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A New Extension Form for Continuous Probability Distributions: Uniform-X Distributions

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ABSTRACT

In this paper, generating extension forms for continuous probability distribution functions is studied. The considered transformer function is applied to three well-known probability distributions- Normal, Kumaraswamy, Weibull- and new extensions of these distributions are obtained. The related functions of the new extensions are defined, random samples are generated from the new extensions and the results are presented. Parameter estimation procedures of the extensions are studied, and likelihood equations are obtained. To demonstrate the modeling performance of the extensions, three different data sets are considered, separately. Each data set is modeled by both the corresponding probability distribution and its extensions. The new extensions give the best fit to the corresponding data over the well-known probability distributions.

Keywords: Extensions of probability distributions, Transformer function, Quantile function, Maximum likelihood estimation, Skewness.

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Introduction

Generalization of the well-known distributions have been widely used to get more flexible statistical models. For instance, the generalized gamma distribution for modeling the distribution of income rate is introduced in [1], proposed inverse Gaussian distribution is proposed in [2], the generalization of Pareto distributions are studied in [3-5] and, the generalized beta of the first and second kind as models for the size distribution of income is introduced in [6]. Beta distribution is used as a generator function by [7] and they propose a new class of distribution, which are called beta-generated (BG) distributions. This generalization proceeds as follows, $F(x)$ is the cumulative distribution function (cdf) of any random variable X, $b(t)$ is the probability density function (pdf) of beta random variable then the cdf of beta- generated random variable $G(x)$ is defined as

$$G(x) = \int_0^{F(x)} b(t)dt \tag{1.1}$$

(1.2) gives the pdf of beta-generated random variable.

$$g(x) = \frac{1}{B(\alpha,\beta)} f(x)[F(x)]^{\alpha-1}[1-F(x)]^{\beta-1} \tag{1.2}$$

In literature, there are various types of studies including beta-generated distributions, see [8-13]. Kumaraswamy generalized distribution (KWG) by using Kumaraswamy distribution instead of beta distribution in (1.2) is introduced in [14]. Many researchers proposed some variations of KWG distributions, see [15-17].

Following the idea of generating BG distributions, a new technique to generate continuous probability distributions is proposed in [18]. This new approach is described as follows:

Let X be a random variable whose pdf is $f(x)$ and cdf is $F(x)$. Let T be a continuous random variable with pdf $h(t)$ defined on the interval $[a,b]$. The cdf of the new family of distribution is defined as

$$G(x) = \int_a^{W[F(x)]} h(t)dt \tag{1.3}$$

where $W[F(x)]$ is differentiable and monotonically non-decreasing in x. It should be also noted that $W[F(x)] \to a$ as $x \to -\infty$ and $W[F(x)] \to b$ as $x \to \infty$. The corresponding pdf of X can be written as

$$g(x) = \left\{\frac{d}{dx} W[F(x)]\right\} h\{W[F(x)]\} \tag{1.4}$$

In this generalization procedure, the random variable T is called "transformed" into a new cdf $G(x)$ through the function $W[F(x)]$, which is called "transformer". So, $G(x)$ is called "Transformed-Transformer" or $T-X$ distribution. Following the technique which is proposed and well defined in [18], we study with the transformer function $W[F(x);\theta]$ which is introduced to generate a new life time distribution in [19]. This transformer function is also used to obtain a new extension of Generalized Extreme Value distribution which is proposed as a model for an earthquake data in [20].

In this paper, we study to obtain new extensions of the three probability distributions-Normal, Kumaraswamy, and Weibull by transforming uniform random variable through the transformer function $W[F(x);\theta]$

The rest of the paper is organized as follows: In Section 2, a brief summary of the methodology is given. In Section 3 the methodology is applied to Normal, Kumaraswamy and Weibull distributions to get new extensions. The properties of new extensions distributions such as moments, quantiles functions and the maximum likelihood (ML) equations are obtained. Some simulations studies are conducted to present how the new extensions change according to the representative values of the parameter (θ) of the generator function, $W[F(x);\theta]$. In Section 4, real data examples are considered to present the performances of the new extensions of the studied probability distributions. Section 5 concludes the paper.

