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Some Identities with Special Numbers

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ABSTRACT

In this paper, we derive new identities which are related to some special numbers and generalized harmonic numbers $H_n(\alpha)$ by using the argument of the generating function given in [3] and comparing the coefficients of the generating functions. Also considering q -numbers involving q -Changhee numbers Ch_{nq} and q-Daehee numbers D_{nq}, some sums are given. For example, for any positive integer n and any positive real number $q > 1$, when$\alpha=\frac{q}{q-1}$, we have the relationship between generalized harmonic numbers and q -Daehee numbers.

Keywords: Harmonic numbers, Cauchy numbers of order r, $q-$ Changhee number, Generating functions

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Introduction

In [1], for any $\alpha \in \mathbb{R}^+$ and $n \in \mathbb{N}$, the generalized harmonic numbers $H_n(\alpha)$ are defined by

$$H_0(\alpha) = 0 \text{ and } H_n(\alpha) = \sum_{i=1}^{n} \frac{1}{i\alpha^i} \quad \text{for } n \geq 1. \qquad (1)$$

For $\alpha = 1$, the usual harmonic numbers are $H_n(1) = H_n$ and the generating function of $H_n(\alpha)$ is

$$\sum_{n=1}^{\infty} H_n(\alpha)t^n = -\frac{\ln\left(1-\frac{t}{\alpha}\right)}{1-t}. \qquad (2)$$

The works of Cauchy numbers of order r C_n^r, Daehee numbers of order r D_n^r, $q-$ Changhee numbers $Ch_{n,q}$, $q-$ Daehee numbers $D_{n,q}$ are given. Their combinatorial identities and relations have received much attention [2-7].

The Cauchy numbers of order r, denoted by C_n^r, are defined by the generating function

$$\sum_{n=0}^{\infty} C_n^r \frac{t^n}{n!} = \left(\frac{t}{\ln(1+t)}\right)^r \quad [13]. \qquad (3)$$

For $r = 1$, $C_n^1 = C_n$ are called Cauchy numbers.

The Daehee numbers of order r, denoted by D_n^r, are defined by the generating function

$$\sum_{n=0}^{\infty} D_n^r \frac{t^n}{n!} = \left(\frac{\ln(1+t)}{t}\right)^r \quad [11-13]. \qquad (4)$$

For $r = 1$, $D_n^1 = D_n$ are called Daehee numbers.

The Stirling numbers of the first kind $S_1(n,k)$ are defined by

$$x^{\underline{n}} = \sum_{k=0}^{n} S_1(n,k)x^k,$$

and the Stirling numbers of the second kind $S_2(n,k)$ are defined by

$$x^n = \sum_{k=0}^{n} S_2(n,k)x^{\underline{k}},$$

where $x^{\underline{n}}$ stands for the falling factorial defined by $x^{\underline{0}} = 1$ and $x^{\underline{n}} = x(x-1)\cdots(x-n+1)$. It is known that $S_1(n,k) = 0$ for $k > n$ and $S_1(n,n) = 1$.

The generating function of the Stirling numbers of the first kind $S_1(n,k)$ is given by

$$\sum_{n=k}^{\infty} S_1(n,k)\frac{t^n}{n!} = \frac{(\ln(1+t))^k}{k!}, k \geq 0, \qquad (5)$$

and the generating function of the Stirling numbers of the second kind $S_2(n,k)$ is given by

$$\sum_{n=k}^{\infty} S_2(n,k)\frac{t^n}{n!} = \frac{(e^t-1)^k}{k!}, k \geq 0 \quad [10]. \qquad (6)$$

Let $|S_1(n,k)|$ be the unsigned Stirling numbers of the first kind given by

$$x^{\overline{n}} = \sum_{k=0}^{n} |S_1(n,k)| x^k,$$

where $x^{\overline{n}}$ stands for the rising factorial defined by $x^{\overline{0}} = 1$ and $x^{\overline{n}} = x(x+1)\cdots(x+n-1)$. It is clear that $S_1(n,k) = (-1)^{n-k}|S_1(n,k)|$ [5].

