Cumhuriyet Sci. J., 43(2) (2022) 299-307
DOI: https://doi.org/10.17776/csj.1054869

Cumhuriyet Science Journal

| csj.cumhuriyet.edu.tr | Founded: 2002 ISSN: 2587-2680 e-ISSN: 2587-246X Publisher: Sivas Cumhuriyet University

Kinematic Analysis in 3-Dimensional Generalized Space

Ümit Ziya Savcı [1,a,*]

[1] *Department of Mathematics and Science Education, Faculty of Education, Kütahya Dumlupınar University, Kütahya, Türkiye.*
**Corresponding author*

Research Article

History
Received: 07/01/2022
Accepted: 22/04/2022

ABSTRACT

In this paper, we have first obtained the derivatives of spherical and spatial motions by using the special matrix group in generalized space $E^3(\alpha,\beta)$. The rotation matrices and tangent operators were found by using derivatives of one- and multi-parameters motions in $E^3(\alpha,\beta)$. Also, we obtained the angular velocity matrix of the moving body and its linear velocity vector. Finally, we gave some examples including applications of tangent operators and rotation matrices in support of our results.

Keywords: Generalized space, Rigid motion, Kinematics, Tangent operators, Matrix group.

[a] ziyasavci@hotmail.com https://orcid.org/0000-0003-2772-9283

Introduction

Kinematics is a Greek word that means 'motion', and it is one of the branches of mechanics that deals with the analysis of the motion of particles and rigid bodies. The rigid body is a set of the points that the distance between two of the points never varies after motion [1].

In order to represent a rigid motion in Euclidean or Lorentzian space equipped with multiple coordinate frames, it is required to determine the concept of a rotation matrix and a translation axis. These concepts are used to construct homogeneous transformation matrices that are used to represent the position and orientation of a coordinate frame relative to the other. These transformations allow us to navigate from one to another coordinate frame [2-6].

Recent studies on robot kinematics are dealing with the establishment of different coordinate systems to represent the positions and orientations of rigid bodies. Also, Robot kinematics is concerned with the transformations between these coordinate systems [7-10].

To obtain frame M from frame F, it is needed to first apply a rotation determined by R and then a translation (with respect to F) given as t. This transformation called coordinates transformation is denoted as $T:F\rightarrow M$, and it is determined as $x'=R\cdot x+t$. In this notation, R is an $n\times n$ orthogonal matrix called a rotation matrix, and t is an n-dimensional vector called a translation. This transformation is denoted by T=(R,t) and defined as a matrix-vector pair [11,12].

The derivative of a motion represents the velocity of a point from the fixed frame F to the moving frame M. Linear velocity is the instantaneous rate of change in the linear position of a point relative to some frame. The angular velocity is ω, which describes the rotational motion of M with respect to F. The relationship between the angular velocity vector ω and time-varying rotation matrix R(t) is defined by $[\Omega]=[\dot{R}R^T]$ [6,11,12].

The rotation matrix, which is used to represent relative orientations between coordinate frames, is an orthogonal matrix in Euclidean or Lorentzian space. In Euclidean and Lorentzian spaces, if A is an orthogonal matrix, detA=1 denotes rotation and detA=-1 denotes reflection [11-14].

The generalized quaternions $H(\alpha,\beta)$ are four-dimensional algebra that is associative but not commutative. This algebra is a pair of sub-algebras of Clifford algebra of three-dimensional generalized space $E^3(\alpha,\beta)$, where $E^3(\alpha,\beta)$ is a real vector space R^3 equipped with the metric $\langle u,v\rangle_G=\alpha u_1 v_1+\beta u_2 v_2+\alpha\beta u_3 v_3$, $\alpha,\beta\in R$. For 3-dimensional non-degenerate vector space, $E^3(\alpha,\beta)$ with an orthonormal basis $\{e_1,e_2,e_3\}$, the Clifford algebra $Cl(E^3(\alpha,\beta))=Cl_{(p,q)}$, $p+q=2$ has the basis $\{1,e_1=i,e_2=j,e_1 e_2=k\}$, where $e_1^2=-\alpha$, $e_2^2=-\beta$ and $e_1 e_2=-e_2 e_1$. General information about generalized space and their algebraic properties can be found from [15-21].

Beggs (1965) gave a derivation for a screw matrix by using two different coordinate systems [3]. By defining the pitch for a finite screw as the ratio of one-half the translation to the tangent of one-half the rotation, Parkin has shown that the finite screw cylindroid can be represented by the linear combinations of two bases screws in 1992 [22]. In 1994, Huang and Roth showed the finite displacement of a rigid body can be represented completely by six independent parameters [23]. Knossow, Ronfard, and Horaud showed that the tangent operator can be used to explain the human body kinematic chain and robotics motion in 2008 [24]. In 2017 Durmaz, Aktaş and Gundogan computed the derivative and the tangent operator of motion in Lorentzian space [25]. In 2021 Ata and Savci obtained the generalized Cayley formula, Rodrigues equation, and Euler parameters of a generalized orthogonal matrix in 3-dimensional generalized space E^3 (α,β)[16].

Due to the definition of the generalized space E^3 (α,β), it is Euclidean space E^3 (1,1) if α=β=1, and it is semi-Euclidean space E^3 (1,-1) if α=1 and β=-1. Therefore, this space E^3 (α,β) allows us to define more general algebraic structures and study them. In this study, for all situations of α and β except zero, derivatives of spherical and spatial motion and tangent operators have been obtained for one- and multi-parameter motions in generalized space. We get ordinary differential equations using these derivatives. The rotation matrix is obtained from the solution of these equations. In addition, Lie product of tangent operators and some properties provided by Lie product is defined.

