Cumhuriyet Sci. J., 43(1) (2022) 132-136
DOI: https://doi.org/10.17776/csj.1052798

Cumhuriyet Science Journal

| csj.cumhuriyet.edu.tr | Founded: 2002 ISSN: 2587-2680 e-ISSN: 2587-246X Publisher: Sivas Cumhuriyet University

Dynamics of the Dirac Particle in an Anisotropic Rainbow Universe

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Research Article

History
Received: 03/01/2022
Accepted: 24/02/2022

ABSTRACT

An alternative way of understanding physical effects in curved space time is to solve the associated particle equation such as the Dirac equation. It is a first-order relativistic wave equation and defines spin-$\frac{1}{2}$ massive particles like electrons and quarks. In this study, we solved the Dirac equation in an anisotropic rainbow universe. Subsequently, the reduced wave equation is obtained by making use of the asymptotic property of the Whittaker function. In the final stage, we calculate each component of the spin current density and then graphically evaluate their behavior according to the rainbow function. According to our results, the spin current density only depends on the z component of the momentum. In addition, the sign of both spin current densities is not changing with time. Finally, the current density amplitude in the high energy state or high scale parameter ($\epsilon = 0.9$) is rapidly decreasing faster than in $\epsilon = 0.6$ and $\epsilon = 0.3$.

Keywords: Dirac equation, Rainbow function, Spin current density.

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Introduction

Generally, a well-known shortcoming of the Klein-Gordon theory is a negative quantum probability that is considered to be physically meaningless[1]. To overcome such a challenge, Dirac proposed a first-order relativistic wave equation that plays an important role in many branches including those in nuclear and high energy physics. It is commonly believed that this idea is the most effective mathematical method to analyze the relativistic quantum mechanical behavior of the spin-$\frac{1}{2}$ particles (electron, proton, and their corresponding antiparticles)[2]. Therefore, one can easily see that there are lots of papers where some solutions of the Dirac equation are illustrated in various spacetimes[3-7].

Investigating the effects of the rainbow functions in different research areas is still a very attractive topic in recent years. The dynamics of the photon equation are discussed in the cosmic string spacetime[8]. Bakke and Mota evaluate the Aharonov–Bohm effect in the context of rainbow gravity[9]. Junior and his co-authors investigate regular black holes in rainbow gravity [10]. Ling derives the kinematics of particles moving in rainbow spacetime[11]. The idea behind it is the Planck scale (or the Planck energy $E_{pl} \approx 10^{19}$ GeV)[12]. It plays a vital role while determining the boundary between the classical and quantum gravity regimes[13-14]. However, the existence of such a scale forces us to consider the Planck energy as an invariant quantity for all observers in momentum space whereas the invariance of light's speed c is valid in Special Relativity (SR). This result points out the existence of a Deformed Special Relativity (DSR)[15-16]. Smolin and Magueijo presented a modified energy and momentum relation as the following formulation[17]:

$$f^2(\epsilon)E^2 - g^2(\epsilon)P^2 = m^2 \tag{1}$$

Here $\epsilon = \frac{E}{E_{pl}}$ is a scale parameter, where E_{pl} represents the Plank energy, and E denotes particle's energy. Furthermore, both $f(\epsilon)$ and $g(\epsilon)$ are called rainbow functions obeying the following relation

$$\lim_{\epsilon\to 0} f(\epsilon) = \lim_{\epsilon\to 0} g(\epsilon) = 1 \tag{2}$$

Thus, the DSR formalism can be reduced to the SR framework with the help of the above condition. In literature, there are several proposals[15,17] for the rainbow functions:

$$1^{st}\text{ Scenario: } f(\epsilon) = \sqrt{1-\epsilon^2} \quad g(\epsilon) = 1 \tag{3}$$

$$2^{st}\text{ Scenario: } f(\epsilon) = 1 \quad g(\epsilon) = 1 + \frac{\epsilon}{2} \tag{4}$$

Following Refs[17,18-20], one can write the rainbow type line-elements by making use of the transformations $dt \to \frac{dt}{f(\epsilon)}$ and $dx^i \to \frac{dx^i}{g(\epsilon)}$. From this point of view, the energy-independent tetrads defined in the Minkowski spacetime are transformed mathematically into energy-dependent ones. For this reason, in the rainbow formalism of gravity (RFG), the metric tensor of a given line-element is rewritten in terms of the energy-dependent tetrad[17]s:

$$g_{\mu\nu} = e_{\mu}^{(i)}(\epsilon) \otimes e_{\nu}^{(j)}(\epsilon) \tag{5}$$

where the Greek and Latin indices show the curved(μ: 0, 1, 2,3) and flat spacetime(i: 0, 1, 2,3), respectively. Therefore, all components of energy-dependent tetrads are divided into two main groups according to time and space components:

$$e_0^{(0)}(\epsilon) = \frac{1}{f(\epsilon)}\tilde{e}_0^{(0)} \tag{6}$$

$$e_i^{(i)}(\epsilon) = \frac{1}{f(\epsilon)}\tilde{e}_i^{(i)} \tag{7}$$

where the tilde denotes the Minkowski spacetime.

