\gamma}(s) \\ \bar{v}(s) \end{pmatrix}.$$

Hence, we conclude that $\{\bar{\gamma}(s), \bar{v}(s), \bar{\mu}(s)\}$ is RM frame along the natural lift curve $\bar{\Gamma}$. The proof is completed.

Natural Lift Curves and Singularities of the Ruled Surfaces According to RM Frame

In this section, firstly, we mention the isomorphism among the subset of tangent bundle of unit 2-sphere, $T\bar{M}$, unit dual sphere, DS^2 and the ruled surface in IR^3. Secondly, exploiting this isomorphism, we examine the singularities of two ruled surfaces, which are obtained by natural lift curves, according to RM frame. Thirdly, some examples are given to support the main results.

The set of dual numbers is defined as

$$ID = \left\{ X = x + \varepsilon x^* : (x, x^*) \in IR \times IR, \varepsilon^2 = 0 \right\}.$$

The combination of $\vec{x}$ and $\vec{x}^*$ is called dual vectors in IR^3. These vectors are real part and dual part of $\vec{X}$, respectively. If $\vec{x}$ and $\vec{x}^*$ are vectors in IR^3, then $\vec{X} = \vec{x} + \varepsilon \vec{x}^*$ is defined as dual vector. Assume that $\vec{X} = \vec{x} + \varepsilon \vec{x}^*$ and $\vec{Y} = \vec{y} + \varepsilon \vec{y}^*$ are dual vectors. The addition, inner product and vector product are presented as follows:

The addition is

$$\vec{X} + \vec{Y} = (\vec{x} + \vec{y}) + \varepsilon(\vec{x}^* + \vec{y}^*)$$

and their inner product is

$$\langle \vec{X}, \vec{Y} \rangle = \langle \vec{x}, \vec{y} \rangle + \varepsilon(\langle \vec{x}^*, \vec{y} \rangle + \langle \vec{x}, \vec{y}^* \rangle).$$

Also, the vector product is given as

$$\vec{X} \times \vec{Y} = \vec{x} \times \vec{y} + \varepsilon(\vec{x} \times \vec{y}^* + \vec{x}^* \times \vec{y}).$$

The norm of $\vec{X} = \vec{x} + \varepsilon \vec{x}^*$ is defined as

$$|\vec{X}| = \sqrt{\langle \vec{x}, \vec{x} \rangle} + \varepsilon \frac{\langle \vec{x}, \vec{x}^* \rangle}{\sqrt{\langle \vec{x}, \vec{x} \rangle}}. \tag{6}$$

The norm of $\vec{X}$ exists only for $\vec{x} \neq 0$. If the norm of $\vec{X}$ is equal to 1, the dual vector is called unit dual vector. The unit dual sphere which consists of the all unit dual vectors is defined as

$$DS^2 = \left\{ \vec{X} = \vec{x} + \varepsilon \vec{x}^* \in D^3 : |\vec{X}| = 1 \right\}. \tag{7}$$

Here, D^3 is called the D-module which consists of the dual vectors. For detailed information for dual vectors in [16]. The correspondence between the unit dual sphere and the subset of the tangent bundle of unit 2-sphere of the natural lift curve is given via Eqs. (3) and (7):

$$T\bar{M} \to DS^2,$$

$$\bar{\Gamma} = (\bar{q}, \bar{v}) \mapsto \bar{\bar{\Gamma}} = \bar{\bar{q}} + \varepsilon \bar{\bar{v}}.$$

Here $\bar{q}$ and $\bar{v}$ are $\bar{q}'$ and $\bar{v}'$, respectively.

Theorem 3 (E. Study mapping) There exists one-to-one correspondence between the oriented lines in IR^3 and the points of DS^2, [16].