Preliminaries

In this section, we provide some fundamental properties of the generalized space, the transformation and the rotation.

Definition 1: Let $u=(u_1,u_2,u_3)$, $v=(v_1,v_2,v_3)$ be two vectors in $\mathbb{R}^3$ and $\alpha,\beta\in\mathbb{R}$. Then the generalized metric tensor product is defined by

$$<u,v>_G=\alpha u_1v_1+\beta u_2v_2+\alpha\beta u_3v_3.$$

This can be written as $<u,v>_G=u^T\begin{bmatrix}\alpha & 0 & 0\\ 0 & \beta & 0\\ 0 & 0 & \alpha\beta\end{bmatrix}v.$

The vector space $\mathbb{R}^3$ equipped with the generalized scalar product, is called as $3-$dimensional generalized space and is denoted by $E^3(\alpha,\beta)=(R^3,<,>_G)$. The generalized cross product in $E^3(\alpha,\beta)$ is defined by

$$u\wedge_G v=\beta(u_2v_3-u_3v_2)i-\alpha(u_1v_3-u_3v_2)j+(u_1v_2-u_2v_1)k,$$

where $i\wedge_G j=k$, $j\wedge_G k=\beta i$, and $k\wedge_G i=-\alpha j$ [18].

If $<u,v>_G$ is a generalized semi-Euclidean inner product, then $E^3(\alpha,\beta)$ is a $3-$dimensional generalized semi-Euclidean space E_2^3. If $<u,v>_G$ is an Euclidean inner product, then $E^3(\alpha,\beta)$ space is known as E^3 Euclidean space.

Definition 2: Let $E^3(\alpha,\beta)$ be a generalized semi-Euclidean space with a generalized semi-Euclidean inner product. A vector $w\in E^3(\alpha,\beta)$ is called
generalized spacelike vector, if $<v,v>_G$>0 or $v=0$,
generalized timelike vector, if $<v,v>_G$ <0,
generalized null vector, if $<v,v>_G$=0 and $v\neq 0$.

$\|v\|=\sqrt{|\alpha v_1+\beta v_2+\alpha\beta v_3|}$ represents the norm of a vector $v\in E^3(\alpha,\beta)$ [13,14,18].

Definition 3: The set of the 3×3 invertible matrices, denoted $GL(\alpha,\beta)(3)$, is an algebraic group under the operation of matrix multiplication in generalized space $E^3(\alpha,\beta)$ [17].

Definition 4: A matrix $C=\begin{bmatrix}0 & \beta s_3 & \beta s_2\\ \alpha s_3 & 0 & -\alpha s_1\\ -s_2 & s_1 & 0\end{bmatrix}$called a generalized skew-symmetric matrix if $C^T\varepsilon=-\varepsilon C$, where

$\varepsilon=\begin{bmatrix}\alpha & 0 & 0\\ 0 & \beta & 0\\ 0 & 0 & \alpha\beta\end{bmatrix}$ and $\alpha,\beta\in\mathbb{R}-\{0\}$ [18].

Definition 5: A matrix R is called a generalized orthogonal matrix if $R^T\varepsilon R=|R|\varepsilon$ where
$\varepsilon=\begin{bmatrix}\alpha & 0 & 0\\ 0 & \beta & 0\\ 0 & 0 & \alpha\beta\end{bmatrix}$ and $\alpha,\beta\in\mathbb{R}-\{0\}$.
The set of all generalized orthogonal matrices with the operation of matrix multiplication is called the rotation group in $E^3(\alpha,\beta)$ [14].

A rotation about the origin can be given with the equation of $x'=R\cdot_G x$, where R is 3×3 G-orthogonal matrix and $x\in E^3(\alpha,\beta)$. Generalized Cayley formula is defined as $R=(I-C)^{-1}\cdot(I+C)=(I+C)\cdot(I-C)^{-1}$, where C is a G-skew symmetric matrix. By using G-Cayley formula, any G-orthogonal matrix can be obtained by a G-skew symmetric matrix C, where

$$C=\begin{bmatrix}0 & -\beta c_3 & \beta c_2\\ \alpha c_3 & 0 & -\alpha c_1\\ -c_2 & c_1 & 0\end{bmatrix}$$

the matrix C obtained the vector $c=(c_1,c_2,c_3)$ and satisfying the equation $C\cdot_G y=c\wedge_G y$ [16].

The rotations in the three-dimensional space are represented by 3×3 rotation matrices, i.e. by means of 9 parameters. Since constrained by the orthogonality conditions $R^T\varepsilon R=|R|\varepsilon$ these parameters are not independent. Only three independent parameters are needed to obtain a minimal representation of rotations in space.

If frame M is obtained from frame F by first applying a rotation specified by R followed by a translation given (with respect to F) by t, then the coordinates are given by

$$x'=R\cdot_G x+t \qquad (1)$$

Since the displacement is not a linear transformation it not be represented by 3×3 matrix transformation.

Definition 6: A transformation of the form given in eq. (1) is called a rigid motion if R is generalized orthogonal matrix.

Since the set of displacements of an 3-dimensional generalized space $E^3(\alpha,\beta)$ is an algebraic group. If $T_1\colon F\to M_1$ and $T_2\colon M_1\to M_2$ are displacements, then $T=T_1T_2\colon F\to M_2$ is also a displacement [26].