Materials and Methods

The transformer function which is introduced by [19] is given and then recalling (1.3), the cdf of Uniform-X distributions is defined.

$$W[F(x);\theta]=\frac{\exp(-\theta F(x))-1}{\exp(-\theta)-1} \tag{2.1}$$

where $\theta \epsilon R$. Furthermore, $W[F(x);\theta]\to 0$ as $x\to-\infty$ and $W[F(x);\theta]\to 1$ as $x\to\infty$.

Definition 2.1 Let X be a random variable whose pdf is $f(x);\ a<x<b$ and cdf is $F(x)$. Let T be a uniform random variable, whose pdf is $h(t)=1; 0<t<1$. Then

$$G(x)=\int_0^{\frac{\exp(-\theta F(x))-1}{\exp(-\theta)-1}} dy=\frac{\exp(-\theta F(x))-1}{\exp(-\theta)-1},\qquad a<x<b \tag{2.2}$$

is a cdf of the Uniform-X (Uni-X) distribution. The pdf of Uni-X distribution is in (2.3).

$$(x)=\frac{\theta f(x)\exp(-\theta F(x))}{1-\exp(-\theta)},\qquad a<x<b, \theta\epsilon R \tag{2.3}$$

The properties of the Uni-X distribution are as follows:

1. For all bounded or unbounded intervals, this new extension of distributions inherits the properties of being a pdf
2. Extensions are defined on the same interval $[a,b]$
3. If θ goes to 0, the $g(x)$ defined in (2.3) converges to the studied pdf distribution $f(x)$

$$\lim_{\theta\to 0}\frac{\theta f(x)\exp(-\theta F(x))}{1-\exp(-\theta)}\to f(x).$$

The moment generating function (mgf) of Uni-X distributions is defined as follows

$$M_X(t)=\int_a^b e^{tx}\frac{\theta f(x)\exp(-\theta F(x))}{1-\exp(-\theta)}dx. \tag{2.4}$$

By substituting $u=F(x)$, Taylor series expansion of $e^{tF^{-1}(u)}e^{-\theta u}$ can be considered to obtain the moments of the Uni-X distributions. However, the closed form of the moments of the studied Uni-X distributions in this study cannot be obtained.

The quantile function of Uni-X distributions in (2.5) is obtained by using the simple inverse cdf technique.

$$Q(p)=F^{-1}\left\{-\frac{1}{\theta}\ln(p(\exp(-\theta)-1)+1)\right\} \tag{2.5}$$

The Uni-X distributions are extensions of the studied distributions firstly denoted by $F(x)$ in (2.1). In this study, we study normal distribution (typical example of symmetric distributions), Weibull distribution (representative of the positively skewed distributions) and Kumaraswamy distribution (symmetric, positively skewed and negatively skewed form) as $F(x)$ and introduce their extensions which are called Uni-Normal, Uni-Weibull, Uni-Kumaraswamy, respectively in the following three sub-sections.

Uniform-Normal Distribution

Consider a normal distribution with pdf $\emptyset(z)$ and cdf Φ(z) where $z=\frac{x-\mu}{\sigma}$, μ and σ are the location and the scale parameters, respectively. Recalling (2.2), the cdf of uniform-normal (Uni-Normal) distribution is defined as

$$G(x)=\frac{\exp(-\theta\Phi(z))-1}{\exp(-\theta)-1},\quad -\infty<x<\infty \tag{3.1}$$

The pdf of Uni-Normal distribution is defined as

$$g(x)=\frac{\theta\emptyset(z)\exp(-\theta\Phi(z))}{\sigma(1-\exp(-\theta))}, -\infty<x<\infty \tag{3.2}$$