The generating function of $|S_1(n,k)|$ is given by

$$\sum_{n=k}^{\infty} |S_1(n,k)| \frac{t^n}{n!} = \frac{(-\ln(1-t))^k}{k!}.$$

The numbers associated with $S_1(n,k)$ are given as follows: For $n < k$,

$$\rho(n,k) = \frac{|S_1(k,k-n)|}{\binom{k-1}{n}},$$

and for $n \geq k$,

$\rho(n,k) = n!\,\sigma_n(k)$,

where $\sigma_n(x)$ is the Stirling polynomial [5]. The generating function of these numbers is

$$\sum_{n=0}^{\infty} \rho(n,k) \frac{t^n}{n!} = \left(\frac{t}{1-e^{-t}}\right)^k. \quad (7)$$

It is clearly that $\rho(n,k) = B_n^{(k)}(k)$ is known as the classical Bernoulli polynomials of order k [9].

Let p be a fixed odd prime number. $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of $p-$adic integers, the field of $p-$adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$. The $p-$adic norm $|.|_p$ is normalized by $|p|_p = \frac{1}{p}$. Let q be an indeterminate in $\mathbb{C}_p$ such that $|1-q|_p < p^{\frac{-1}{p-1}}$. The $q-$ extension of number x is defined as $[x]_q = \frac{1-q^x}{1-q}$. It is clear that $\lim_{q\to 1}[x]_q = x$.

The $q-$ Changhee polynomials $Ch_{n,q}(x)$[4] are defined by the generating function

$$\sum_{n=0}^{\infty} Ch_{n,q}(x) \frac{t^n}{n!} = \frac{1+q}{1+q(1+t)}(1+t)^x. \quad (8)$$

When $x = 0$, $Ch_{n,q} = Ch_{n,q}(0)$ are called $q-$ Changhee numbers and when $q = 1$, $Ch_n = Ch_{n,1}(0)$ are called Changhee numbers.

The $q-$Daehee polynomials $D_{n,q}(x)$[7] are defined by the generating function

$$\sum_{n=0}^{\infty} D_{n,q}(x) \frac{t^n}{n!} = \frac{1-q+\frac{1-q}{\ln q}\ln(1+t)}{1-q-qt}(1+t)^x. \quad (9)$$

In the special case, when $q = 1$, $D_n(x) = D_{n,1}(x)$ are called Daehee polynomials and when $x = 0$, $D_{n,q} = D_{n,q}(0)$ are called $q-$Daehee numbers.

Let $f(t)$ be a generating function (a power series) for a sequence $\{A_n\}$, the sequence of coefficients of the expansion of $f(t)^r$ is defined by $A_n^{(r)}$, where r is a fixed real nonzero number:

$$f(t) = \sum_{n=0}^{\infty} A_n \frac{t^n}{n!}, \quad f(t)^r = \sum_{n=0}^{\infty} A_n^{(r)} \frac{t^n}{n!} \quad (10)$$

absolutely convergent in a neighborhood of the origin.

Suppose $f(t)$ has a subsidiary generating function $g(t)$ so that

$$f(t) = \left(1+g(t)\right)^{-1}, \quad |g(t)| < 1 \quad \text{and} \quad g(t)^n = \sum_{m=M(n)}^{\infty} a_m^{(n)} \frac{t^m}{m!}, \quad (11)$$

where $M(n)$ is a non-negative integer. Note that $g(t) = \sum_{m=0}^{\infty} a_m \frac{t^m}{m!}$ [8].