Theorem 4 Let $\overline{\Gamma}(s)=\overline{q}(s)+\varepsilon\overline{v}(s)$ be a natural lift curve on DS^2. In IR^3, the ruled surface obtained by the natural lift curve $\overline{\Gamma}(s)$ can be represented as

$$\overline{\phi}_{(\overline{\beta},\overline{\gamma})}(s,u)=\overline{\beta}(s)+u\overline{\gamma}(s), \quad (8)$$

where

$$\overline{\beta}(s)=\overline{\gamma}(s)\times\overline{v}(s) \quad (9)$$

is the base curve of $\overline{\phi}$. Consequently, the isomorphism among $T\overline{M}$, DS^2 and IR^3 can be given as

$$T\overline{M}\to DS^2\to IR^3,$$

$$\overline{\Gamma}(s)=(\overline{q}(s),\overline{v}(s))\mapsto\overline{\overline{\Gamma}}(s)=\vec{\overline{q}}(s)+\varepsilon\vec{\overline{v}}(s)\mapsto\overline{\phi}_{(\overline{\beta},\overline{\gamma})}(s,u)=\overline{\beta}(s)+u\overline{\gamma}(s).$$

Here $\overline{\phi}_{(\overline{\beta},\overline{\gamma})}(s,u)$ is the ruled surface in $\mathbb{R}^3$ corresponding to the dual curve $\overline{\overline{\Gamma}}(s)=\vec{\overline{q}}(s)+\varepsilon\vec{\overline{v}}(s)\in DS^2$ (or to the natural lift curve $\overline{\Gamma}(s)\in T\overline{M}$) in [19]. Considering the ruled surface given in Eq. (8), the striction curve of ruled surface is expressed as

$$\hat{\beta}(s)=(\overline{\gamma}(s)\times\overline{v}(s))+\frac{\langle\overline{\gamma}(s)\times\overline{v}(s),\overline{\gamma}'(s)\rangle}{\langle\overline{\gamma}'(s),\overline{\gamma}'(s)\rangle}\overline{\gamma}(s).$$

The ruled surface $\overline{\phi}_{(\overline{\beta},\overline{\gamma})}(s,u)=\overline{\beta}(s)+u\overline{\gamma}(s)$, is called developable if $\det(\overline{\beta}(s),\overline{\gamma}(s),\overline{\gamma}'(s))=0$. Exploitting the RM frame for μ- direction curve $\overline{\beta}(s)$, the following six ruled surfaces may be defined as

$$\overline{\phi}_{(\overline{\beta}_{1i},\overline{\gamma}_{1i})}(s,u)=\overline{\beta}_{1i}(s)+u\overline{\gamma}_{1i}(s), i=1,2,...,6. \quad (10)$$

Corollary 5
All ruled surfaces $\overline{\phi}_{(\overline{\beta}_{1i},\overline{\gamma}_{1i})}(s,u)=\overline{\beta}_{1i}(s)+u\overline{\gamma}_{1i}(s)$
generated by natural lift curves are developable for $i=1,2,...,6$.
In Corollary 5, the dral of these ruled surfaces equals to zero. That is,

$$P_{\overline{\phi}}=\frac{\det(\frac{d\overline{\beta}'_{1i}}{ds},\overline{\gamma}_{1i},\frac{d\overline{\gamma}'_{1i}}{ds})}{\left|\frac{d\overline{\phi}}{ds}\right|^2}=0.$$

This verifies that these ruled surfaces are developable.

Let's mention the parametric equations of three type surfaces in IR^3, [21]:

(i) Cuspidal edge; $C\times IR=\{(x_1,x_2):x_1^2=x_2^3\}\times IR$.

(ii) Swallowtail;
$$SW=\{(x_1,x_2,x_3):x_1=3u^4+u^2v, x_2=4u^3+2uv, x_3=v\}.$$

(iii) Cuspidal crosscap;
$$CCR=\{(x_1,x_2,x_3):x_1=u^3, x_2=u^3v^3, x_3=v^2\}.$$

Now, we investigate the local classification of singularities of the ruled surfaces in the following theorem:

Theorem 6 Let $\overline{\Gamma}(s)=(\overline{q}(s),\overline{v}(s))$ be the natural lift curve on $UT\overline{M}$. Then, we have the following assertions with the RM frame $\{\overline{\mu},\overline{\gamma},\overline{v}\}$ for $\overline{\mu}$- direction curve $\overline{\beta}(s)$:

