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Investigation of the Asymptotic Behavior of Generalized Baskakov-Durrmeyer-Stancu Type Operators

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ABSTRACT

In this manuscript, we firstly find the Korovkin test functions for the Baskakov operators, secondly, we find the generalized Baskakov-Durrmeyer-Stancu type operators. Thirdly, we give the modulus of continuity for the generalized Baskakov-Durrmeyer-Stancu type operators. Then, the asymptotic approach of these operators has been studied by using the Voronovskaja-type theorem. Finally, it is demonstrated that the generalized Baskakov-Durrmeyer-Stancu type operators converge to the considered function by plotting the graphs. Moreover, the convergence of the generalized Baskakov-Durrmeyer-Stancu type operators is compared with that of some other operators to the same function.

Keywords: Voronovskaja type theorem, Baskakov Durrmeyer Stancu type operators, Baskakov operators

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Introduction

Weierstrass approximation theorem has played a key role in the development of approximation theory [1]. With the help of this theorem, the approximation theory of linear positive operators has emerged by using suitable sequences defined by several mathematicians.
In [2], Bernstein defined the linear positive operators and showed that these operators converged smoothly to a continuous function in a closed interval.
In [3], for $f \in C[0,1]$, Stancu introduced the following linear positive operators

$$S_n^{\alpha,\beta}(f,x) = \sum_{k=0}^{n} f\left(\frac{k+\alpha}{n+\beta}\right)\binom{n}{k} x^k (1-x)^{n-k} \tag{1}$$

where $x \in [0,1]$, the parameters α and β satisfy the conditions $0 \leq \alpha \leq \beta$. He examined the convergence properties of the operators (1), which are called Bernstein-Stancu type operators, in the interval [0,1].
In [4], for $f \in C[0,\infty)$, $n \in \mathbb{N}$, Baskakov defined the linear positive operators as follows:

$$B_n(f,x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) P_{n,k}(x) \tag{2}$$

where $P_{n,k}(x) = \binom{n+k-1}{k} x^k (1+x)^{n-k}$, $x \in [0,\infty)$ is the core of the Baskakov operators.

The convergence theorems of the bounded and continuous functions for the operators (2) were studied by Baskakov.

In [5], Mihesan introduced the generalized Baskakov operators with a constant $a \geq 0$ independent of n and defined as follows:

$$B_n^a(f,x) = \sum_{k=0}^{n} W_{n,k}^a(x) f\left(\frac{k}{n}\right) \tag{3}$$

where

$$W_{n,k}^a(x) = e^{-\frac{ak}{1+x}} \frac{P_k(n,a)}{k!} x^k (1+x)^{-n-k} \tag{4}$$

and

$$P_k(n,a) = \sum_{j=0}^{k} \binom{k}{j} (n)_j \, a^{k-j} \tag{5}$$

with $(n)_0 = 1$, $(n)_j = n(n+1)(n+2)\dots(n+j-1)$ for $j \geq 1$.

He proved that these operators converged uniformly on $[0,b]$ for functions that had exponential growth. Also, he discussed a pointwise estimate. In addition, Wafi and Khatoon [6] calculated the rate of convergence of the operators (3) and obtained the Voronovskaja-type theorem. Erencin and Başcanbaz-Tunca [7] studied the weighted approximation properties and estimated the order of approximation in terms of the usual modulus of continuity for the operators (3). They derived a recurrence relation for the moments of these operators.

To approximate the space of the integrable functions, Durrmeyer [8] defined Durrmeyer operators, which is an integral type generalization of Bernstein operators, and Lupaş [9] developed these operators independently.