In [2], let

$$a(m,k) = (-1)^k \sum_{n=k}^{M^{-1}(m)} \frac{1}{n!} S_1(n,k)\, a_m^{(n)}, \quad (12)$$

where $M^{-1}(m)$ indicates the inverse function of M (in most cases, it is simply $M^{-1}(m) = m$). Then

$$A_m^{(r)} = \sum_{k=1}^{M^{-1}(m)} a(m,k) r^k, m \geq 1. \quad (13)$$

Also Liu gave the sum as follows:

$$A_m^{(r)} = \sum_{i=0}^{M^{-1}(m)} \binom{-r}{i} a_m^{(i)}. \quad (14)$$

In [3], Kim et. al. gave obvious formula for coefficients of the expansion of given generating function, when that function has a suitable form, the coefficients can be represented by the Daehee numbers of order r and the Changhee numbers of order r. By the classical method of comparing the coefficients of the generating function, some identities related to these numbers were shown. For example,

$$D_n^r = \sum_{m=0}^{n} B_m^{(r)} S_1(n,m),$$

where $B_n^{(r)}$ are the Bernoulli numbers of order r.

In this paper, we derive new identities which are related to some special numbers by using the argument of the generating function given in [2]. For example, for any positive integer n and any positive real number $q \neq 1$,

$$\sum_{i=0}^{n-1} \left(\frac{1-q}{q}\right)^{i+1} \frac{D_i}{i!} = \ln q \left(D_{n,q} \frac{(1-q)^n}{n!\,q^n} - 1\right),$$

and for any positive integers n and r,

$$C_n^r=\sum_{m=0}^{n}\sum_{k=0}^{m}(-1)^k\binom{m}{k}\binom{r+m-1}{m}D_n^k.$$

Some identities with special numbers

In this section, we will give some identities involving generalized harmonic numbers, Cauchy numbers of order r, $q-$Changhee numbers and $q-$Daehee numbers.

Theorem 1. For any positive integer n and any positive real number $q>1$, we have

$$H_n\left(\frac{q}{q-1}\right)=\ln q\left(1-D_{n,q}\frac{(1-q)^n}{n!\,q^n}\right).$$

Proof. From (2) and (9), we have

$$\sum_{n=0}^{\infty}(-1)^nD_{n,q}\frac{t^n}{n!}=\frac{1-q}{1-q+qt}+\frac{1-q}{\ln q}\frac{1-t}{1-q+qt}\frac{\ln(1-t)}{1-t}$$

$$=\frac{1-q}{1-q+qt}-\frac{1-q}{\ln q}\frac{1-t}{1-q+qt}\sum_{k=0}^{\infty}H_k\,t^k$$

$$=\frac{1-q}{1-q+qt}+\frac{1}{\ln q}\frac{1-q}{1-q+qt}\left(\sum_{k=0}^{\infty}H_k\,t^{k+1}-\sum_{k=0}^{\infty}H_k\,t^k\right)$$

and by $\sum_{k=0}^{\infty}x^k=\frac{1}{1-x}$, equals to

$$\sum_{n=0}^{\infty}(-1)^n\frac{q^n}{(1-q)^n}t^n+\frac{1}{\ln q}\sum_{n=0}^{\infty}(-1)^n\frac{q^n}{(1-q)^n}t^n\left(\sum_{k=1}^{\infty}H_{k-1}\,t^k-\sum_{k=0}^{\infty}H_k\,t^k\right)$$

$$=\sum_{n=0}^{\infty}(-1)^n\frac{q^n}{(1-q)^n}t^n-\frac{1}{\ln q}\sum_{n=0}^{\infty}(-1)^n\frac{q^n}{(1-q)^n}t^n\sum_{k=0}^{\infty}H_k\,t^k+\frac{1}{\ln q}\sum_{n=0}^{\infty}(-1)^n\frac{q^n}{(1-q)^n}t^n\sum_{k=1}^{\infty}H_{k-1}\,t^k$$

and by some combinatoric operations,

$$\sum_{n=0}^{\infty}(-1)^nD_{n,q}\frac{t^n}{n!}$$

$$=\sum_{n=0}^{\infty}(-1)^n\frac{q^n}{(1-q)^n}t^n-\frac{1}{\ln q}\sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^k\frac{q^k}{(1-q)^k}H_{n-k}t^n+\frac{1}{\ln q}\sum_{n=1}^{\infty}\sum_{k=0}^{n-1}(-1)^k\frac{q^k}{(1-q)^k}H_{n-k-1}t^n$$

$$=\sum_{n=0}^{\infty}\left((-1)^n\frac{q^n}{(1-q)^n}+\frac{1}{\ln q}\sum_{k=0}^{n-1}(-1)^{k+1}\frac{q^k}{(1-q)^k}\frac{1}{n-k}\right)t^n.$$