A combination of those two displacements with the matrices identity;

$$\begin{bmatrix}R_1 & t_1\\ 0 & 1\end{bmatrix}\cdot\begin{bmatrix}R_2 & t_2\\ 0 & 1\end{bmatrix}=\begin{bmatrix}R_1\cdot R_2 & R_1\cdot t_2+t_1\\ 0 & 1\end{bmatrix}$$

where 0 denotes the row vector $(0,0,0)$, shows that the rigid motions can be represented by the set of matrices of the form;

$$G-H(4)=\begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix};\ R\in \mathrm{SO}(\alpha,\beta)(3) \tag{2}$$

Transformation matrices of the form eq. (2) are called homogeneous transformation in $E^3(\alpha,\beta)$ space and denote by $[g]=[R,t]$. The displacement is not a linear transformation, but the homogeneous transformation $[g]=[R,t]$ is a linear transformation. The homogeneous representation eq. (2) is a special case of homogeneous coordinates, which have been extensively used in the field of computer graphics. The most general homogeneous transformation takes the form;

$$G-H(4)=\begin{bmatrix} R_{3\times 3} \mid t_{3\times 1} \\ f_{1\times 3} \mid s_{1\times 1} \end{bmatrix}=\begin{bmatrix} \text{Rotation} \quad \mid \text{Translation} \\ \text{perspective} \mid \text{scale factor} \end{bmatrix}$$

From the definition of the metric tensor, all possible selections of α and β can be covered by two conditions given above. From now on we take these two cases into consideration.

The Derivative of a Motion and G-Tangent Operators in Generalized Space

We will use G-tangent operator instead of tangent operator in generalized space for appropriate notation. The continuous motion of a rigid body is the parametrized set of linear transformations, $[g(s)]:\mathbb{R}\to GL(\alpha,\beta)(3)$. In particular, spherical motion define by $[g(s)]:\mathbb{R}\to SO(\alpha,\beta)(3)$ and spatial motion define by $[g(s)]:\mathbb{R}\to G-H(4)$ in generalized space $E^3(\alpha,\beta)$.

Generally, since elements s_{ij} of $[g(s)]:\mathbb{R}\to GL(\alpha,\beta)(3)$ are continuous functions of a real parameter, derivative of this matrix function is the matrix of derivatives of its elements and defined by $[\dot{g}(s)]$. The tangent direction of the motion at $[g(s_0)]$ defined by $[\dot{g}(s_0)]$.

The matrix function $[g(s)]:\mathbb{R}\to GL(\alpha,\beta)(3)$ generates a continuous set of points

$$B(s)=[g(s)]\cdot b$$

is called the trajectory of b. The direction of the tangent to the trajectory $B(s)$ at $s=s_0$ is the derivative

$$\dot{B}(s_0)=[\dot{g}(s_0)]\cdot b=[\dot{g}\cdot g^{-1}(s_0)]\cdot B(s_0)$$

From the equation above we can see that $[\dot{g}\cdot g^{-1}]$ calculates the derivative of $\dot{B}(s)$ by using the the trajectory $B(s)$. Also, from the following equation, we can see that $[\dot{g}\cdot g^{-1}]$ also computes the derivative of $[g(s)]$.

$$[\dot{g}(s)]=[\dot{g}\cdot g^{-1}]\cdot[g(s)].$$

Definition 7: $[\dot{g}\cdot g^{-1}]$ matrix is called G-tangent operator on $GL(\alpha,\beta)(3)$. We now determine the motion $[A(s)]$ that has a constant matrix $[w]$ as its tangent operator. As the matrix $[w]$ calculates the derivative $[\dot{A}(s)]$ at every point $[A(s)]$ we obtain the matrix differential equation

$$[\dot{A}(s)]=[w]\cdot[A(s)]$$

If the initial condition is $[A(0)]=[A_0]$, then it has the solution

$$[A(s)]=[A_0]\cdot e^{s[w]}=[A_0](I+s[w]+\frac{(s[w])^2}{2!}+\frac{(s[w])^3}{3!}+\cdots).$$

The last equation has the initial condition $[A(0)]=[I]$, becomes simplified as

$$[R(t)]=e^{s[w]}.$$

Notice that in this case, the tangent operator $[w]$ is the derivative of $[A(s)]$ at $[A(0)]=[I]$. Thus, the set of tangent operators on $GL(\alpha,\beta)(3)$ is identical to the tangent directions at the identity $[I]$.

If the set of tangent directions at the identity $[A(0)]=[I]$ is $[\dot{A}(0)]$ then,

$$[\dot{A}(0)]=[w]\cdot e^{s[w]}=[w]\cdot[A(s)]$$
$$[\dot{A}(0)]=[w]\cdot[A(0)]=[w].$$

Lie product is defined for the tangent operators $[g]$ and $[w]$ as $[g]\wedge[w]$ is also tangent operator

$$[g]\wedge[w]=[g\cdot w-w\cdot g] \tag{3}$$

where $g\cdot w$ denotes the matrix product in generalized space.

The Tangent Operators of SO(α,β)(3)

The condition defining the tangent operators of $SO(\alpha,\beta)(3)$ is obtained from the relation $[R^T\varepsilon R]=[\varepsilon]$ which must be satisfied by all rotation matrices in $E^3(\alpha,\beta)$. Differentiating both sides, we obtain

$$[\dot{R}^T\varepsilon R]+[R^T\varepsilon\dot{R}]=[0]$$

which can be written as

$$[R^T\varepsilon\dot{R}]=-[R^T\varepsilon\dot{R}]^T$$

the last equation shows that $[R^T\varepsilon\dot{R}]=[\Theta]$ is a skew-symmetric matrix that is called G-angular velocity matrix of the rotation $[A(s)]$ in $E^3(\alpha,\beta)$. If we can calculate that $[\dot{R}]=[R\varepsilon^{-1}\Theta]$, let $[\varepsilon^{-1}\Theta]=[\Phi]$, then $[\dot{R}]=[R]\cdot[\Phi]$.
Note that $[\Phi]$ is a G-skew-symmetric matrix.
For a given matrix $[\Phi]$ we obtain a one parameter group of rotations from the matrix differential equation

$$[\dot{R}(s)]=[R(s)]\cdot[\Phi]$$

the equation has solution

$[R(s)] = e^{s[\Phi]}$

so $[R(s)] = e^{s[\Phi]} = \sum_{n=0}^{\infty} \frac{(s[\Phi])^n}{n!}$ note, we assume $[R(0)] = [I]$.