One of the most important scenarios, where the effects of Quantum Gravity (QG) are tested in an anisotropic universe, is associated with the subsequent metric[21]:

$$ds^2 = -dt^2 + t^2(dx^2 + dy^2) + dz^2 \tag{8}$$

Applying the procedure of RFG to the metric gives

$$ds^2 = -\frac{1}{f^2}dt^2 + \frac{t^2}{g^2}(dx^2 + dy^2) + \frac{1}{g^2}dz^2 \tag{9}$$

As a result, we can easily write the covariant and contravariant forms of the metric tensor as given below

$$g_{\mu\nu} = \begin{pmatrix} -\frac{1}{f^2} & 0 & 0 & 0 \\ 0 & \frac{t^2}{g^2} & 0 & 0 \\ 0 & 0 & \frac{t^2}{g^2} & 0 \\ 0 & 0 & 0 & \frac{1}{g^2} \end{pmatrix} \tag{10.a}$$

$$g_{\mu\nu} = \begin{pmatrix} -f^2 & 0 & 0 & 0 \\ 0 & \frac{g^2}{t^2} & 0 & 0 \\ 0 & 0 & \frac{g^2}{t^2} & 0 \\ 0 & 0 & 0 & g^2 \end{pmatrix} \tag{10.b}$$

According to the general transport theory, spin current, one of the most significant physical quantities in quantum physics, is accompanied by a spin continuity equation which includes additional torque terms like the spin operator $\hat{s} = \frac{\hbar}{2}\hat{\sigma}$ which are the Pauli matrices. If we compare $\rho_e = \Psi^\dagger e\Psi$ (charge density) with $\rho_s = \Psi^\dagger \hat{s}\Psi$ (spin density), $\hat{s}$ is thought of as the spin-charge. In the fermion state, the spin current is converted into the Dirac current and it consists of the Gordon decomposition and spin magnetization. Further, both currents are precisely discussed in [22-24]. From the point of the physical interpretation, the Gordon current arises from a moving point charge while the spin magnetization one is created by the spin of the elementary particles.

The layout of the study is as follows: in the second section, we solve the Dirac equation for the anisotropic universe. In the third section, we rewrite the asymptotic wave function. Then, the Dirac current is calculated with the help of the rainbow function and is graphically illustrated. The last section is devoted to the conclusion part.

The Solution of Dirac Equation

The Dirac equation in curved spacetime is given by [25]

$$\left[i\gamma^\lambda(\partial_\lambda + \Gamma_\lambda) - m\right]\Psi(t,\vec{r}) = 0 \tag{11}$$

where m is the particle's mass, $\Psi(t,\vec{r})$is the particle's wave function, and Γ_λ is the spinor affine connections as given below

$$\Gamma_\lambda = -\frac{1}{8}g_{\mu\alpha}\Gamma^\alpha_{\mu\lambda}[\gamma^\mu, \gamma^\nu] \tag{12}$$

$\Gamma^\alpha_{\mu\lambda}$ denotes the Christoffel symbols and are defined by the following equation

$$\Gamma^\alpha_{\mu\lambda} = \frac{1}{2}g^{\alpha\sigma}(\partial_\nu g_{\lambda\sigma} + \partial_\lambda g_{\nu\sigma} - \partial_\sigma g_{\nu\lambda}) \tag{13}$$

The non-zero components of the Christoffel symbol for the given line element are expressed in terms of time and the rainbow functions as follows

$$\begin{aligned} \Gamma^0_{11} &= \Gamma^0_{22} = \frac{tf^2}{g^2} \\ \Gamma^1_{01} &= \Gamma^1_{10} = \frac{1}{t} \\ \Gamma^2_{02} &= \Gamma^2_{20} = \frac{1}{t} \end{aligned} \tag{14}$$

Substituting Eq.14 into Eq.12 gives

$$\Gamma_0 = \frac{f}{2g}\gamma^{(0)}\gamma^{(2)}, \quad \Gamma_1 = \frac{f}{2g}\gamma^{(0)}\gamma^{(1)} \tag{15}$$

where the brackets () represents the Minkowski spacetime. If Eq.15 is inserted into Eq.11, the Dirac equation becomes

$$\left[\gamma^{(0)}\left(\partial_0 + \frac{f}{t}\right) + \frac{g}{tf}\left(\gamma^{(1)}\partial_1 + \gamma^{(2)}\partial_2\right) + \frac{g}{f}\gamma^{(3)}\partial_3 + \frac{im}{f}\right]\Psi = 0 \tag{16}$$