In [10], for $f \in C_B[0,\infty)$, $n \in \mathbb{N}$, Erencin introduced the Durrmeyer-type modification of the operators (3) as follows:

$$L_n^a(f,x) = \sum_{k=0}^{\infty} W_{n,k}^a(x) \frac{1}{B(k+1,n)} \int_0^{\infty} \frac{t^k}{(1+t)^{n+k+1}} f(t)dt \quad (6)$$

where $C_B[0,\infty)$ stands for the space of all bounded-continuous functions on the interval $[0,\infty)$, and this space is equipped with the norm $\|f\| = \max_{x\in[0,\infty)} |f(x)|$, and the beta function $B(k+1,n)$ is given by

$$B(x,y) = \int_0^{\infty} \frac{t^{x-1}}{(1+t)^{x+y}} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad x,y > 0 \quad (7)$$

In this study, Erencin gave some approximation properties of the operators (6).

Furthermore, the approximation properties of the modified forms of the operators (6) have been reviewed by Agrawal et al. [11].

In [12], Kumar et al. defined the following Stancu-type generalization of the Durrmeyer-type modification of the operators (6) for $f \in \mathcal{L}$ and $n \in \mathbb{N}$,

$$G_{n,a}^{\alpha,\beta}(f,x) = \sum_{k=0}^{\infty} W_{n,k}^a(x) \frac{1}{B(k+1,n)} \int_0^{\infty} \frac{t^k}{(1+t)^{n+k+1}} f\left(\frac{nk+\alpha}{n+\beta}\right) dt \quad (8)$$

where α and β are non-negative numbers with $0 \le \alpha \le \beta$, and $\mathcal{L}$ denotes the class of all Lebesque measurable function such that $n > m$ with $\int_0^{\infty} \frac{|f(t)|}{(1+t)^m} dt < \infty$, $m \in \mathbb{Z}^+$.

They studied some direct local approximation properties of the operators (8). They obtained local direct results in terms of the second-order modulus of smoothness, the rate of convergence in terms of the modulus of continuity. Several studies have been carried out some approximation properties for these types of operators are given in [14-17].

In this study, we examined the asymptotic behavior of the generalized Baskakov-Durrmeyer-Stancu type operators defined by (8) with the help of the Voronovskaja-type theorem.

Some Auxiliary Lemmas

In this section, some lemmas will be given for examining the approximation properties of the generalized Baskakov-Durrmeyer-Stancu type operators defined by (8). The proofs of Lemma 2.1 and Lemma 2.2 given below are routine

Lemma 2.1 For $W_{n,k}^a(x)$ given by (4), we have the following equation:

$$\sum_{k=0}^{\infty} W_{n,k}^a(x) = 1.$$

The Korovkin test functions for the Baskakov operators expressed in (3) are given below.

Lemma 2.2 Let $e_m(t) = t^m$, for $m = 0,1,2,3,4$. For $n \in \mathbb{N}$ and a is a non-negative integer, we have the following equations:

(i) $B_n^a(e_0(t),x) = 1.$

(ii) $B_n^a(e_1(t),x) = \frac{x}{n}\left\{\frac{a}{1+x} + n\right\}.$

(iii) $B_n^a(e_2(t),x) = \frac{x^2}{n^2}\left\{\frac{a^2}{(1+x)^2} + \frac{2an}{1+x} + n(n+1)\right\} + \frac{x}{n^2}\left\{\frac{a}{1+x} + n\right\}.$

(iv) $$B_n^a(e_3(t),x) = \frac{x^3}{n^3}\left\{\frac{a^3}{(1+x)^3} + \frac{3a^2n}{(1+x)^2} + \frac{3a(n+1)}{1+x} + n(n+1)(n+2)\right\} + \frac{3x^2}{n^3}\left\{\frac{a^2}{(1+x)^2} + \frac{2an}{1+x} + n(n+1)\right\} + \frac{x}{n^3}\left\{\frac{a}{1+x} + n\right\}.$$

(v) $$B_n^a(e_4(t),x)=\frac{x^4}{n^4}\Big\{\frac{a^4}{(1+x)^4}+\frac{4a^3n}{(1+x)^3}+\frac{6a^2n(n+1)}{(1+x)^2}+\frac{4an(n+1)(n+2)}{1+x}+n(n+1)(n+2)(n+3)\Big\}+\frac{6x^3}{n^4}\Big\{\frac{a^3}{(1+x)^3}+\frac{3a^2n}{(1+x)^2}+\frac{3a(n+1)}{1+x}+n(n+1)(n+2)\Big\}+\frac{7x^2}{n^4}\Big\{\frac{a^2}{(1+x)^2}+\frac{2an}{1+x}+n(n+1)\Big\}+\frac{x}{n^4}\Big\{\frac{a}{1+x}+n\Big\}.$$

Now, we give the following lemmas that give the Korovkin test functions and continuity modules for the generalized Baskakov-Durrmeyer-Stancu type operators defined in (8).