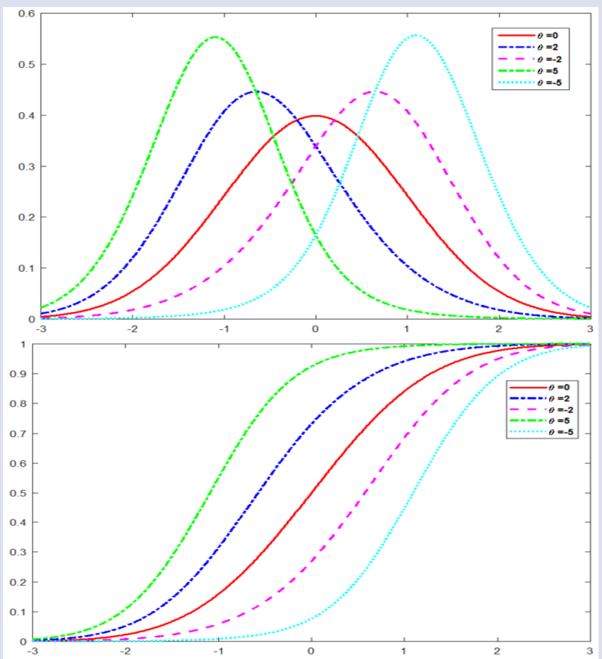

Figure 1. The pdfs and cdfs of Uni-Normal distribution, $\mu=0$ and $\sigma=1$

Figure 1 illustrates the pdfs and the cdfs of the Uni-Normal distribution for different θ values. When θ tends to 0, the pdf reduces to the well-known normal distribution, when $\theta > 0$, then the pdf becomes positively skewed and $\theta < 0$ then the pdf becomes negatively skewed.

The quantile function $Q(p)$, $0 < p < 1$ and the median of the Uni-Normal distribution are defined as

$$Q(p) = \mu + \sigma\Phi^{-1}\left\{-\frac{1}{\theta}\ln(1 - p(exp(-\theta) - 1))\right\} \tag{3.3}$$

$$Q\left(\frac{1}{2}\right) = \mu + \sigma\Phi^{-1}\left\{-\frac{1}{\theta}\ln(1 - \frac{1}{2}(1 + exp(-\theta)))\right\} \tag{3.4}$$

respectively.

Considering (3.3), a simulation study is conducted to present how to change sample mean, variance, skewness values (SV) and kurtosis values (KV) according to the different values of θThe random samples with size 100 are generated 100.000 times and the results are listed in Table 1.

Table 1. The simulation results for the Uni-Normal distribution, $\mu = 0$ and $\sigma = 1$

θ	Mean	Variance	SV	KV
-5	1.082	0.767	-0.288	3.583
-2	0.534	0.938	-0.218	3.180
-0.001	0.001	0.997	-0.002	2.936
0	0.000	1.000	0	3.000
0.001	-0.001	0.997	0.001	2.938
2	-0.534	0.937	0.224	3.185
5	-1.081	0.767	0.289	3.534

Table 1 indicates that θ tends to 0 the Uni-Normal distribution converges to the well-known normal distribution and θ increases, the SV and the KV increase and the variance decreases.

Now, suppose $Z_1, Z_2, \ldots, Z_n$ are random variables from a Uni-Normal distribution defined in (3.1), then the likelihood function and the log-likelihood function are defined in (3.5) and (3.6), respectively.

$$L(\mu,\sigma,\theta) = \theta^n\sigma^{-n}(1 - exp(-\theta))^{-n}\prod_{i=1}^{n}\emptyset(z_i)\prod_{i=1}^{n}\exp(-\theta\emptyset(z_i)) \tag{3.5}$$

$$l(\mu,\sigma,\theta) = nln\theta - nln\sigma - \ln(1 - exp(-\theta)) + \sum_{i=1}^{n}\ln(\emptyset(z_i)) - \theta\sum_{i=1}^{n}\Phi(z_i)\,. \tag{3.6}$$

By differentiating the log-likelihood function with respect to the unknown parameters and equating them to zero, we obtain the following likelihood equations.

$$\frac{\partial lnL}{\partial\mu} = \sum_{i=1}^{n} z_i + \theta\sum_{i=}^{n}\emptyset(z_i) = 0$$

$$\frac{\partial lnL}{\partial\sigma} = -n + \sum_{i=1}^{n} z_i^2 + \theta\sum_{i=}^{n}\emptyset(z_i) = 0 \tag{3.7}$$

$$\frac{\partial lnL}{\partial\theta} = \frac{n}{\theta} - \frac{nexp(-\theta)}{1 - exp(-\theta)} - \sum_{i=}^{n}\emptyset(z_i) = 0$$

Solutions of (3.7) are called ML estimates. The equations need to be solved with numerical methods such as Newton Raphson or iteratively reweighting algorithm.