1. The ruled surface $\overline{\phi}_{(\overline{\beta},\overline{\gamma})}(s,u)=\overline{\beta}(s)+u\overline{\gamma}(s)$, where $\overline{\beta}(s)=\overline{\gamma}(s)\times\overline{v}(s)$ is the base curve of $\overline{\phi}$ which is generated by the natural lift curve is locally diffeomorphic to;

(i) Cuspidal edge at $\overline{\phi}_{(\overline{\beta},\overline{\gamma})}(s_0,u_0)$ if and only if $u_0=-\overline{m}(s_0)^{-1}\neq 0$ and $\overline{m}'(s_0)\neq 0$.

(ii) Swallowtail at $\overline{\phi}_{(\overline{\beta},\overline{\gamma})}(s_0,u_0)$ if and only if $u_0=-\overline{m}(s_0)^{-1}\neq 0, \overline{m}'(s_0)=0$ and $(\overline{m}''(s_0))^{-1}=0$.

2. The ruled surface $\overline{\phi}_{(\overline{\beta},\overline{v})}(s,u)=\overline{\beta}(s)+u\overline{v}(s)$, where $\overline{\beta}(s)=\overline{\gamma}(s)\times\overline{v}(s)$ is the base curve of $\overline{\phi}$ which is generated by the natural lift curve is locally diffeomorphic to;

(i) Cuspidal edge at $\overline{\phi}_{(\overline{\beta},\overline{v})}(s_0,u_0)$ if and only if $u_0=-\overline{n}(s_0)^{-1}\neq 0$ and $u_0'(s_0)\neq 0$.

(ii) Swallowtail at $\overline{\phi}_{(\overline{\beta},\overline{v})}(s_0,u_0)$ if and only if $u_0=-\overline{n}(s_0)^{-1}\neq 0, \overline{n}'(s_0)=0$ and $(\overline{n}''(s_0))^{-1}=0$.

3. The ruled surface $\overline{\phi}_{(\overline{\beta},\overline{\gamma})}(s,u)=\overline{\beta}(s)+u\overline{\gamma}(s)$ (resp. $\overline{\phi}_{(\overline{\beta},\overline{v})}(s,u)=\overline{\beta}(s)+u\overline{v}(s)$) which is generated by the natural lift curve that is a cone surface if and only if $\overline{m}$ (resp. $\overline{n}$) is constant.

Proof. Let $\bar{\Gamma}(s) = (\bar{q}(s), \bar{v}(s))$ be the natural lift curve on $UT\bar{M}$. with the RM frame $\{\bar{\mu}, \bar{\gamma}, \bar{v}\}$ for $\bar{\mu}$ direction curve $\bar{\beta}(s)$. Then, we have

$$\frac{\partial \bar{\phi}_{(\bar{\beta},\bar{\gamma})}(s,u)}{\partial s} = (1 + u\bar{m}(s))\bar{\mu},$$

$$\frac{\partial \bar{\phi}_{(\bar{\beta},\bar{\gamma})}(s,u)}{\partial u} = \bar{\gamma}.$$

The vector product of $\frac{\partial \bar{\phi}_{(\bar{\beta},\bar{\gamma})}(s,u)}{\partial s}$ and $\frac{\partial \bar{\phi}_{(\bar{\beta},\bar{\gamma})}(s,u)}{\partial u}$ is calculated as $(1 + u\bar{m}(s))\bar{v}$. Singularities of the normal vector field of $\bar{\phi}_{(\bar{\beta},\bar{\gamma})} = \bar{\phi}_{(\bar{\beta},\bar{\gamma})}(s,u)$ is