For the sake of shortness, the following abbreviations will be used in the next steps,

$$M_\beta(n)=(n+\beta)\quad,\qquad U_s(n)=\prod_{i=1}^{s}(n-i).$$

Lemma 2.3 Let $e_m(t)=t^m$ for $m=0,1,2,3,4,$ and we get the following equations for the $G_{n,a}^{\alpha,\beta}(f(t),x)$ operators defined in (8):

(i) $$G_{n,a}^{\alpha,\beta}(e_0(t),x)=1.$$

(ii) $$G_{n,a}^{\alpha,\beta}(e_1(t),x)=x\Big\{\frac{n}{M_\beta(n)U_1(n)}\Big(\frac{a}{1+x}+n\Big)\Big\}+\frac{n}{M_\beta(n)U_1(n)}+\frac{a}{M_\beta(n)}.$$

(iii) $$G_{n,a}^{\alpha,\beta}(e_2(t),x)=x^2\Big\{\frac{n^2}{[M_\beta(n)]^2U_2(n)}\Big(\frac{a^2}{(1+x)^2}+\frac{2an}{1+x}+n(n+1)\Big)\Big\}+x\Big\{\frac{4n^2+2na(n-2)}{[M_\beta(n)]^2U_2(n)}\Big(\frac{a}{1+x}+n\Big)\Big\}+\Big\{\frac{2n^2+2na(n-2)}{[M_\beta(n)]^2U_2(n)}+\frac{\alpha^2}{[M_\beta(n)]^2}\Big\}.$$

(iv) $$G_{n,a}^{\alpha,\beta}(e_3(t),x)=x^3\Big\{\frac{n^3}{[M_\beta(n)]^3U_3(n)}\Big(\frac{a^3}{(1+x)^3}+\frac{3a^2n}{(1+x)^2}+\frac{3an(n+1)}{1+x}+n(n+1)(n+2)\Big)\Big\}+x^2\Big\{\frac{9n^3+3n^2a(n-3)}{[M_\beta(n)]^3U_3(n)}\Big(\frac{a^2}{(1+x)^2}+\frac{2an}{1+x}+n(n+1)\Big)\Big\}+x\Big\{\frac{18n^3+12n^2a(n-3)+3na^2(n-2)(n-3)}{[M_\beta(n)]^3U_3(n)}\Big(\frac{a}{1+x}+n\Big)\Big\}+\Big\{\frac{6n^3+6n^2a(n-3)+3na^2(n-2)(n-3)}{[M_\beta(n)]^3U_3(n)}+\frac{a^3}{[M_\beta(n)]^3}\Big\}.$$

(v) $$G_{n,a}^{\alpha,\beta}(e_4(t),x)=x^4\Big\{\frac{n^4}{[M_\beta(n)]^4U_4(n)}\Big(\frac{a^4}{(1+x)^4}+\frac{4a^3n}{(1+x)^3}+\frac{6a^2n(n+1)}{(1+x)^2}+\frac{4an(n+1)(n+2)}{1+x}+n(n+1)(n+2)(n+3)\Big)\Big\}+x^3\Big\{\frac{16n^4+4n^3\alpha(n-4)}{[M_\beta(n)]^4U_4(n)}\Big(\frac{a^3}{(1+x)^3}+\frac{3a^2n}{(1+x)^2}+\frac{3an(n+1)}{1+x}+n(n+1)(n+2)\Big)\Big\}+x^2\Big\{\frac{72n^4+36n^3\alpha(n-4)+6n^2\alpha^2(n-3)(n-4)}{[M_\beta(n)]^4U_4(n)}\Big(\frac{a^2}{(1+x)^2}+\frac{2an}{1+x}+n(n+1)\Big)\Big\}+x\Big(\frac{a}{1+x}+n\Big)\Big\{\frac{96n^4+72n^3\alpha(n-4)}{[M_\beta(n)]^4U_4(n)}+\frac{24n^2\alpha^2(n-3)(n-4)+4n\alpha^3(n-2)(n-3)(n-4)}{[M_\beta(n)]^4U_4(n)}\Big\}+\frac{24n^4+24n^3\alpha(n-4)+12n^2\alpha^2(n-3)(n-4)}{[M_\beta(n)]^4U_4(n)}+\frac{4n\alpha^3(n-2)(n-3)(n-4)}{[M_\beta(n)]^4U_4(n)}+\frac{a^4}{[M_\beta(n)]^4}.$$