Let $k = (c_1, c_2, c_3) \in E^3(\alpha, \beta)$ be vector corresponding to G-skew-symmetric matrix in $E^3(\alpha, \beta)$, then

$$[\Phi] = \begin{bmatrix} 0 & -\beta c_3 & \beta c_2 \\ \alpha c_3 & 0 & -\alpha c_1 \\ -c_2 & c_1 & 0. \end{bmatrix}$$

The norm of $k = (c_1, c_2, c_3)$ is $= \| k\| = \sqrt{\alpha c_1^2 + \beta c_2^2 + \alpha\beta c_3^2}$. We can obtain the unit vector in direction of ϕ as $t = \frac{k}{\phi} = (t_1, t_2, t_3)$. Thus, we get $\frac{[\Phi]}{\phi} = [T]$

$$[T] = \begin{bmatrix} 0 & -\beta t_3 & \beta t_2 \\ \alpha t_3 & 0 & -\alpha t_1 \\ -t_2 & t_1 & 0. \end{bmatrix}$$

$$\begin{aligned} [R(s)] &= e^{(\phi s)[T]} = \sum_{n=0}^{\infty} \frac{((\phi s)[T])^n}{n!} \\ &= I + \frac{(\phi s)[T]}{1!} + \frac{((\phi s)[T])^2}{2!} + \frac{((\phi s)[T])^3}{3!} + \frac{((\phi s)[T])^4}{4!} + \frac{((\phi s)[T])^5}{5!} + \frac{((\phi s)[T])^6}{6!} + \cdots \qquad (4) \\ &= I + \frac{(\phi s)[T]}{1!} + \frac{(\phi s)^2[T^2]}{2!} + \frac{(\phi s)^3[T^3]}{3!} + \frac{(\phi s)^4[T^4]}{4!} + \frac{(\phi s)^5[T^5]}{5!} + \frac{(\phi s)^6[T^6]}{6!} + \cdots \end{aligned}$$

Let $t = (t_1, t_2, t_3) \in E^3(\alpha, \beta)$ be unit vector corresponding to G-skew-symmetric matrix in $E^3(\alpha, \beta)$, then

$$[T] = \begin{bmatrix} 0 & -\beta t_3 & \beta t_2 \\ \alpha t_3 & 0 & -\alpha t_1 \\ -t_2 & t_1 & 0. \end{bmatrix}$$

We get $[T^2]$ and $[T^3]$ as;

$$[T^2] = \begin{bmatrix} -\beta t_2^2 - \alpha\beta t_3^2 & \beta t_1 t_2 & \alpha\beta t_1 t_3 \\ \alpha t_1 t_2 & -\alpha t_1^2 - \alpha\beta t_3^2 & \alpha\beta t_2 t_3 \\ \alpha t_1 t_3 & \beta t_2 t_3 & -\alpha t_1^2 - \beta t_2^2 \end{bmatrix}$$

and

$$[T^3] = \begin{bmatrix} 0 & \beta t_3(\alpha t_1^2 + \beta t_2^2 + \alpha\beta t_3^2) & -\beta t_2(\alpha t_1^2 + \beta t_2^2 + \alpha\beta t_3^2) \\ -\alpha t_3(\alpha t_1^2 + \beta t_2^2 + \alpha\beta t_3^2) & 0 & \alpha t_1(\alpha t_1^2 + \beta t_2^2 + \alpha\beta t_3^2) \\ t_2(\alpha t_1^2 + \beta t_2^2 + \alpha\beta t_3^2) & -t_1(\alpha t_1^2 + \beta t_2^2 + \alpha\beta t_3^2) & 0 \end{bmatrix}$$

case 1: Let $\alpha > 0$ and $\beta > 0$. Since norm of unit vector $t = (t_1, t_2, t_3)$ is $\|t\| = 1$, we have

$$\begin{aligned} [\mathrm{T}^3] &= -[\mathrm{T}] \\ [\mathrm{T}^4] &= [\mathrm{T}] \cdot [\mathrm{T}^3] = -[\mathrm{T}] \cdot [\mathrm{T}] = -[T^2] \\ [\mathrm{T}^5] &= [\mathrm{T}] \cdot [\mathrm{T}^4] = -[\mathrm{T}^3] = [\mathrm{T}] \qquad (5) \\ [\mathrm{T}^6] &= [\mathrm{T}] \cdot [\mathrm{T}^5] = [\mathrm{T}] \cdot [\mathrm{T}] = [T^2]. \end{aligned}$$