Hence, by comparing the coefficients of t^n above gives

$$\frac{D_{n,q}}{n!}=\frac{q^n}{(1-q)^n}+\frac{1}{\ln q}\sum_{k=0}^{n-1}(-1)^{n+k+1}\frac{q^k}{(1-q)^k}\frac{1}{n-k}.$$

Thus, from (1), the desired result is obtained.

Corollary 1. For any positive integer n and any positive real number $q\neq 1$, we have

$$\sum_{i=0}^{n-1}\left(\frac{1-q}{q}\right)^{i+1}\frac{D_i}{i!}=\ln q\left(D_{n,q}\frac{(1-q)^n}{n!\,q^n}-1\right).$$

Proof. From Theorem 1, we obtain

$$\ln q\left(1-D_{n,q}\frac{(1-q)^n}{n!\,q^n}\right)=\sum_{i=1}^{n}\frac{(-1)^i(1-q)^i}{iq^i}=-\sum_{i=0}^{n-1}\frac{(-1)^i(1-q)^{i+1}}{q^{i+1}}\frac{i!}{(i+1)!},$$

and by Daehee number $D_n=\frac{(-1)^n}{n+1}n!$,

$$\ln q\left(D_{n,q}\frac{(1-q)^n}{n!\,q^n}-1\right)=\sum_{i=0}^{n-1}\left(\frac{1-q}{q}\right)^{i+1}\frac{D_i}{i!},$$

as claimed.

Theorem 2. For any positive integers n and r, we have

$$\rho(n,r)=\sum_{i=0}^{n}\sum_{m=0}^{n}\sum_{k=0}^{i}(-1)^{k+n}\binom{r+i-1}{i}\binom{i}{k}S_2(n,m)C_m^k.$$

Proof. For $f(t)=\frac{t}{1-e^{-t}}$, by (11) and Binomial theorem, we have

$$g(t)^i=\left(\frac{e^{-t}-1}{\ln\left(1+(e^{-t}-1)\right)}-1\right)^i=\sum_{k=0}^{i}(-1)^{i-k}\binom{i}{k}\left(\frac{e^{-t}-1}{\ln\left(1+(e^{-t}-1)\right)}\right)^k.$$

From (3) and (6), we have

$$\begin{aligned}g(t)^i&=\sum_{k=0}^{i}(-1)^{i-k}\binom{i}{k}\sum_{m=0}^{\infty}C_m^k\frac{(e^{-t}-1)^m}{m!}\\&=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{k=0}^{i}(-1)^{i-k+n}\binom{i}{k}C_m^kS_2(n,m)\frac{t^n}{n!},\end{aligned}$$

and by (11),

$$a_n^{(i)}=\sum_{m=0}^{n}\sum_{k=0}^{i}(-1)^{i-k+n}\binom{i}{k}C_m^kS_2(n,m).$$

Note that for integers $r\geq 1$ and $j\geq 0$,

$$\binom{-r}{j}=(-1)^j\binom{r+j-1}{j}. \tag{15}$$

Then, by (14), we have

$$A_n^{(r)}=\sum_{i=0}^{n}\sum_{m=0}^{n}\sum_{k=0}^{i}(-1)^{k+n}\binom{r+i-1}{i}\binom{i}{k}S_2(n,m)C_m^k.$$

(7) and (10) give that

$$\sum_{n=0}^{\infty}A_n^{(r)}\frac{t^n}{n!}=\left(\frac{t}{1-e^{-t}}\right)^r=\sum_{n=0}^{\infty}\rho(n,r)\frac{t^n}{n!}.$$

Thus, comparing the coefficients of $\frac{t^n}{n!}$, the desired result is obtained.