Subsequently, Eq. 16 is reduced to the following form by setting $\Psi = t^{-1}\psi$ to eliminate the contribution from the spinor connections

$$\left[\gamma^{(0)}\partial_0 + \frac{g}{tf}\left(\gamma^{(1)}\partial_1 + \gamma^{(2)}\partial_2\right) + \frac{g}{f}\gamma^{(3)}\partial_3 + \frac{im}{f}\right]\psi = 0 \tag{17}$$

By using $\gamma^{(3)}\gamma^{(0)}\gamma^{(3)}\gamma^{(0)} = 1$, Eq. 17 can be easily expressed as a sum of two first-order differential operators after some mathematical steps

$$[L_1 + L_2]\Phi = 0 \tag{18}$$

Where

$$L_1 = t\left(\gamma^{(3)}\partial_0 + + \frac{g}{f}\gamma^{(0)}\partial_3 + \frac{im}{f}\right)$$

$$L_2 = \frac{g}{f}\left(\gamma^{(1)}\partial_1 + \gamma^{(2)}\partial_2\right) \tag{19}$$

$$\Phi = \gamma^{(3)}\gamma^{(0)}\psi$$

We chose the following representation of the Dirac matrices[21]:

$$\gamma^{(0)} = \begin{pmatrix} -i\sigma^1 & 0 \\ 0 & i\sigma^1 \end{pmatrix}, \quad \gamma^{(1)} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \tag{20.a}$$

$$\gamma^{(2)} = \begin{pmatrix} \sigma^2 & 0 \\ 0 & -\sigma^2 \end{pmatrix}, \quad \gamma^{(3)} = \begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix} \tag{20.b}$$

With

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{21.a}$$

$$\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{21.b}$$

By using a separation constant λ, Eq.18 can be written as

$$L_1\Phi = -\lambda\Phi \tag{22}$$

$$L_2\Phi = \lambda\Phi \tag{23}$$

Since the particle moves freely in x, y, and z coordinates, the spinor Φ is given by

$$\Phi = e^{i(k_1x + k_2y + k_3z)}\chi(t) \tag{24}$$

where

$$\chi(t) = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \tag{25}$$

If we insert Eq.25 and Eq.24 into Eq.23 and apply some mathematical steps, we get the following relation between components of the bispinor

$$\chi_2 = \frac{k_1\sigma^2}{ik_2 + \frac{f}{g}\lambda}\chi_1 \tag{24}$$

where $\lambda = \mp i\frac{g}{f}\sqrt{k_1^2 + k_2^2}$. Then, if the same procedure is applied for Eq. 22, we find two coupled differential equations as below

$$\frac{\partial\Phi_2}{\partial t} + \frac{\lambda}{t}\Phi_2 - \left(\frac{gk_3}{f} + \frac{m}{f}\right)\Phi_1 = 0 \tag{27}$$

$$\frac{\partial\Phi_1}{\partial t} - \frac{\lambda}{t}\Phi_1 + \left(\frac{gk_3}{f} + \frac{m}{f}\right)\Phi_2 = 0 \tag{28}$$

where the bispinor χ_1 is chosen by the following form

$$\chi_1 = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} \tag{29}$$

If we perform some algebra between these coupled equations, we get the following second-order differential equation:

$$\left[\frac{\partial^2}{\partial t^2} + \frac{\lambda - \lambda^2}{t^2} - \left\{\left(\frac{m}{2}\right)^2 - \left(\frac{gk_3}{f}\right)^2\right\}\right]\Phi_1 = 0 \tag{30}$$

Defining a new variable $t = \alpha\, u$ takes Eq.30 to the form of the well-known Whittaker Equation and thus, the corresponding solutions are written as [26]

$$\Phi_1 = N_1 M_{\kappa,\mu}\left({}^{t}/_{\alpha}\right) + N_2 W_{\kappa,\mu}\left({}^{t}/_{\alpha}\right) \tag{31}$$

which $M_{\kappa,\mu}$ and $W_{\kappa,\mu}$ are called as Whittaker functions, $\kappa = 0$, $\mu = \mp\left(\lambda - \frac{1}{2}\right)$ and $\alpha = \frac{f}{2gk_3}\left(1 - \left(\frac{m}{gk_3}\right)^2\right)^{-1}$. If we analyze the asymptotic behavior of Whittaker functions as $t \to \infty$[26], N_1 must be zero since $M_{\kappa,\mu}$ leads to the divergence of Φ_1 in that limit. Therefore, Eq.31 is reduced to the following form

$$\Phi_1 = W_{0,\mu}\left({}^{t}/_{\alpha}\right) \tag{32.a}$$

$$\Phi_2 = \frac{f}{m - gk_3}\left(\partial_t + \frac{\lambda}{t}\right)W_{0,\mu}\left({}^{t}/_{\alpha}\right) \tag{32.b}$$