$$u = -\frac{1}{\bar{m}(s)}.$$

In [5], if there is a parameter s_0 such that $u_0 = -\frac{1}{\bar{m}(s_0)} \neq 0$ and $u_0' = -\frac{\bar{m}'(s_0)}{\bar{m}^2(s_0)} \neq 0$, $\bar{\phi}_{(\bar{\beta},\bar{\gamma})}(s,u)$ is locally diffeomorphic to $C \times R$ at $\bar{\phi}_{(\bar{\beta},\bar{\gamma})}(s_0, u_0)$. This completes the assertions of 1. (i). Again from [10], if there is a parameter s_0 such that $u_0 = -\frac{1}{\bar{m}(s_0)} \neq 0$, $u_0' = -\frac{\bar{m}'(s_0)}{\bar{m}^2(s_0)} = 0$, and $(\bar{m}(s_0)^{-1})'' = 0$, $\bar{\phi}_{(\bar{\beta},\bar{\gamma})}$ is locally diffeomorphic to SW at $\bar{\phi}_{(\bar{\beta},\bar{\gamma})}(s_0, u_0)$. This completes the assertions of 1. (ii). Likewise, proof of assertion 2 could be done as proof of assertion 1. The proof of assertion 3 is that the singular point equals to the striction curve of $\bar{\phi}$. Then, we write

$$\bar{\phi}_{(\bar{\beta},\bar{\gamma})}(s,u) = \bar{\phi}_{(\bar{\beta},\bar{\gamma})}(s, -\frac{1}{\bar{m}(s)}) = \bar{\beta}(s) - \frac{1}{\bar{m}(s)}\bar{\gamma}(s)$$

(resp. $\bar{\phi}_{(\bar{\beta},\bar{v})}(s,u) = \bar{\phi}_{(\bar{\beta},\bar{v})}(s, -\frac{1}{\bar{m}(s)}) = \bar{\beta}(s) - \frac{1}{\bar{m}(s)}\bar{v}(s)$).

Hence, we get

$$\bar{\phi}'_{(\bar{\beta},\bar{\gamma})}(s,u) = (-\frac{1}{\bar{m}(s)})'\bar{\gamma}(s)$$

(resp. $\bar{\phi}'_{(\bar{\beta},\bar{v})}(s,u) = (-\frac{1}{\bar{m}(s)})'\bar{v}(s)$).

It is concluded that if $\bar{m}(s)$ is constant, $\bar{\phi}'_{(\bar{\beta},\bar{\gamma})}(s,u) = \bar{\phi}'_{(\bar{\beta},\bar{v})}(s,u) = 0$. Finally, we say that $\bar{\phi}_{(\bar{\beta},\bar{\gamma})}(s,u)$ (or $\bar{\phi}_{(\bar{\beta},\bar{v})}$) has only one singularity point. It is a cone surface. Hence, the proof is completed.

Example 7. Let us consider

$\bar{\gamma}(s) = (\frac{-1}{\sqrt{5}}\sin(\frac{s}{\sqrt{5}}), \frac{1}{\sqrt{5}}\cos(\frac{s}{\sqrt{5}}), \frac{2}{\sqrt{5}})$ and

$\bar{v}(s) = (\frac{2}{\sqrt{5}}\sin(\frac{s}{\sqrt{5}}), \frac{-2}{\sqrt{5}}\cos(\frac{s}{\sqrt{5}}), \frac{1}{\sqrt{5}})$. Since $|\bar{\gamma}(s)| = |\bar{v}(s)| = 1$ and $\langle \bar{\gamma}(s), \bar{v}(s) \rangle = 0$, the natural lift curve $\bar{\Gamma}(s) = (\bar{\gamma}(s), \bar{v}(s)) \in UT\bar{M}$. The ruled surface generated by $\bar{\Gamma}(s)$ is

$$\bar{\phi}_{(\bar{v},\bar{\beta})}(s,u) = \frac{1}{\sqrt{5}}((2-u)\sin(\frac{s}{\sqrt{5}}), (u-2)\cos(\frac{s}{\sqrt{5}}), (1+2u)),$$

where the base curve is

$$\bar{\beta}(s) = (\frac{2}{\sqrt{5}}\sin(\frac{s}{\sqrt{5}}), \frac{-2}{\sqrt{5}}\cos(\frac{s}{\sqrt{5}}), \frac{1}{\sqrt{5}}).$$