Uniform-Kumaraswamy Distribution

Let $f(x)$ and $F(x)$ be the pdf and the cdf of Kumaraswamy distribution, respectively. Based on the Definition 2.1, the cdf of the Uniform-Kumaraswamy (Uni-Kums) distribution is obtained

$$K(x) = \frac{\exp(-\theta F(x)) - 1}{\exp(-\theta) - 1} = \frac{\exp(-\theta[1 - (1 - x^a)^b]) - 1}{\exp(-\theta) - 1} \tag{3.9}$$

$$k(x) = \frac{\theta f(x)exp(-\theta F(x))}{1-\exp(-\theta)} = \frac{\theta abx^{a-1}(1-x^a)^{b-1}\exp(-\theta[1-(1-x^a)^b])}{1-\exp(-\theta)} \tag{3.10}$$

where $0 < x < 1, \quad a, b > 0$

Figure 2 demonstrates the pdfs and the cdfs of Uni-Kums distribution for different values of θ. The Uni-Kums distribution becomes more skewed (positively or negatively) according to the parameter θ values.

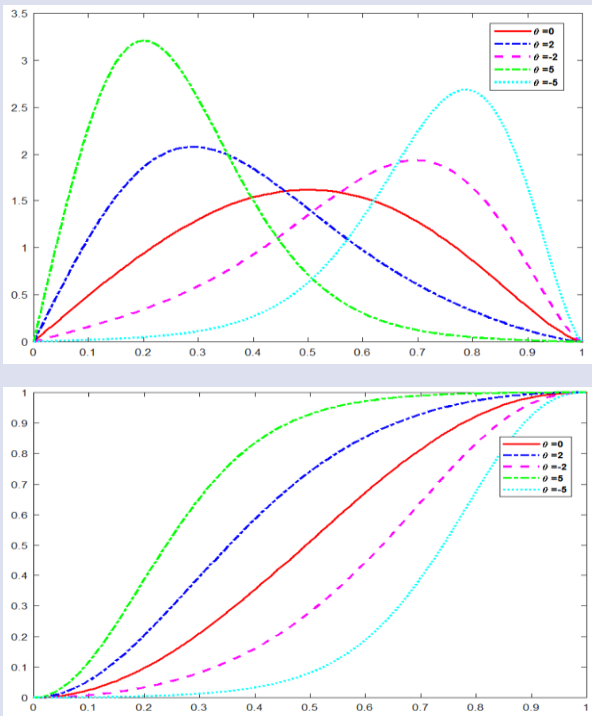

Figure 2. The pdfs and cdfs of Uni-Kums distribution, $a = 2$ and $b = 2.5$

The quantile function $Q(p)$, 0<p<1 and the median of the Uni-Kums distribution are defined as respectively.

$$Q(p)=\left(1-\left[1+\frac{\ln(1+p(\exp(-\theta)-1))}{\theta}\right]^{\frac{1}{b}}\right)^{\frac{1}{a}} \quad (3.11)$$

$$Q\left(\frac{1}{2}\right)=\left(1-\left[1+\frac{\ln(1+0.5(\exp(-\theta)-1))}{\theta}\right]^{\frac{1}{b}}\right)^{\frac{1}{a}} \quad (3.12)$$

Considering (3.11), the simulation procedure in sub-section 3.1 is applied to Uni-Kums distribution with the parameters $a=2$ and $b=2$. The results are listed in Table 2 indicate that θ tends to 0, the Uni-Kums distribution converges to the well-known Kumaraswamy distribution and when θ values increases the SV and the KV also increases.

Table 2. The simulation results for the Uni-Kums distribution, a=2, b=2

θ	Mean	Variance	SV	KV
-5	0.766	0.143	-1.056	4.348
-2	0.652	0.199	-0.640	2.840
-0.001	0.533	0.221	-0.123	2.193
0	0.533	0.221	-0.123	2.193
0.001	0.533	0.221	-0.123	2.193
2	0.412	0.208	0.375	2.466
5	0.290	0.159	0.763	3.745

Suppose $X_1, X_2, \dots, X_n$ are random variables from the Uni-Kums distribution defined in (3.9) and the likelihood function and the log-likelihood functions of this random sample are obtained as follows, respectively.