Theorem 3. For any positive integers n and r, we have

$$C_n^r = \sum_{i=0}^{n} \sum_{k=0}^{i} (-1)^k \binom{r+i-1}{i} \binom{i}{k} D_n^k.$$

Proof. We take $f(t) = \frac{t}{\ln(1+t)}$ for using (11). From Binomial theorem and (4), we have

$$g(t)^i = \left(\frac{\ln(1+t)}{t} - 1\right)^i = \sum_{k=0}^{i} (-1)^{i-k} \binom{i}{k} \left(\frac{\ln(1+t)}{t}\right)^k$$
$$= \sum_{k=0}^{i} (-1)^{i-k} \binom{i}{k} \sum_{n=0}^{\infty} D_n^k \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{k=0}^{i} (-1)^{i-k} \binom{i}{k} D_n^k \frac{t^n}{n!},$$

which equals by (11),

$$a_n^{(i)} = \sum_{k=0}^{i} (-1)^{i-k} \binom{i}{k} D_n^k.$$

From here, by (14) and (15), we obtain that

$$A_n^{(r)} = \sum_{i=0}^{n} \sum_{k=0}^{i} (-1)^k \binom{i}{k} \binom{r+i-1}{i} D_n^k,$$

and from (7) and (10),

$$\sum_{n=0}^{\infty} A_n^{(r)} \frac{t^n}{n!} = \sum_{n=0}^{\infty} C_n^r \frac{t^n}{n!}.$$

Thus, we have the proof.

Theorem 4. For any positive integers n and r, we have

$$\sum_{i=0}^{n} (-1)^n S_2(n,i) C_i^r = \sum_{i=0}^{n} \sum_{k=0}^{i} (-1)^k \binom{i}{k} \binom{r+i-1}{i} \rho(n,k).$$

Proof. By (11), we note that

$$f(t) = \frac{e^{-t}-1}{\ln\left(1+(e^{-t}-1)\right)} \text{ and } g(t) = \frac{t-1+e^{-t}}{1-e^{-t}}.$$

From Binomial theorem, (6) and (7) , we have

$$g(t)^i = \left(\frac{t}{1-e^{-t}} - 1\right)^i = \sum_{k=0}^{i} (-1)^{i-k} \binom{i}{k} \left(\frac{t}{1-e^{-t}}\right)^k$$
$$= \sum_{k=0}^{i} (-1)^{i-k} \binom{i}{k} \sum_{n=0}^{\infty} \rho(n,k) \frac{t^n}{n!}$$
$$= \sum_{n=0}^{\infty} \sum_{k=0}^{i} (-1)^{i-k} \binom{i}{k} \rho(n,k) \frac{t^n}{n!},$$

and using (11),

$$a_n^{(i)} = \sum_{k=0}^{i} (-1)^{i-k} \binom{i}{k} \rho(n,k).$$

Hence, (14) and (15) yield that

$$A_n^{(r)}=\sum_{i=0}^{n}\sum_{k=0}^{i}(-1)^k\binom{i}{k}\binom{r+i-1}{i}\rho(n,k).$$

From (3), (6) and (10), we obtain that

$$\sum_{n=0}^{\infty}A_n^{(r)}\frac{t^n}{n!}=f(t)^r=\left(\frac{e^{-t}-1}{\ln\left(1+(e^{-t}-1)\right)}\right)^r$$
$$=\sum_{i=0}^{\infty}C_i^r\frac{(e^{-t}-1)^i}{i!}=\sum_{i=0}^{\infty}C_i^r\sum_{n=i}^{\infty}(-1)^nS_2(n,i)\frac{t^n}{n!}$$
$$=\sum_{n=0}^{\infty}\sum_{i=0}^{n}(-1)^nS_2(n,i)C_i^r\frac{t^n}{n!}.$$

Thus, comparing the coefficients of $\frac{t^n}{n!}$, we have the proof.