Proof: (i) Taking $e_m(t)=t^m$, $m=0$ in operators $G_{n,a}^{\alpha,\beta}(e_0(t),x)$ and using the beta function in (7) and Lemma 2.1, we can write the following equation:

$$G_{n,a}^{\alpha,\beta}(e_0(t),x)=\sum_{k=0}^{\infty}W_{n,k}^{a}(x)\frac{1}{B(k+1,n)}\int_0^\infty\frac{t^k}{(1+t)^{n+k+1}}dt,$$

$$=\sum_{k=0}^{\infty}W_{n,k}^{a}(x)\frac{1}{B(k+1,n)}B(k+1,n)=1.$$

(ii) Substituting $e_m(t)=t^m$ for $m=1$ in operators $G_{n,a}^{\alpha,\beta}(e_m(t),x)$, we have

$$G_{n,a}^{\alpha,\beta}(e_1(t),x)=\sum_{k=0}^{\infty}W_{n,k}^{a}(x)\frac{1}{B(k+1,n)}\int_0^\infty\frac{t^k}{(1+t)^{n+k+1}}\left(\frac{nt+\alpha}{n+\beta}\right)dt.$$

Using the beta function in (7), we get

$$G_{n,a}^{\alpha,\beta}(e_1(t),x)=\sum_{k=0}^{\infty}W_{n,k}^{a}(x)\frac{1}{B(k+1,n)}\frac{1}{(n+\beta)}\left\{n\frac{(k+1)}{(n-1)}B(k+1,n)+\alpha B(k+1,n)\right\}.$$

Hence, we find

$$G_{n,a}^{\alpha,\beta}(e_1(t),x)=\frac{1}{(n+\beta)}\left\{\frac{n}{(n-1)}\sum_{k=0}^{\infty}W_{n,k}^{a}(x)(k+1)+\alpha\sum_{k=0}^{\infty}W_{n,k}^{a}(x)\right\},$$

$$=\frac{1}{M_\beta(n)}\left\{\frac{n}{U_1(n)}[nB_n^a(t,x)+1]+\alpha\right\},$$

$$=x\left\{\frac{n}{M_\beta(n)U_1(n)}\left(\frac{a}{1+x}+n\right)\right\}+\frac{n}{M_\beta(n)U_1(n)}+\frac{a}{M_\beta(n)}.$$

Similarly, (iii)-(v) equations are obtained.

To obtain approximation velocities of generalized Baskakov-Durrmeyer-Stancu type operators with Voronovskaja type theorem, Lemma 2.4, which gives the continuity modules of these operators, will be given first as follows:

Lemma 2.4 We have the following limits;

(i) $\lim\limits_{n\to\infty} n\,G_{n,a}^{\alpha,\beta}(t-x,x)=\left(\frac{a}{1+x}+1-\beta\right)x+a+1,$

(ii) $\lim\limits_{n\to\infty} n\,G_{n,a}^{\alpha,\beta}((t-x)^2,x)=2x^2+x\,,$

(iii) $\lim\limits_{n\to\infty} n^2\,G_{n,a}^{\alpha,\beta}((t-x)^4,x)=12x^4-\left(\frac{12\alpha^2}{1+x}-24\right)x^3-(6\alpha^2-6\alpha-12)x^2.$