$$L=(a,b,\theta)=\Bigg((1-\exp(-\theta))^{-n}\theta^n a^n b^n \exp\left(-\theta\sum_{i=1}^{n}1-(1-x_i^a)^b\right)\Bigg)\prod_{i=1}^{n}x_i^{a-1}\prod_{i=1}^{n}(1-x_i^a)^{b-1} \quad (3.13)$$

$$l(a,b,\theta)=-n\ln(1-\exp(-\theta))+n\ln\theta+n\ln a+n\ln b+\sum_{i=1}^{n}(a-1)\ln x_i+\sum_{i=1}^{n}(b-1)\ln(1-x_i^a)-\theta\sum_{i=1}^{n}[1-(1-x_i^a)^b] \quad (3.14)$$

By differentiating the log-likelihood function with respect to the unknown parameters and equating them to zero we obtain the following likelihood equations

$$\frac{\partial lnL}{\partial a}=\frac{n}{a}+\sum_{i=1}^{n}lnx_i-(b-1)\sum_{i=1}^{n}\frac{1}{1-x_i^a}x_i^a lnx_i+\theta\sum_{i=1}^{n}b(1-x_i^a)^{b-1}x_i^a\, lnx_i$$

$$\frac{\partial lnL}{\partial b}=\frac{n}{b}+\sum_{i=1}^{n}\ln(1-x_i^a)+\theta\sum_{i=1}^{n}(1-x_i^a)^b\ln(1-x_i^a) \quad (3.15)$$

$$\frac{\partial lnL}{\partial \theta}=\frac{n}{\theta}-\frac{nexp(-\theta)}{1-exp(-\theta)}-\sum_{i=1}^{n}[1-(1-x_i^a)^b]$$

Solutions of (3.16) are called ML estimates and which are obtained by solving the equations with numerical methods.

Uniform-Weibull Distribution

In this sub-section, we consider $h(x)$ and $H(x)$ as the pdf and cdf of Weibull distribution, respectively. Considering the Definition 2.1, the cdf and pdf of the Uniform-Weibull (Uni-Weib) distribution are obtained as, respectively.

$$V(x)=\frac{\exp(-\theta H(x))-1}{\exp(-\theta)-1}=\frac{e^{-\theta\left[1-\exp(-(\frac{x}{\lambda})^k)\right]}-1}{\exp(-\theta)-1},\quad x,\lambda,k>0 \quad (3.16)$$

$$v(x)=\frac{\theta h(x)\exp(-\theta H(x))}{1-\exp(-\theta)}=\frac{\theta\frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1}\exp\left(-\left(\frac{x}{\lambda}\right)^k\right)\exp\left(-\theta\left[1-\exp\left(e^{-\left(\frac{x}{\lambda}\right)^k}\right)\right]\right)}{\exp(-\theta)-1},\quad x,\lambda,k>0 \quad (3.17)$$

The quantile function and the median of the Uni-Weib are defined, respectively.

$$Q(p)=\lambda\left[-ln\left(1+\frac{ln(p(\exp(-\theta)-1)+1)}{\theta}\right)\right]^{\frac{1}{k}} \quad (3.18)$$

$$Q\left(\frac{1}{2}\right)=\lambda\left[-ln\left(1+\frac{ln\left(\frac{1}{2}(\exp(-\theta)-1)+1\right)}{\theta}\right)\right]^{\frac{1}{k}} \quad (3.19)$$

Figure 3 illustrates the pdfs and the cdfs of the Uni-Weib distribution for some skewness parameter θ.

Same simulation procedure in the previous sub-sections is applied to Uni-Weib distribution with parameters $\lambda=1$ and $k=1.5$. The results for representative θ values are listed in Table 3. When θ tends to 0, the Uni-Weib distribution converges to the well-known Weibull distribution, as Figure 3 supports the results.