Now, for any positive integers r, we have $q-$ numbers $\binom{n+r-1}{r-1}Ch_{n,q}$ given by

$$\left(\frac{1+q}{q(1+t)+1}\right)^r=\sum_{n=0}^{\infty}\binom{n+r-1}{r-1}Ch_{n,q}\frac{t^n}{n!}. \tag{16}$$

Theorem 5. For any positive integers n and r, we have

$$\binom{r}{n}\sum_{i=0}^{r}\binom{r-n}{i-n}q^i=\frac{(1+q)^r}{n!}Ch_{n,q}\sum_{i=0}^{n}\sum_{k=0}^{i}(-1)^k\binom{i}{k}\binom{r+i-1}{i}\binom{n+k-1}{k-1}.$$

Proof. For $f(t)=\frac{q(1+t)+1}{1+q}$, by (11), we have

$$g(t)=\frac{-qt}{q(1+t)+1}.$$

From Binomial theorem, we have

$$f(t)^r=\left(\frac{q(1+t)+1}{1+q}\right)^r=\frac{1}{(1+q)^r}(q(1+t)+1)^r$$
$$=\frac{1}{(1+q)^r}\sum_{i=0}^{r}\binom{r}{i}q^i(1+t)^i=\frac{1}{(1+q)^r}\sum_{n=0}^{\infty}\sum_{i=0}^{r}\binom{r}{i}\binom{i}{n}q^it^n \tag{17}$$

which, by Binomial theorem and (16), we write

$$g(t)^i=\left(\frac{1+q}{q(1+t)+1}-1\right)^i=\sum_{k=0}^{i}(-1)^{i-k}\binom{i}{k}\left(\frac{1+q}{q(1+t)+1}\right)^k$$
$$=\sum_{n=0}^{\infty}\sum_{k=0}^{i}(-1)^{i-k}\binom{i}{k}\binom{n+k-1}{k-1}Ch_{n,q}\frac{t^n}{n!}.$$

Hence, with the help of (11), by comparing coefficients of t^n, we obtain that

$$a_n^{(i)}=\sum_{k=0}^{i}(-1)^{i-k}\binom{i}{k}\binom{n+k-1}{k-1}Ch_{n,q}.$$

By (10), (14) and (15), we get

$$A_n^{(r)} = \sum_{i=0}^{n}\sum_{k=0}^{i}(-1)^k\binom{i}{k}\binom{r+i-1}{i}\binom{n+k-1}{k-1}Ch_{n,q},$$

and

$$f(t)^r = \sum_{n=0}^{\infty}\sum_{i=0}^{n}\sum_{k=0}^{i}(-1)^k\binom{i}{k}\binom{r+i-1}{i}\binom{n+k-1}{k-1}Ch_{n,q}\frac{t^n}{n!}. \tag{18}$$

Finally, (17) and (18) give that

$$\sum_{i=0}^{r}\binom{r}{i}\binom{i}{n}q^i = \frac{(1+q)^r}{n!}Ch_{n,q}\sum_{i=0}^{n}\sum_{k=0}^{i}(-1)^k\binom{i}{k}\binom{r+i-1}{i}\binom{n+k-1}{k-1}.$$

By the equality $\binom{r}{i}\binom{i}{n} = \binom{r}{n}\binom{r-n}{i-n}$, we have the proof.

Theorem 6. For any positive integers n and r, we have

$$\sum_{i=1}^{n}\sum_{k=0}^{i}\sum_{j=0}^{k}(-1)^k\binom{i}{k}\binom{k}{j}\binom{r+i-1}{i}\binom{n+j-1}{j-1}\frac{q^j}{(1+q)^k} = (1+q)^{r-n}\sum_{k=0}^{r}(-1)^k\binom{r}{k}\binom{n+k-1}{k-1}\frac{q^k}{(1+q)^k}.$$

Proof. The proof is similar to the proof of above theorems, taking $f(t) = (1+q)\frac{1+t}{1+q+t}$ and using the generating function

$$\sum_{n=0}^{\infty}\binom{n+r-1}{r-1}\frac{(-1)^n}{(1+q)^n}t^n = \frac{(1+q)^r}{(1+q+t)^r}.$$

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Conflicts of interest.

There are no conflicts of interest in this work.

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