if we use eq. (5) in eq. (4), we obtain

$$\begin{aligned} [R(s)] &= I + \frac{(\phi s)[T]}{1!} + \frac{(\phi s)^2[T^2]}{2!} - \frac{(\phi s)^3[T]}{3!} - \frac{(\phi s)^4[T^2]}{4!} + \frac{(\phi s)^5[T]}{5!} + \frac{(\phi s)^6[T^2]}{6!} + \cdots \\ &= I + (\frac{(\phi s)}{1!} - \frac{(\phi s)^3}{3!} + \frac{(\phi s)^5}{5!} - \cdots)[T] + (\frac{(\phi s)}{2!} - \frac{(\phi s)^4}{4!} + \frac{(\phi s)^6}{6!} - \cdots)[T^2] \\ &= I + (\sum_{n=0}^{\infty} \frac{(-1)^n(\phi s)^{2n+1}}{(2n+1)!})[T] + (1 - \sum_{n=0}^{\infty} \frac{(-1)^n(\phi s)^{2n}}{(2n)!})[T^2] \end{aligned}$$

$$[R(s)] = I + \sin(\phi s)[\mathrm{T}] + (1 - \cos(\phi s))[T^2] \qquad (6).$$

case 2: Let $\alpha > 0$ and $\beta < 0$. We should consider two subcases such that the unit vector t is a spacelike or timelike vector. If t is a spacelike, then we have the same result in case 1. If t is a timelike, then norm of the unit timelike vector is $\|t\| = -1$. Thus, we get

$$[\mathrm{T}^3] = [\mathrm{T}]$$

$$[\mathrm{T}^4] = [\mathrm{T}]\cdot[\mathrm{T}^3] = [\mathrm{T}]\cdot[\mathrm{T}] = [T^2]$$

$$[\mathrm{T}^5] = [\mathrm{T}]\cdot[\mathrm{T}^4] = [\mathrm{T}^3] = [\mathrm{T}] \tag{7}$$

$$[\mathrm{T}^6] = [\mathrm{T}]\cdot[\mathrm{T}^5] = [\mathrm{T}]\cdot[\mathrm{T}] = [T^2].$$

if we use eq. (7) in eq. (4), we have

$$\begin{aligned}[R(s)] &= I + \frac{(\phi s)[T]}{1!} + \frac{(\phi s)^2[T^2]}{2!} + \frac{(\phi s)^3[T]}{3!} + \frac{(\phi s)^4[T^2]}{4!} + \frac{(\phi s)^5[T]}{5!} + \frac{(\phi s)^6[T^2]}{6!} + \cdots \\ &= I + (\frac{(\phi s)}{1!} + \frac{(\phi s)^3}{3!} + \frac{(\phi s)^5}{5!} + \cdots)[T] + (\frac{(\phi s)}{2!} + \frac{(\phi s)^4}{4!} + \frac{(\phi s)^6}{6!} + \cdots)[T^2] \\ &= I + (\sum_{n=0}^{\infty} \frac{(\phi s)^{2n+1}}{(2n+1)!})[T] + (\sum_{n=0}^{\infty} \frac{(\phi s)^{2n}}{(2n)!} - 1)[T^2]\end{aligned}$$

$$[R(s)] = I + \sin h(\phi s)[\mathrm{T}] + (\cos h(\phi s) - 1)[T^2] \tag{8}$$

Example 1: Let G-skew symmetric matrix C given as;

$$C = \begin{bmatrix} 0 & 0 & \beta t \\ 0 & 0 & 0 \\ -t & 0 & 0 \end{bmatrix}$$

We can obtain G-orthogonal matrix R from the matrix C using by G-Cayley formula as

$$R = (I - C)^{-1}(I + C) = \frac{1}{1 + \beta t^2}\begin{bmatrix} 1 - \beta t^2 & 0 & 2\beta t \\ 0 & 1 + \beta t^2 & 0 \\ -2t & 0 & 1 - \beta t^2 \end{bmatrix}$$

Let $cos\theta = \frac{1-\beta t^2}{1+\beta t^2}$ and $sin\theta = \frac{2\sqrt{\beta}t}{1+\beta t^2}$, then

$$R = \begin{bmatrix} \cos\theta & 0 & \sqrt{\beta}sin\theta \\ 0 & 1 & 0 \\ -\frac{sin\theta}{\sqrt{\beta}} & 0 & \cos\theta \end{bmatrix}$$

we see that R is a G-orthogonal matrix that it is the θ −degree rotation about y −axis in $E^3(\alpha, \beta)$ space. If we calculate $[\Theta] = [R^T \varepsilon \dot{R}]$

$$[\Theta] = [R^T \varepsilon \dot{R}] = \begin{bmatrix} \frac{1 - \beta t^2}{1 + \beta t^2} & 0 & -\frac{2t}{1 + \beta t^2} \\ 0 & 1 & 0 \\ \frac{2\beta t}{1 + \beta t^2} & 0 & \frac{1 - \beta t^2}{1 + \beta t^2} \end{bmatrix}\begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \alpha\beta \end{bmatrix}\begin{bmatrix} -\frac{4\beta t}{(1 + \beta t^2)^2} & 0 & \frac{2\beta - 2\beta^2 t^2}{(1 + \beta t^2)^2} \\ 0 & 0 & 0 \\ \frac{-2 + 2\beta t^2}{(1 + \beta t^2)^2} & 0 & -\frac{4\beta t}{(1 + \beta t^2)^2} \end{bmatrix}$$

we get the tangent operator of $R(t)$;

$$[\Theta] = \begin{bmatrix} 0 & 0 & \frac{2\alpha\beta}{(1 + \beta t^2)^2} \\ 0 & 0 & 0 \\ -\frac{2\alpha\beta}{(1 + \beta t^2)^2} & 0 & 0 \end{bmatrix}.$$

Example 2: Let $\alpha > 0$ and $\beta < 0$. The G-skew symmetric matrix C corresponding to the vector $c = (0,3,0)$;

$$C = \begin{bmatrix} 0 & 0 & 3\beta \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix}$$