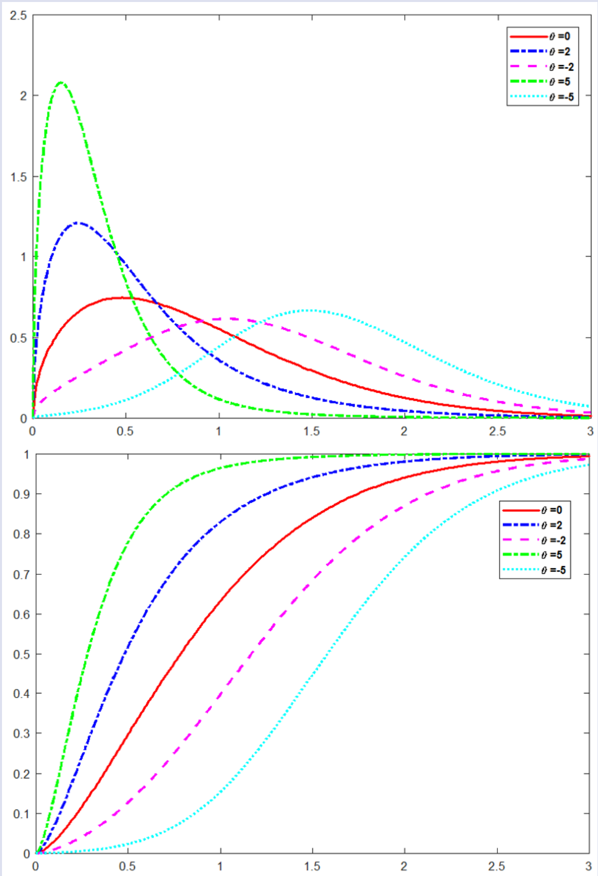

Figure 3. The pdfs and cdfs of Uni-Weib distribution, λ=1 and k=1.5

Same simulation procedure in the previous sub-sections is applied to Uni-Weib distribution with parameters λ=1 and $k = 1.5$. The results for representative θ values are listed in Table 3. When θ tends to 0, the Uni-Weib distribution converges to the well-known Weibull distribution, as Figure 3 supports the results.

Table 3. The simulation results for the Uni-Weib distribution, λ=1 and k=1.5

θ	Mean	Variance	SV	KV
-5	1.631	0.636	0.486	3.438
-2	1.238	0.661	0.632	3.376
-0.001	0.903	0.610	1.003	4.044
0	0.904	0.610	1.003	4.036
0.001	0.904	0.610	1.006	4.052
2	0.606	0.484	1.512	5.953
5	0.352	0.291	1.862	8.406

Suppose $X_1, X_2, \ldots, X_n$ are random variables from the Uni-Weib distribution defined in (3.20). The likelihood and log-likelihood functions are given, respectively in (3.22) and (3.23).

$$L(k,\lambda,\theta) = (1 - \exp(-\theta))^{-n}\theta^n k^n \lambda^{-n}\exp\left(-\theta\sum_{i=1}^{n}\left[1 - \exp(-(\tfrac{x_i}{\lambda})^k)\right]\right)\prod_{i=1}^{n}\left(\frac{x_i}{\lambda}\right)^{k-1}\exp\left(-\sum_{i=1}^{n}\left(\frac{x_i}{\lambda}\right)^{k-1}\right) \quad (3.22)$$

$$l(k,\lambda,\theta) = -nln(1-\exp(-\theta)) + nln\theta + nlnk - nln\lambda + \sum_{i=1}^{n}(k-1)ln\left(\frac{x_i}{\lambda}\right) - \sum_{i=1}^{n}\left(\frac{x_i}{\lambda}\right)^k - \theta\sum_{i=1}^{n}\left[1 - \exp(-(\tfrac{x_i}{\lambda})^k)\right]. \quad (3.23)$$

After differentiating the log-likelihood function with respect to the parameters and equating them to zero following likelihood equations are obtained.

$$\frac{\partial lnL}{\partial k} = \frac{n}{k} + \sum_{i=1}^{n}\left(\frac{x_i}{\lambda}\right) - \sum_{i=1}^{n}\left(\frac{x_i}{\lambda}\right)^k ln\left(\frac{x_i}{\lambda}\right) + \theta\sum_{i=1}^{n}\exp(-(\tfrac{x_i}{\lambda})^k)\left(\frac{x_i}{\lambda}\right)ln\left(\frac{x_i}{\lambda}\right).$$

$$\frac{\partial lnL}{\partial \lambda} = -n - \sum_{i=1}^{n}\frac{(k-1)}{x_i} + k\sum_{i=1}^{n}\left(\frac{x_i}{\lambda}\right)^{k-1} + \theta k\sum_{i=1}^{n}\exp(-(\tfrac{x_i}{\lambda})^k)\left(\frac{x_i}{\lambda}\right)^{k-1}. \quad (3.24)$$

$$\frac{\partial lnL}{\partial \theta} = -\frac{n}{\theta} - \frac{n\exp(-\theta)}{1-\exp(-\theta)} - \sum_{i=1}^{n}\left(1 - \exp(-(\tfrac{x_i}{\lambda})^k)\right).$$

Results and Discussion

In this section, we compare the results which are obtained by fitting distributions to the real data sets. Each extension which are introduced in Section 3 and their original distributions are fitted to the data, separately. Akaike Information Criterion (AIC) and log-likelihood (loglik) values are calculated to compare fitting performances of the original distribution and their extensions.