Proof: (i) Using the linearity property of the $G_{n,a}^{\alpha,\beta}(t;x)$ operators from Lemma 2.3, we have

$$\lim_{n\to\infty} n\,G_{n,a}^{\alpha,\beta}(t-x,x)=\lim_{n\to\infty} n\left\{x\left(\frac{na}{(1+x)M_\beta(n)U_1(n)}+\frac{n^2}{M_\beta(n)U_1(n)}-1\right)+\frac{n}{M_\beta(n)U_1(n)}+\frac{a}{M_\beta(n)}\right\},$$

$$\lim_{n\to\infty} n\,G_{n,a}^{\alpha,\beta}(t-x,x)=\left(\frac{a}{1+x}+1-\beta\right)x+a+1.$$

(ii) Similarly, using linearity and Lemma 2.3, we obtain

$$G_{n,a}^{\alpha,\beta}((t-x)^2,x)=\left[\left\{\frac{1}{[M_\beta(n)]^2U_2(n)}\left(\frac{a^2n^2}{(1+x)^2}+\frac{2an^3}{(1+x)}+n^3(n+1)\right)-\frac{1}{M_\beta(n)U_1(n)}\left(\frac{2na}{1+x}+2n^2\right)+1\right\}x^2\right.$$
$$\left.+\left\{\frac{4n^2+2n(n-2)\alpha}{[M_\beta(n)]^2U_2(n)}\left(\frac{a}{1+x}+n\right)-\frac{2n}{M_\beta(n)U_1(n)}-\frac{2\alpha}{M_\beta(n)}\right\}x+\frac{2n^2+2n(n-2)\alpha}{[M_\beta(n)]^2U_2(n)}+\frac{\alpha^2}{[M_\beta(n)]^2}\right].$$

When necessary arrangements are made in the last equation, the equation is multiplied by n, and the limit is taken as n approaches infinity, we get

$$\lim_{n\to\infty} n\, G_{n,a}^{\alpha,\beta}((t-x)^2,x) = \lim_{n\to\infty} \frac{n}{[M_\beta(n)]^2 U_2(n)}\Big[\Big\{\frac{1}{(1+x)^2}(a^2n^2) + \frac{1}{(1+x)}(4an^2+4an\beta-2an^2\beta)+2n^3+n^2\beta^2+2n^2\beta +2n^2-3n\beta^2-4n\beta+2\beta^2\Big\}x^2+\Big\{\frac{1}{(1+x)}(4an^2-4an\alpha+2an^2\alpha) +4\alpha\beta-4n\beta-4n\alpha+2n^2\alpha+2n^2\beta+4n^2+2n^3-6n\alpha\beta+2n^2\alpha\beta\Big\}x +n^2\alpha^2+2n^2\alpha+2n^2-3n\alpha^2-4n\alpha+2\alpha^2\}\Big]=2x^2+2x.$$

(iv) Equation (iii) is easily obtained when the operations in (i) and (ii) are similarly performed in $G_{n,a}^{\alpha,\beta}((t-x)^4,x)$.

We give the weighted Korovkin- type theorems which were proved by Gadzhiev [13]. Let $B_\sigma[0,\infty)$ be the space of all g functions with real values, where function g satisfies the growth condition $|g(x)| \le N_g\sigma(x)$ and $\sigma(x)=1+x^2$, N_g is a constant dependent on g. According to the $\|g\|_\sigma = \sup\left\{\frac{|g(x)|}{\sigma(x)} : x\in\mathbb{R}\right\}$ norm, $B_\sigma[0,\infty)$ is a normed space. It is a subspace of $B_\sigma[0,\infty)$ space, with $C_\sigma^*[0,\infty)$ being a space of continuous functions satisfying the condition $\lim_{|x|\to\infty}\frac{|g(x)|}{\sigma(x)}$.

Now, using Lemma 2.3 and Lemma 2.4 we give the following Voronovskaja-type theorem for $G_{n,a}^{\alpha,\beta}(g(t),x)$.