The norm of $c = (0,3,0)$ is $\|c\| = 3\sqrt{-\beta}$. The unit vector in direction of c is $t = \frac{c}{3\sqrt{-\beta}} = (0, \frac{1}{\sqrt{-\beta}}, 0)$. So G-skew symmetric matrix T corresponding to the vector s;

$$T = \begin{bmatrix} 0 & 0 & \sqrt{-\beta} \\ 0 & 0 & 0 \\ -\frac{1}{\sqrt{-\beta}} & 0 & 0 \end{bmatrix}.$$

We can obtain the rotation matrix R from eq. (7)

$$\begin{aligned} R &= I + \sinh(3\sqrt{-\beta}t)[S] + (\cosh(3\sqrt{-\beta}t) - 1)[S^2] \\ &= \begin{bmatrix} \cosh(3\sqrt{-\beta}t) & 0 & \sqrt{-\beta}\sinh(3\sqrt{-\beta}t) \\ 0 & 1 & 0 \\ \frac{\sinh(3\sqrt{-\beta}t)}{\sqrt{-\beta}} & 0 & \cosh(3\sqrt{-\beta}t) \end{bmatrix} \end{aligned}$$

The Tangent Operators of $G - H(4)$

In this chapter, we study the operations of rotation and translation and introduce the notion of homogeneous transformations.
The tangent operators of $G - H(4)$ must satisfy the relation

$$\begin{aligned} [\dot{g} \cdot g^{-1}] &= \begin{bmatrix} \dot{R} & \dot{t} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R^{-1} & -R^{-1} \cdot t \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \dot{R}R^{-1} & -\dot{R} \cdot R^{-1} \cdot t + \dot{t} \\ 0 & 0 \end{bmatrix} \end{aligned}$$

or

$$[\dot{g} \cdot g^{-1}] = \begin{bmatrix} \Omega & v \\ 0 & 0 \end{bmatrix} = [\Omega, v]$$

where $[\Omega] = [\dot{R}R^{-1}]$ is the 3×3 G-angular velocity matrix of the moving body and $v = -[R] \cdot t + \dot{t}$ is its 3-dimensional G-linear velocity vector.
Let us consider a special case of the equation $[\dot{g}(s) \cdot g^{-1}(s)] = [B(s)]$ when the $[B(s)]$ is a constant matrix. Thus, the one parameter subgroup of $G - H(4)$ can be obtained from ordinary differential equation

$$[\dot{g}(s)] = [\mathrm{B}] \cdot [g(s)] \tag{9}$$

where $[B] = [A, s]$. Let us consider a special case of the eq. (9) when the $[B]$ is a constant matrix. Assuming that a fixed frame and a moving frame coincide at the moment $t = 0$, so $[g(0)] = [I]$. We may conclude that:

$$[g(s)] = e^{s[B]}.$$

We can decompose G-linear velocity vector v into components. Let c be a point on the G-screw axis, then $v^{\perp} = c \wedge w$ and $kw = v - v^{\perp}$ perpendicular and G-angular velocity vector w, respectively.
We consider the case $v^{\perp} = 0$, then $v = pw$. Since $[\Omega] \cdot w = 0$, we have

$$[B^2] = \begin{bmatrix} \Omega^2 & 0 \\ 0 & 0 \end{bmatrix}$$

therefore,

$$[g(s)] = e^{s[B]} = \begin{bmatrix} e^{s[\Omega]} & pws \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R(s) & pws \\ 0 & 1 \end{bmatrix}$$

where $[R(s)]$ is the rotation matrix.

Vector Associated with Tangent Operators

The tangent operator of $SO(\alpha,\beta)(3)$ is 3×3 G-skew symmetric matrix has only three independent elements of nine elements, likewise, the tangent operator of $G-H(4)$ is 4×4 has only six independent elements of sixteen elements.
Definition 8: The components of a tangent operator of $[R]=[\Psi,v]$ are assembled into the $6-$dimensional vector $R=(\psi,v)$, called a screw.
Lie product defined in eq. (3) provides a product operation for the vectors associated with each of these tangent operators. For $SO(\alpha,\beta)(3)$, ψ and ϕ are corresponding vectors of the G-skew symmetric matrices $[\Psi]$ and $[\Phi]$, then we find that vectors corresponding vectors the G-skew symmetric matrices obtained from the Lie product

$$[\Psi]\wedge[\Phi] = [\Psi\cdot\Phi - \Phi\cdot\Psi]$$

is $\psi \times_G \phi$.
For homogeneous transformation $G-H(4)$, Lie product of the tangent operators $[R]=[\Psi,r]$ and $[S]=[\Phi,s]$ is defined by

$$\begin{aligned}[R]\wedge[S] &= [R\cdot S - S\cdot R] \\ &= [\Psi,r][\Phi,s]-[\Phi,s][\Psi,r] \\ &= [\Psi\cdot\Phi-\Phi\cdot\Psi, [\Psi]\cdot s-[\Phi]\cdot r] \\ &= (\psi\wedge\phi, \psi\wedge s-\phi\wedge r),\end{aligned}$$

where $[R]$ and $[S]$ as follow respectively;

$$[R]=\begin{bmatrix} 0 & -\beta\psi_3 & \beta\psi_2 & r_1 \\ \alpha\psi_3 & 0 & -\alpha\psi_1 & r_2 \\ -\psi_2 & \psi_1 & 0. & r_3 \\ 0 & 0 & 0 & 0\end{bmatrix} \text{ and } \quad [S]=\begin{bmatrix} 0 & -\beta\phi_3 & \beta\phi_2 & s_1 \\ \alpha\phi_3 & 0 & -\alpha\phi_1 & s_2 \\ -\phi_2 & \phi_1 & 0. & s_3 \\ 0 & 0 & 0 & 0\end{bmatrix}.$$