FG Scores Data

The first data set refers the Ferriman–Gallwey (FG) scores which are studied by [21]. FG score is a method of evaluating and quantifying hirsutism in women. The data set consists of FG scores of the women living in different areas of Turkey. The data set is 28.774; 27.958; 39.751; 22.659; 31.232; 32.990; 34.408; 34.920; 35.822; 23.685; 41.101; 35.879; 9.811; 24.689; 13.217; 22.343; 28.273; 27.340; 25.214; 14.960; 39.724; 35.557; 37.173; 25.412; 46.286; 31.564; 13.321; 29.606; 25.112; 18.158; 33.057; 22.683; 36.380; 31.451; 37.919; 25.729. The descriptive statistics of the FG Scores data are listed in Table 4.

Table 4. Descriptive Statistics for FG Scores Data

Mean	Std. Dev.	Skewness	Kurtosis
29.004	8.509	-0.374	2.816

Maximum likelihood estimates of Normal and Uni-Normal distributions are listed in Table 5. The Uni-Normal distribution gives best fit over Normal distribution according to the calculated log-likelihood and AIC values (see Table 5). Figures 4 also illustrates this conclusion.

Table 5. The ML Estimates, Values of Loglik and AIC for Normal and Uni-Normal Distributions

	$\hat{\mu}$	$\hat{\sigma}$	$\hat{\theta}$	loglik	AIC
Normal	21.004	8.509	-	-127.66	259.32
Uni-Normal	28.941	11.866	0.581	-109.27	224.54

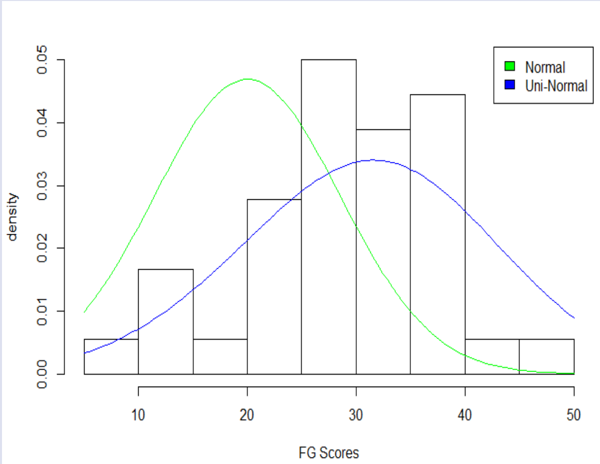

Figure 4. The histogram of FG Scores data with fitted pdfs

Failure Times Data

The second data refers the times of failure and running times for a sample of devices from a field-tracking study of a larger system. At a certain point in time, 30 units were installed in normal service conditions. Two causes of failure were observed for each unit that failed: the failure caused by an accumulation of randomly occurring damage from power-line voltage spikes during electric storms and failure caused by normal product wear. The data is studied by [222]. They proposed to fit Weibull distribution to the data. The times are 275, 13, 147, 23, 181, 30, 65, 10, 300, 173, 106, 300, 300, 212, 300, 300, 300, 2, 261, 293, 88, 247, 28,143, 300, 23, 300, 80, 245, 266. The descriptive statistics of the Failure Times data are summarized in Table 6.

Table 6. Descriptive Statistics for Failure Times Data

Mean	Std. Dev.	Skewness	Kurtosis
178.142	112.971	-0.323	1.449

The ML estimates of Weibull and its extension Uni-Weibull are obtained, and values of log-likelihood and AIC are listed in Table 7.