Theorem 2.5: For any $g\in C_\sigma^*[0,\infty)$ such that $g', g''\in C_\sigma^*[0,\infty)$ we have the following limit:

$$\lim_{n\to\infty} n(G_{n,a}^{\alpha,\beta}(g(t);x)-g(x)) = g'(x)\left\{\left(\frac{a}{1+x}+1-\beta\right)x+a+1\right\}+\frac{1}{2}g''(x)\{2x^2+x\}.$$

Proof: From the Taylor's expansion of g, we get

$$g(t)=g(x)+g'(x)(t-x)+\frac{1}{2}g''(x)(t-x)^2+\delta(t,x)(t-x)^2 \tag{9}$$

where $\delta(t,x)\to 0$ as $t\to x$. If we apply operators $G_{n,a}^{\alpha,\beta}$ to equation (9) using the linearity property of the operators $G_{n,a}^{\alpha,\beta}$, then we obtain

$$G_{n,a}^{\alpha,\beta}(g(t);x)-g(x)=g'(x)G_{n,a}^{\alpha,\beta}((t-x);x)+\frac{1}{2}g''(x)G_{n,a}^{\alpha,\beta}((t-x)^2;x)+G_{n,a}^{\alpha,\beta}(\delta(t,x)(t-x)^2;x) \tag{10}$$

Then, if the $G_{n,a}^{\alpha,\beta}(\delta(t,x)(t-x)^2;x)$ term of the equation (10) is multiplied by n and the Cauchy- Schwarz inequality is applied, we find

$$nG_{n,a}^{\alpha,\beta}(\delta(t,x)(t-x)^2;x)\le\left(G_{n,a}^{\alpha,\beta}\delta(t,x)^2;x\right)^{\frac{1}{2}}\left(n^2G_{n,a}^{\alpha,\beta}((t-x)^4;x)\right)^{\frac{1}{2}}. \tag{11}$$

We have $\lim_{n\to\infty} G_{n,a}^{\alpha,\beta}(\delta(t,x)^2;x)=0$, and from (iii) of Lemma 4, we have

$\lim_{n\to\infty} n^2G_{n,a}^{\alpha,\beta}((t-x)^4;x)$ is finite.

Then, taking the limit of the inequality (11) while n approaching infinity, we get

$\lim_{n\to\infty} n\,G_{n,a}^{\alpha,\beta}(\delta(t,x)(t-x)^2;x)=0.$

Therefore, when the limit of both sides of (10) is taken for n approaching infinity, we get

$$\lim_{n\to\infty} n(G_{n,a}^{\alpha,\beta}(g(t);x)-g(x)) = g'(x)\left\{\left(\frac{a}{1+x}+1-\beta\right)x+a+1\right\}+\frac{1}{2}g''(x)\{2x^2+x\}.$$

As a result, it is seen that the proof is complete.

Some Graphical Analysis

In this section, the graphs below show the convergence of the of the Generalized Baskakov-Durmeyer- Stancu type Operators to the considered function
$g(x) = \sqrt{x}e^{-2x}$
For different values of n, k, a, α and β.