The normal vector of $\bar{\phi}_{(\bar{\beta},\bar{v})}$ is

$$(\frac{1}{5\sqrt{5}}(u-2)\sin(\frac{s}{\sqrt{5}}), \frac{-2}{5\sqrt{5}}(2-u)\cos(\frac{s}{\sqrt{5}}), \frac{1}{5\sqrt{5}}(2-u)).$$

Hence, the ruled surface $\bar{\phi}_{(\bar{\beta},\bar{v})}$ is a cone surface. Additionally, $u = 2$ is a singular point.

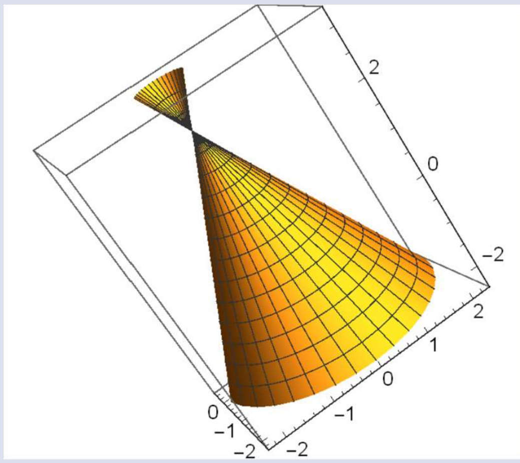

Figure 1. The ruled surface is the cone surface with one singularity point

Theorem 8. Let $\bar{\Gamma}(s) = (\bar{q}(s), \bar{v}(s))$ be the natural lift curve on $UT\bar{M}$. Then, we have the following assertions with the RM frame $\{\bar{\mu}, \bar{\gamma}, \bar{v}\}$ for $\bar{\mu}$- direction curve $\bar{\beta}(s)$:

1. The ruled surface $\bar{\phi}_{(\bar{\gamma},\bar{\beta})}(s,u)=\bar{\gamma}(s)+u\bar{\beta}(s)$ which is generated by the natural lift curve is locally diffeomorphic to;

(i) Cuspidal edge at $\bar{\phi}_{(\bar{\gamma},\bar{\beta})}(s_0,u_0)$ if and only if $u_0=-\bar{m}(s_0)\neq 0$ and $\bar{m}'(s_0)\neq 0$.

(ii) Swallowtail at $\bar{\phi}_{(\bar{\gamma},\bar{\beta})}(s_0,u_0)$ if and only if $u_0=-\bar{m}(s_0)=0, \bar{m}'(s_0)\neq 0$.

(iv) Cuspidial crosscap at $\bar{\phi}_{(\bar{\gamma},\bar{\beta})}(s_0,u_0)$ if and only if $u_0=-\bar{m}(s_0)\neq 0, \bar{m}'(s_0)=0$ and $\bar{m}''(s_0)\neq 0$.

2. The ruled surface $\bar{\phi}_{(\bar{\nu},\bar{\beta})}(s,u)=\bar{\nu}(s)+u\bar{\beta}(s)$ which is generated by the natural lift curve is locally diffeomorphic to;

(i) Cuspidal edge at $\bar{\phi}_{(\bar{\nu},\bar{\beta})}(s_0,u_0)$ if and only if $u_0=-\bar{n}(s_0)\neq 0$ and $\bar{n}'(s_0)\neq 0$.

(ii) Swallowtail at $\bar{\phi}_{(\bar{\nu},\bar{\beta})}(s_0,u_0)$ if and only if $u_0=-\bar{n}(s_0)=0, \bar{n}'(s_0)=0$ and $\bar{n}''(s_0)\neq 0$.