With the help of the differential definition of the Whittaker function[26], the explicit form of Eq.32 can be written as

$$\Phi_1 = W_{0,\mu}\left({}^{t}/_{\alpha}\right) \tag{33.a}$$

$$\Phi_2 = \frac{f}{m - gk_3}\left[\left(\frac{1}{2\alpha} - \frac{\lambda}{t}\right)W_{0,\mu}\left({}^{t}/_{\alpha}\right) - \frac{1}{t}W_{1,\mu}\left({}^{t}/_{\alpha}\right)\right] \tag{33.b}$$

Similarly, if we use the asymptotic property of the Whittaker function[26] for Eq. 33, we get

$$\Phi_1 = e^{-\frac{t}{2\alpha}} \tag{34.a}$$

$$\Phi_2 = \beta e^{-\frac{t}{2\alpha}} \tag{34.b}$$

where

$$\beta = \sqrt{\frac{k_3 + \frac{m}{g}}{k_3 - \frac{m}{g}}} \tag{35}$$

Thus, the total associated Dirac wave function is in the following form:

$$\Psi(t, \vec{r}) = N\begin{pmatrix} 1 \\ \beta \\ -\gamma\beta \\ \gamma \end{pmatrix}\frac{e^{\left(i\vec{k}.\vec{r} - \frac{t}{2\alpha}\right)}}{t} \tag{36}$$

where $\gamma = \frac{igk_1}{igk_2 + f\,\lambda}$ and N is a normalization coefficient as follows

$$N = i\sqrt{\frac{g^3}{\alpha(1+\beta^2)(1+|\gamma|^2)}} \quad (37)$$

Spin Current Density

From the quantum mechanical perspective, the spin current density associated with the flow of its probability is known as the Dirac current and is given by [27,28]

$$J_D^\mu = J_G^\mu + J_M^\mu \quad (38)$$

With

$$J_G^\mu = ie[\overline{\Psi}\partial^\mu\Psi - \Psi\partial^\mu\overline{\Psi}] \quad (39)$$

$$J_M^\mu = ie\partial_\nu[\overline{\Psi}\gamma^\mu\gamma^\nu\Psi - \overline{\Psi}\gamma^\nu\gamma^\mu\Psi] \quad (40)$$

where $\overline{\Psi} = \Psi^\dagger\gamma^{(0)}$, G, and M denote the Gordon and spin magnetization current. If we calculate the spin current for the line element given in Eq.9, we obtain

$$J_D^\mu = e\left(\frac{g}{f}\right)^2 \frac{e^{-\frac{g}{f}k_3 t}}{t^2} k_{sum} \quad (41)$$

where $k_{sum} = k_1 + k_2 + k_3 > 0$ and also k_3 must be positive because this enables us to avoid divergence of spin current in high time. If making a plot of Eq.41 in terms of charge value(e = 1 and e = −1) by considering two rainbow scenarios, we get the following figüre:

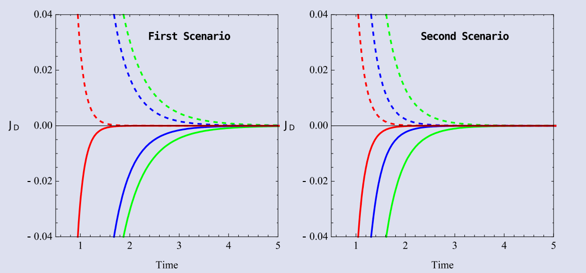

Figure 1. In the both figures, dashed line shows e=1(positron) state while line indicates e=−1(electron) one at k3=0.5 and ksum=1. Furthermore, red(high scale), green(medium one), and blue(low one) color of both dashed and line represent ϵ = 0.9, ϵ = 0.6 and ϵ = 0.3 and respectively

It can be seen that the current density defined for both positron and electron decreases over time. However, the second scenario's slope is much bigger than the first one since Eq.40 is linearly dependent on g while it is inversely proportional to f. Also, any particle creation is not observed in either of the scenarios since they do not have any critical turning point whose sign-magnitudes lead to change through the time axis in both current densities.

Conclusion

In this work, we propose a solution to the Dirac equation in the anisotropic rainbow universe. By rewriting the Dirac wave function considering the asymptotic property of the Whitaker function, we calculate the spin current density using the asymptotic limit. According to our results, no particle creation event occurred in any of the rainbow scenarios. To understand their effects on particle creation, we, therefore, need to discuss the different rainbow scenarios listed in the literature.

Acknowledgment

I would like to thank the anonymous reviewer for giving such constructive comments which substantially helped improving the quality of the paper.

Conflicts of interest

The authors declares no conflict of interest.

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