Multi-Parameter Motion in Generalized Space

The matrix function $[g(s)]:\mathbb{R}\to GL(\alpha,\beta)(3)$ defines a motion of a body that is parameterized by a single variable, we now consider the motions parameterized by n variables $\theta=(\theta_1,\ldots,\theta_n)$, denoted $[F(\theta)]:\mathbb{R}^n\to GL(\alpha,\beta)(3)$.
The partial derivative of $[F(\theta)]=[f_1(\theta_1,\ldots,\theta_n), f_2(\theta_1,\ldots,\theta_n), f_3(\theta_1,\ldots,\theta_n)]$ with respect to a variable θ_i is the $3\times n$ matrix

$$\left[\frac{\partial F}{\partial\theta_i}\right] = \left[\frac{\partial f_1}{\partial\theta_i},\frac{\partial f_2}{\partial\theta_i},\frac{\partial f_3}{\partial\theta_i}\right].$$

If the variables θ_i are functions of a variable ι, that $\theta=\theta(\iota)$, then the chain rule the partial derivatives $\frac{\partial F}{\partial\theta_i}$ to the derivative $\left[\dot{F}\right]$ by the relation

$$\left[\dot{F}\right] = \left[\frac{\partial F}{\partial\theta_1}\right]\dot{\theta}_1+\ldots+\left[\frac{\partial F}{\partial\theta_n}\right]\dot{\theta}_n.$$

Multiplying on the right by $[F^T\varepsilon]$, we obtain the tangent operator

$$\left[F^T\varepsilon\dot{F}\right] = \left[F^T\varepsilon\frac{\partial F}{\partial\theta_1}\right]\dot{\theta}_1+\ldots+\left[F^T\varepsilon\frac{\partial F}{\partial\theta_n}\right]\dot{\theta}_n.$$

The matrices $\left[F^T\varepsilon\dot{F}\right]$ are partial tangent operators associated with each of the parameters θ_i, individually.
Now, let's obtain the tangent operator of two parameters motion.

Example 3: Let G-skew symmetric matrix C given as

$$C=\begin{bmatrix}0 & 0 & \beta t\\ 0 & 0 & -\alpha s\\ -t & s & 0\end{bmatrix}$$

We can obtain rotation matrix R from the matrix C using by G-Cayley formula as,

$$R=(I+C)(I+C)^{-1}=\frac{1}{1+\alpha s^2+\beta t^2}\begin{bmatrix}1+\alpha s^2-\beta t^2 & 2\beta st & 2\beta t\\ 2\alpha st & 1-\alpha s^2-\beta t^2 & 0\\ -2t & 0 & 1-\alpha s^2-\beta t^2\end{bmatrix}$$

The derivatives of the rotation matrix $[R]$ with respect to t and s, respectively;

$$\frac{\partial F}{\partial \mathrm{t}}=\begin{bmatrix}\frac{-4\beta \mathrm{t}(1+\alpha s^2)}{(1+\alpha s^2+\beta t^2)^2} & \frac{2\beta s(1+\alpha s^2-\beta t^2)}{(1+\alpha s^2+\beta t^2)^2} & \frac{2\beta(1+\alpha s^2-\beta t^2)}{(1+\alpha s^2+\beta t^2)^2}\\ \frac{2\alpha s(1+\alpha s^2-\beta t^2)}{(1+\alpha s^2+\beta t^2)^2} & \frac{4\alpha\beta s^2\mathrm{t}}{(1+\alpha s^2+\beta t^2)^2} & \frac{4\alpha\beta s\mathrm{t}}{(1+\alpha s^2+\beta t^2)^2}\\ \frac{-2(1+\alpha s^2-\beta t^2)}{(1+\alpha s^2+\beta t^2)^2} & \frac{-4\beta s\mathrm{t}}{(1+\alpha s^2+\beta t^2)^2} & \frac{-4\beta \mathrm{t}}{(1+\alpha s^2+\beta t^2)^2}\end{bmatrix}$$

and

$$\frac{\partial F}{\partial \mathrm{s}}=\begin{bmatrix}\frac{4\alpha\beta s\mathrm{t}^2}{(1+\alpha s^2+\beta t^2)^2} & \frac{2\beta t(1-\alpha s^2+\beta t^2)}{(1+\alpha s^2+\beta t^2)^2} & \frac{-4\alpha\beta s\mathrm{t}}{(1+\alpha s^2+\beta t^2)^2}\\ \frac{2\alpha t(1-\alpha s^2+\beta t^2)}{(1+\alpha s^2+\beta t^2)^2} & \frac{-4\alpha s(1+\beta t^2)}{(1+\alpha s^2+\beta t^2)^2} & \frac{-2\alpha(1-\alpha s^2+\beta t^2)}{(1+\alpha s^2+\beta t^2)^2}\\ \frac{4\alpha s\mathrm{t}}{(1+\alpha s^2+\beta t^2)^2} & \frac{2(1-\alpha s^2+\beta t^2)}{(1+\alpha s^2+\beta t^2)^2} & \frac{-4\alpha s}{(1+\alpha s^2+\beta t^2)^2}\end{bmatrix}$$