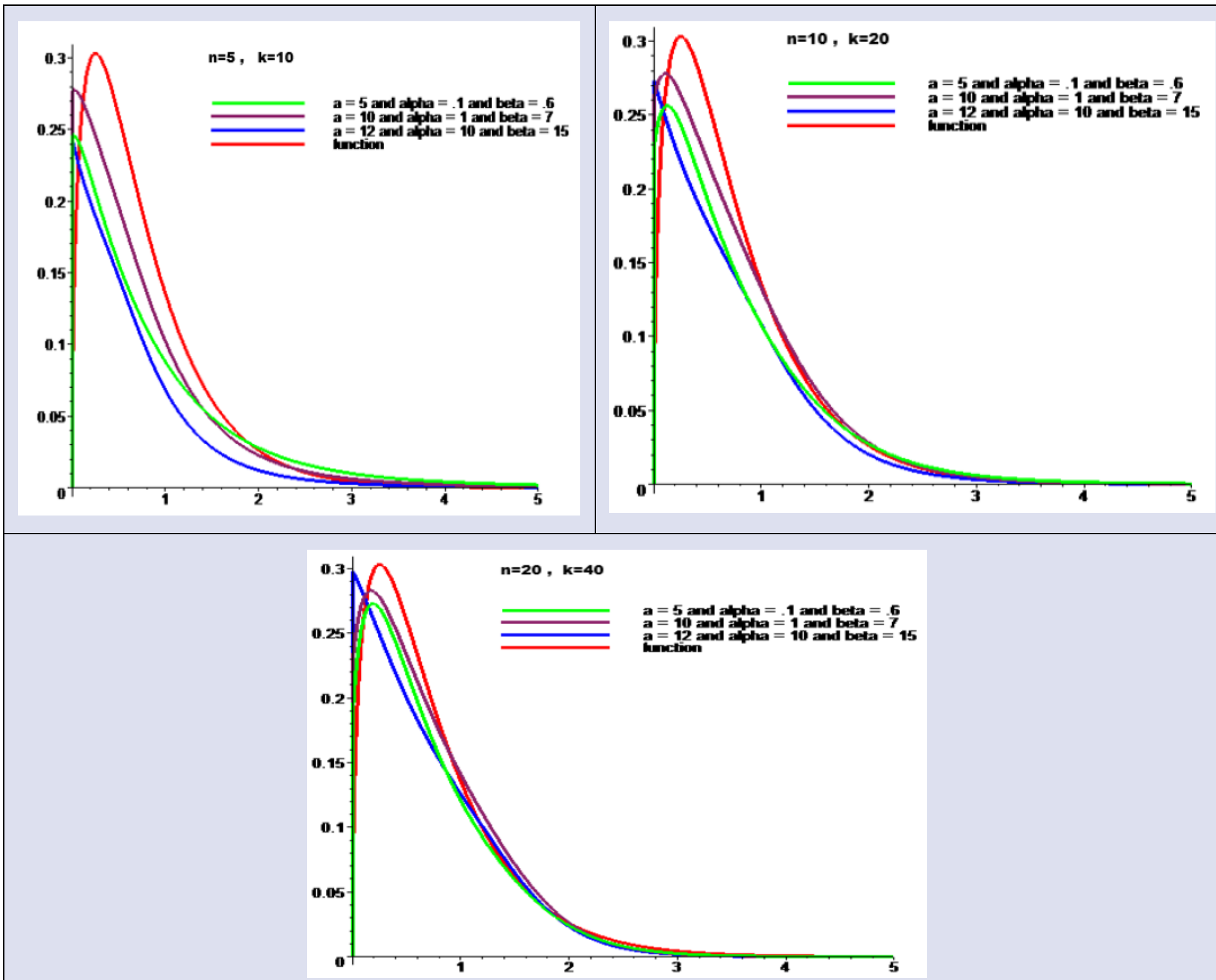

Figure 1. Convergence of $G_{n,a}^{\alpha,\beta}(g;x)$ for different values of n, k, a, α, and β

The graph below shows the convergence of $B_n^{a}(g,x)$ (BO), $L_n^{a}(g,x)$ (BDO) and $G_{n,a}^{\alpha,\beta}(g;x)$ (BDSO) to the $g(x)$ function for $n = 20, k = 40, a = 10, \alpha = 1$, and $\beta = 7$.