(iii) Cuspidial crosscap at $\bar{\phi}_{(\bar{\nu},\bar{\beta})}(s_0,u_0)$ if and only if $u_0=-\bar{n}(s_0)=0, \bar{n}'(s_0)\neq 0$

3.The ruled surface $\bar{\phi}_{(\bar{\gamma},\bar{\beta})}(s,u)=\bar{\gamma}(s)+u\bar{\beta}(s)$ (resp. $\bar{\phi}_{(\bar{\nu},\bar{\beta})}(s,u)=\bar{\nu}(s)+u\bar{\beta}(s)$) which is generated by the natural lift curve that is a cone surface if and only if $\bar{m}$ (resp. $\bar{n}$) is constant.

Proof. Similarly, this theorem can be simply proved by using the method of the proof in Theorem 6.

Example 9.

Let us consider $\bar{\gamma}(s)=(\frac{-1}{\sqrt{2}}\sin(\frac{s}{\sqrt{2}}),\frac{1}{\sqrt{2}}\cos(\frac{s}{\sqrt{2}}),\frac{1}{\sqrt{2}})$ and $\bar{\nu}(s)=(\frac{1}{\sqrt{2}}\sin(\frac{s}{\sqrt{2}}),\frac{1}{\sqrt{2}}\cos(\frac{s}{\sqrt{2}}),\frac{1}{\sqrt{2}})$. Since $\left|\bar{\gamma}(s)\right|=\left|\bar{\nu}(s)\right|=1$ and $\left\langle\bar{\gamma}(s),\bar{\nu}(s)\right\rangle=0$, the natural lift curve $\bar{\Gamma}(s)=(\bar{\gamma}(s),\bar{\nu}(s))\in UT\bar{M}$. The ruled surface generated by $\bar{\Gamma}(s)$ is

$$\bar{\phi}_{(\bar{\beta},\bar{\nu})}(s,u)=\frac{1}{\sqrt{2}}((1-u)\sin(\frac{s}{\sqrt{2}}),(u+1)\cos(\frac{s}{\sqrt{2}}),(1+u)),$$

where the base curve is

$$\bar{\beta}(s)=(\frac{1}{\sqrt{2}}\sin(\frac{s}{\sqrt{2}}),\frac{1}{\sqrt{2}}\cos(\frac{s}{\sqrt{2}}),\frac{1}{\sqrt{2}}).$$

The normal vector of $\bar{\phi}_{(\bar{\beta},\bar{\nu})}$ is

$$(\frac{1}{2\sqrt{2}}(u+1)\sin(\frac{s}{\sqrt{2}}),\frac{-1}{2\sqrt{2}}(1-u)\cos(\frac{s}{\sqrt{2}}),\frac{1}{2\sqrt{2}}\cos(\frac{2s}{\sqrt{2}})).$$

Hence, the ruled surface $\bar{\phi}_{(\bar{\beta},\bar{\nu})}$ is a cone surface. Additionally, $u=1$ is a singular point.

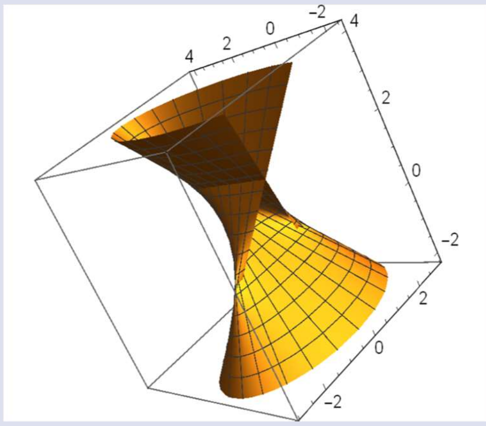

Figure 2. The ruled surface is the cone surface with one singularity point

Theorem 10. Let $\bar{\Gamma}(s)=(\bar{q}(s),\bar{\nu}(s))$ be the natural lift curve on $UT\bar{M}$. Then, we have the following assertions with the RM frame $\{\bar{\mu},\bar{\gamma},\bar{\nu}\}$ for the curvature functions $\bar{m}$ and $\bar{n}$:

1. The ruled surface $\bar{\phi}_{(\bar{\gamma},\bar{\nu})}(s,u)=\bar{\gamma}(s)+u\bar{\nu}(s)$ which is generated by the natural lift curve is locally diffeomorphic to;

(i) Cuspidal edge at $\bar{\phi}_{(\bar{\gamma},\bar{\nu})}(s_0,u_0)$ if and only if $u_0=-\frac{\bar{m}}{\bar{n}}(s_0)=0$ and $(\frac{\bar{m}}{\bar{n}})'(s_0)\neq 0$.