If we calculate components of the tangent operator;

$$[\Phi]_t=\left[R^T\varepsilon\frac{\partial R}{\partial \mathrm{t}}\right]=\begin{bmatrix}0 & \frac{2\alpha\beta s}{1+\alpha s^2+\beta t^2} & \frac{2\alpha\beta}{1+\alpha s^2+\beta t^2}\\ \frac{-2\alpha\beta s}{1+\alpha s^2+\beta t^2} & 0 & 0\\ \frac{-2\alpha\beta}{1+\alpha s^2+\beta t^2} & 0 & 0\end{bmatrix}$$

and

$$[\Phi]_s=\left[R^T\varepsilon\frac{\partial R}{\partial s}\right]=\begin{bmatrix}0 & \frac{-2\alpha\beta t}{1+\alpha s^2+\beta t^2} & 0\\ \frac{2\alpha\beta t}{1+\alpha s^2+\beta t^2} & 0 & \frac{-2\alpha\beta}{1+\alpha s^2+\beta t^2}\\ 0 & \frac{2\alpha\beta}{1+\alpha s^2+\beta t^2} & 0\end{bmatrix}$$

we get the tangent operator of the two parameters motion as;

$$\left[R^T\varepsilon\dot{F}\right]=\left[R^T\varepsilon\frac{\partial R}{\partial t}\right]\frac{\partial}{\partial t}+\left[R^T\varepsilon\frac{\partial R}{\partial s}\right]\frac{\partial}{\partial s}.$$

Conclusion

Since the solutions obtained for generalized space cover both Lorentzian and Euclidean spaces, it is an undeniable fact that the results obtained in generalized space are valid in both spaces. Therefore, the data obtained as a result of the situations examined within the scope of this study provide the necessary conditions for both spaces. It is important in terms of enabling researchers to perform calculations in generalized space and then go to specific instead of making separate calculations for two different spaces. The generalization of the space studied in this study to n dimensions is also foreseen as an advanced research topic.

Conflicts of interest

The author state that did not have conflict of interests.

References

[1] Awrejcewicz J., Classical Mechanics; Kinematics and Statics, New York: Springer, (2012).

[2] Agrawal O.P., Hamilton Operators and Dual-number-quaternions in Spatial Kinematics, *Mech Mach Theory*, 22 (1987) 569-575.

[3] Beggs J.S., Advanced Mechanisms, New York: The Macmillan Company Collier-Macmillan, London (1965).

[4] Herve J.M., The mathematical group structure of the set of displacements*, Mech. Mach. Theory*, 29 (1994) 73-81.

[5] Hiller M., Woernle C., A Unified Representation of Spatial Displacements*, Mech. Mach. Theory*, 19(1984) 477-486.

[6] Spong M.W., Hutchison S., Vidyasagar M., Robot Modeling and Control. Hoboken: NJ John Wiley & Sons, (2006).

[7] Altmann S.L., Rotations, Quaternions and Double Groups. Oxford: Oxford University Press, (1986).

[8] Aragon G., Aragon J.L., Rodriguez M.A., Clifford Algebras and Geometric Algebra, *Adv Appl Clifford Al.*, 7(2) (1997) 91-102.

[9] Rosenfeld B., Geometry of Lie Groups. Dordrecht: Kluwer Academic Publishers, (1997).

[10] Uicker J.J., Pennock G.R., Shigley J., Theory of Machines and Mechanisms. New York: Oxford University Press, (2011).

[11] Bottema O., Roth, B., Theoretical Kinematics. New York: North-Holland Press, (1979).

[12] McCarthy J.M., An Introduction to Theoretical Kinematics. Cambridge: MIT Press, (1990).

[13] O'Neill B., Semi-Riemannian Geometry With Applications to Relativity. New York: Academic Press Inc., (1983).

[14] Ryan P.J., Euclidean and non-Euclidean geometry; an analytic approach. Cambridge, New York: Cambridge Univ. Press, (1986).

[15] Ata E., Yıldırım Y.A., Different Polar Representation for Generalized and Generalized Dual Quaternions, *Adv Appl Clifford Al.*, 28 (2010) 193-202.

[16] Ata E., Savcı Ü.Z., Spherical Kinematics in 3-Dimensional Generalized Space *Int J Geom Meth Mod Phys.,* 18(3) (2020) 2150033.

[17] Erdmann K., Skowronski A., Algebras of generalized quaternion type, *Advances in Mathematics*, 349 (2019) 1036-1116.

[18] Jafari M., Yaylı Y., Generalized Quaternions and Rotation in 3-Space $E^3_{\alpha\beta}$, *TWMS J. Pure Appl. Math.*, 6(2) (2015) 224-232.

[19] Lam T.Y., Introduction to Quadratic Forms Over Fields, *American Mathematical Society*, USA (2005).

[20] Pottman H., Wallner J., Computational line geometry. Berlin Heidelberg, New York: Springer-Verlag, (2000).

[21] Savcı Ü.Z., Generalized Dual Quaternions and Screw Motion in Generalized Space, *Konuralp Journal of Mathematics,* 10 (1) (2022) 197-202.

[22] Parkin I.A., A third conformation witht the screw systems: finite twist displacements of a directed line and point, *Mech. Mach. Theory*, 27(2) (1992) 177- 188.

[23] Huang C., Roth B., Analytic expressions for the finite screw systems, Mech. Mach. Theory, 29(2) (1994) 207-222.

[24] Knossow D., Ronfard R., Horaud R., Human motion tracking with a kinematic parameterization of extremal contours, *International Journal of Computer Vision*, 79 (2008) 247-269.

[25] Durmaz O., Aktaş B., Gündogan H., The derivative and tangent operators of a motion in Lorentzian space, *Int. J. Geom. Meth. Mod. Phys.,* 14(4) (2017) 1750058.

[26] Ward J.P., Quaternions and Cayley numbers algebra and applications. London: Kluwer Academic Publishers, (1997).