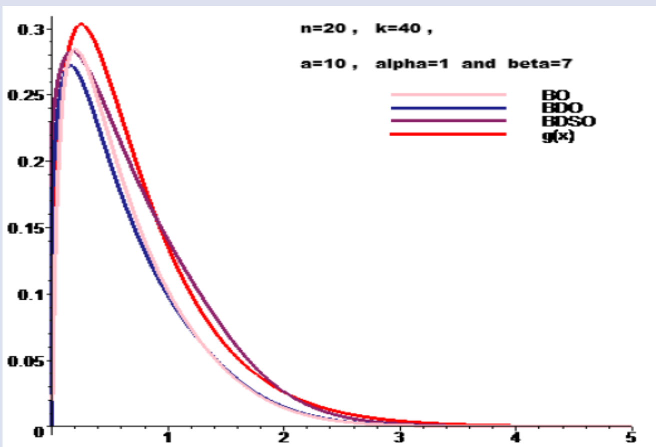

Figure 2. Convergence of $B_n^{a}(g,x), L_n^{a}(g,x), G_{n,a}^{\alpha,\beta}(g;x)$ to $g(x)$ for $n = 20, k = 40, a = 10, \alpha = 1$ and $\beta = 7$

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Conflicts of Interest

The authors declared there is no conflict of interest associated with this work.

References

[1] Weierstrass K., Uber die analytische Darstellbarkeit sogenannter willkurlicher Functionen einer reellen Veranderlichen Sitzungsberichteder, *Koniglich Preussischen Akademie der Wissenschcaften zu Berlin*, (1885) 633-639, 789-805.

[2] Bernstein S. N., Démonstration du théoréme deWeierstrass fondée sur la calcul des probabilitiés, *Commun. Soc.Math. Charkow Sér.* 13(2) (1912) 1-2 .

[3] Stancu D.D., Approximation of functions by a new class of linear polynomial operators, *Rev. Roum. Math. Pures Appl.* 13 (1968) 1173-1194.

[4] Baskakov V. A., An instance of a sequence of positive linear operators in the space of continuous functions, *Doklady Akademii Nauk SSSR*, 113:2 (1957) 249–251,

[5] Mihesan V., Uniform approximation with positive linear operators generated by generalized Baskakov method, *Automat. Comput. Appl. Math*. 7 (1998).34-37.

[6] Wafi A., Khatoon S., On the order of approximation of functions by generalized Baskakov operators, *Indian J. Pure Appl. Math*. (2004) 35347-358.

[7] Erencin A., Bascanbaz-Tunca G., Approximation properties of a class of linear positive operators in weighted spaces. *C. R. Acad. Bulg. Sci*. 63(10), (2010)1397–1404.

[8] Durrmeyer J. L., Une formule d'inversion de la transform´ee de Laplace: applications`a la th´eorie des moments, These de 3e cycle, Facult´e des Sciences de l'Universit´e de Paris, (1967).

[9] Lupas A., Die Folge der Betaoperatoren, Dissertation, Universit¨at Stuttgart, (1972).

[10] Erencin A., Durrmeyer type modification of generalized Baskakov operators, *Appl. Math. Comput.,* 218 (2011) 4384-4390.

[11] Agrawal P. N., Gupta V., Kumar A. S., Generalized Baskakov-Durrmeyer type operators, *Rend. Circ. Mat. Palermo*, 63 (2014) 193–209.

[12] Kumar A. S., Finta Z., Agrawal P. N., On generalized Baskakov-Durrmeyer-Stancu type operators, *Demonstr. Math.,* 50 (2017) 144–155.

[13] Gadhziev A.D., Theorems of the type of P.P. Korovkin type theorems, *Matematicheskie Zametki*, 20(5) (1976) 781-786.

[14] Erençin A., Olgun A., Taşdelen F., Generalized Baskakov type operators, *Mathematica Slovaca*, 67(5) (2017) 1269-1277.

[15] Yesildal FT., Bodur M., Bivariate Baskakov type operators, *Revısta De La Real Academıa De Cıencıas Exactas Fısıcas Y Naturales Serıe A-Matematıcas,* 113(4) (2019) 3269-3281.

[16] Aktaş R., Söylemez D., Taşdelen F., Stancu type generalization of Szász-Durrmeyer operators involving Brenke-type polynomials, *Filomat*, 33(3) (2019) 855-868.

[17] Bodur M., Yılmaz O. G., Aral A., Approximation by Baskakov-Szász-Stancu Operators Preserving Exponential Functions, *Constr. Math. Anal.,* 1(1) (2018) 1-8.