(ii) Swallowtail at $\bar{\phi}_{(\bar{\gamma},\bar{\nu})}(s_0,u_0)$ if and only if $u_0=-\frac{\bar{m}}{\bar{n}}(s_0)\neq 0$, $(\frac{\bar{m}}{\bar{n}})'(s_0)=0$ and $(\frac{\bar{m}}{\bar{n}})'(s_0)\neq 0$.

(iii) Cuspidial crosscap at $\bar{\phi}_{(\bar{\gamma},\bar{\nu})}(s_0,u_0)$ if and only if $u_0=\frac{\bar{m}}{\bar{n}}(s_0)=0$ and $(\frac{\bar{m}}{\bar{n}})'(s_0)\neq 0$.

2. The ruled surface $\bar{\phi}_{(\bar{\nu},\bar{\gamma})}(s,u)=\bar{\nu}(s)+u\bar{\gamma}(s)$ which is generated by the natural lift curve is locally diffeomorphic to;

(i) Cuspidal edge at $\bar{\phi}_{(\bar{v},\bar{\gamma})}(s_0,u_0)$ if and only if $u_0 = -\frac{\bar{n}}{\bar{m}}(s_0) \neq 0$ and $(\frac{\bar{n}}{\bar{m}})'(s_0) \neq 0$.

(ii) Swallowtail at $\bar{\phi}_{(\bar{v},\bar{\gamma})}(s_0,u_0)$ if and only if $u_0 = -\frac{\bar{n}}{\bar{m}}(s_0) \neq 0, (\frac{\bar{n}}{\bar{m}})'(s_0) = 0$ and $(\frac{\bar{n}}{\bar{m}})''(s_0) \neq 0$.

(iii) Cuspidial crosscap at $\bar{\phi}_{(\bar{v},\bar{\gamma})}(s_0,u_0)$ if and only if $u_0 = -\bar{n}(s_0) = 0, \bar{n}'(s_0) \neq 0$.

3.The ruled surface $\bar{\phi}_{(\bar{\gamma},\bar{v})}(s,u) = \bar{\gamma}(s) + u\bar{v}(s)$ (resp. $\bar{\phi}_{(\bar{v},\bar{\gamma})}(s,u) = \bar{v}(s) + u\bar{\gamma}(s)$) which is generated by the natural lift curve that is a cone surface if and only if $\frac{\bar{n}}{\bar{m}}(s)$ (resp. $\frac{\bar{m}}{\bar{n}}(s)$) is constant.

Proof. Similarly, this theorem can be simply proved by using the method of the proof in Theorem 6.

Example 11. Let us consider $\alpha : [0,B] \to R^3 (0 < B < 2\pi)$ as smooth curve defined by

$$\bar{\gamma}(s) = (\frac{\sqrt{3}}{2}\cos 2s, \frac{\sqrt{3}}{2}\sin 2s, \frac{-1}{2}),$$
$$\bar{v}(s) = (-\sqrt{3}\sin 2s, \sqrt{3}\cos 2s, 0),$$
$$\bar{\mu}(s) = (\frac{\sqrt{3}}{2}\cos 2s, \frac{-\sqrt{3}}{2}\sin 2s, \frac{3}{2}).$$

Since $|\bar{\gamma}(s)| = |\bar{v}(s)| = 1$ and $\langle \bar{\gamma}(s), \bar{v}(s) \rangle = 0$, the natural lift curve $\bar{\Gamma}(s) = (\bar{\gamma}(s), \bar{v}(s)) \in UT\bar{M}$. Moreover, $\bar{\Gamma}(s) = (\bar{\gamma}(s), \bar{v}(s))$ is the natural lift curve with the curvature $\bar{m}(s) = \frac{-3}{2}\sin 4s$.