Table 7. The ML Estimates, Values of Loglik and AIC for Weibull and Uni-Weib Distributions

	$\hat{\lambda}$	$\hat{k}$	$\hat{\theta}$	loglik	AIC
Weibull	188.041	1.265	-	-184.041	372.08
Uni-Weib	184.855	1.264	-0.5182	-165.835	337.67

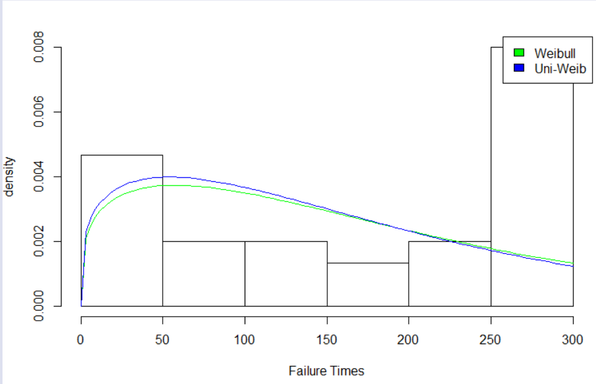

Figure 5. The histogram of Failure Times data with fitted pdfs

Table 7 and Figure 5 indicate that Uni-Weibull distribution gives best fit to the studied data over the well-known Weibull distribution.

Petroleum Reservoir Data

The last example has been studied by [23] and is about the shape measurements of 48 rock samples from a petroleum reservoir. The data set is: 0.09032, 0.14862, 0.18331, 0.11706, 0.12241, 0.16704, 0.18965, 0.16412, 0.20365, 0.16239, 0.15094, 0.14814, 0.22859,0.23162, 0.17256, 0.15348, 0.20431, 0.26272, 0.20007, 0.14481, 0.11385,0.29102, 0.24007, 0.16186, 0.28088, 0.17945, 0.19180, 0.13308, 0.22521, 0.34127, 0.31164, 0.27601, 0.19765, 0.32663, 0.15419, 0.27601,0.17696,0.43871,0.16358,0.25383,0.32864,0.23008, 0.46412, 0.42047, 0.20074, 0.26265, 0.18245, 0.20044.

Table 8 summarizes the descriptive statistics of the Petroleum Reservoir data.

Table 8. Descriptive Statistics for Petroleum Reservoir Data

Mean	Std. Dev.	Skewness	Kurtosis
0.218	0.083	1.208	4.372

Table 9. The ML Estimates, Values of Loglik and AIC for Kumaraswamy and Kums Distributions

	$\hat{a}$	$\hat{b}$	$\hat{\theta}$	loglik	AIC
Kumaraswamy	2.710	44.04	-	52.49	-100.98
Uni-Kums	2.729	24.701	-0.771	60.56	-115.13

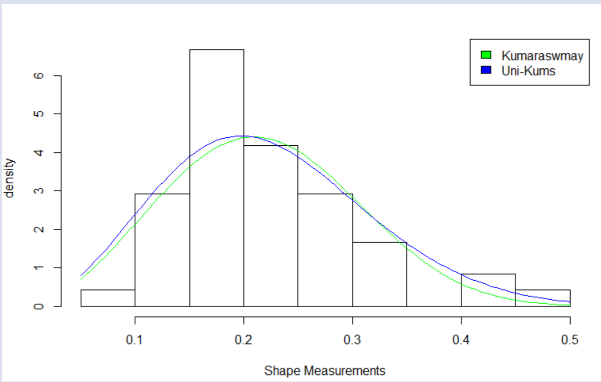

Figure 6. The histogram of Shape Measurements data with fitted pdfs

According to the results in Table 9 and the demonstration by Figure 6, the Uni-Kumaraswamy distribution can be proposed to model shape measurements data over the Kumaraswamy distribution.

Discussion and Conclusion

In this paper, we proposed a general extension form of T - X family of distributions with an additional parameter. We consider -T- as Uniform distribution, then call the new extensions of distributions as Uniform-X distributions. Three examples of Uniform-X distributions which are Uniform-Normal (Uni-Normal), Uniform-Kumaraswamy (Uni-Kums) and Uniform-Weibull (Uni-Weib) distributions are introduced. The properties of these distributions such as the density functions, the medians and the quantile functions are examined. Simulation studies are conducted for demonstrating the sample behavior for the mean, the variance, the skewness and the kurtosis for new distributions. Simulation results show that if the additional parameter θ tends to 0, the Uniform-X distributions converges to the original distributions. In the application section of the paper, the considered probability distributions and their extensions are compared in point of the fitting performances. For all the considered data sets, new extensions give better fit over the considered well-known probability distributions.

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Conflicts of Interest

The authors declared no conflict of interest.

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