Hence, we have the following assertions:

1. If $B = \frac{\pi}{8}$, we write $\bar{m}(\frac{\pi}{8}) = \frac{-3}{2} \neq 0, \bar{m}'(\frac{\pi}{8}) = 0$ and $\bar{m}''(\frac{\pi}{8}) = \frac{3}{2} \neq 0$. Thus, the ruled surface is

$$\bar{\phi}_{(\bar{\beta},\bar{\gamma})}(s,u) = \bar{\beta}(s) + u\bar{\gamma}(s),$$

$$\bar{\phi}_{(\bar{\beta},\bar{\gamma})}(s,u) = (\frac{\sqrt{3}}{4}\sin 2s, \frac{\sqrt{3}}{4}\cos 2s, \frac{-s}{2}) + u(\frac{\sqrt{3}}{2}\cos 2s, \frac{\sqrt{3}}{2}\sin 2s, \frac{-1}{2}).$$

Furthermore, this ruled surface is locally diffeomorphic to the cuspidal edge at $\bar{\phi}_{(\bar{\beta},\bar{\gamma})}(\frac{\pi}{8},\frac{2}{3})$.

2. If $B = \pi$, we write $u_0 = \bar{m}^{-1}(s_0) \neq 0$ and $(\bar{m}^{-1})'(s_0) \neq 0$. The ruled surface is given above is locally diffeomorphic to swallowtail at $\bar{\phi}_{(\bar{\beta},\bar{\gamma})}(\frac{\pi}{8},u_0)$.

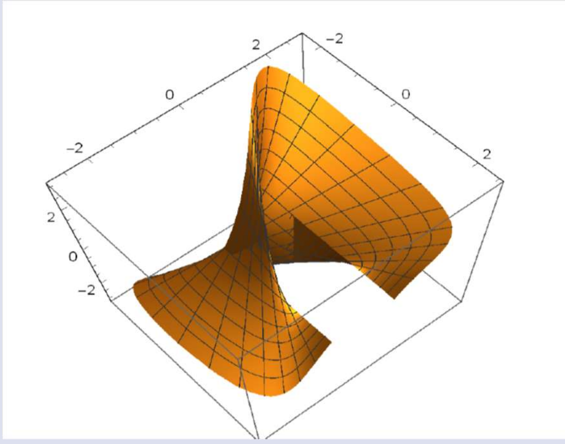

Figure 3. The cuspidal edge at $\bar{\emptyset}_{\bar{\beta},\bar{\gamma}}(\frac{\pi}{8},\frac{8}{3})$

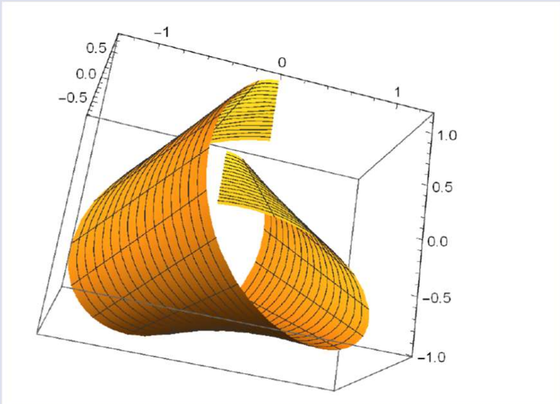

Figure 4. The cuspidal edge at $\bar{\emptyset}_{\bar{\beta},\bar{\gamma}}(\frac{\pi}{8},\mu_0)$

Conclusion

In this paper, we have dealt with the singularities of the ruled surface generated by the natural lift curve. Moreover, we classify the singularities and examine the conditions of being locally diffeomorphic to cuspidal edge, swallowtail or cuspidal crosscap. Additionally, some theorems are verified by giving examples. This study opens new horizons to mathematicians who study the singularities of special curves.

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Conflicts of interest

The authors state that did not have conflict of interests.

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