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Period-doubling Bifurcation and Stability in a Two Dimensional Discrete Prey-predator Model with Allee Effect and Immigration Parameter on Prey

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ABSTRACT

This article is about the dynamics of a discrete-time prey-predator system with Allee effect and immigration parameter on prey population. Particularly, we study existence and local asymptotic stability of the unique positive fixed point. Furthermore, the conditions for the existence of bifurcation in the system are derived. In addition, it is shown that the system goes through period-doubling bifurcation by using bifurcation theory and center manifold theorem. Eventually, numerical examples are given to illustrate theoretical results.

Keywords: Prey-predator model, Stability analysis, Fixed point, Allee effect, Period-doubling bifurcation.

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Introduction

Modelling the prey-predator interaction of a simple ecosystem is one of the important applications of the nonlinear system of differential equations in mathematical biology and ecology. The dynamics of such a system are observed by using the population data which is obtained by the interaction between a pair of prey-predator. The classical predator-prey system is known as Lotka-Volterra model which is first studied by Lotka [1] and Volterra [2]. For implementing more realistic assumptions in prey-predator model, a lot of modifications and extensions were introduced by several researchers [3-10].

As is well known, differential and difference equations are used in the theory of dynamical population models. Differential equations are used to describe continuous-time models while the discrete-time models are described by difference equations. Recent works showed that, researchers are more interested in discrete-time models than continuous-time models since the dynamics of discrete time models can produce a richer set of patterns. Additionally, many studies have proposed that the mathematical model of population dynamics becomes more suitable and practical when discrete-time equations are used for modelling. Besides they have the basic characteristics of the corresponding continuous-time models, they also provide a significant decrease of numerical simulation duration. Moreover, the discrete time models are more suitable for populations with nonoverlapping generations. In fact, nonlinear continuous models are discretized since nonlinear systems generally do not have analytic solutions expressible in terms of a finite representation of elementary functions. Authors in [5-11] analyzed dynamical analysis of different types of discrete-time predator-prey systems.

In [12], the discrete-time prey-predator model represented by the following nonlinear system of difference equations is studied:

$$\begin{aligned} x_{n+1} &= \mu x_n(1-x_n) - x_n y_n \\ y_{n+1} &= y_n(1-\alpha) + \beta x_n y_n \end{aligned} \qquad (1)$$

In (1), x_n and y_n represent prey and predator population densities in the n^{th} generation, respectively. The parameter μ is the intrinsic growth rate of the prey population with carrying ability one in the absence of predator. While α reflects the predators death rate; β denotes the growth rate of predator in the presence of the prey. All the parameters α, β and μ have positive values.

In this study, system (1) is improved with Allee effect and immigration on prey species and the following nonlinear system of difference equations is held:

$$\begin{aligned} x_{t+1} &= \delta x_t(1-x_t) - x_t y_t \frac{x_t}{x_t+m} + s \\ y_{t+1} &= y_t(1-\alpha) + \beta x_t y_t \end{aligned} \qquad (2)$$

In (2), δ is the intrinsic growth rate of prey population x_t, α and β are the death rate of predator and the growth rate of predator in the presence of the prey.

The parameter $s > 0$ represents the immigration parameter. Prey immigration is the number of individuals of the same species added to the prey population from another place in a certain period of time and it increases the size of the population of prey. The immigration factor is an effect that makes the predator-prey population model more realistic [13-17]. So, many researchers studied the role of immigration and its impact on population dynamics. Detailed investigations relating to immigration may be found in the papers [18-22].

The term

$$\frac{x_t}{x_t + m}$$

is called Allee effect where $m > 0$ is Allee constant. Allee effects are encountered among many species such as mammals, plants, insects etc. It describes a positive correlation between the density of the population and the per capita growth rate. It means that, as the population gets smaller survivals of individuals and reproductive diversity decrease. In [23], it is pointed out that on different prey predator systems according to different mechanisms the impact of the Allee effects can vary, too. In [24] and [25], it is shown that a ratio dependent prey predator model including Allee effect removes the possibility of population cycles. In [26] and [27], a new population model and a Lotka-Volterra commensal symbiosis model with Allee effect are studied, respectively. Allee effect and the immigration parameter have an important role in increasing the realism of the population models, besides they help to gain a more accurate description of the model.

This study is organized as follows: In Section 2, we discuss the existence and stability of fixed points of the system (2) and we give some numerical examples. In section 3, the existence of period-doubling bifurcation is shown with the help of bifurcation theory and center manifold theorem. The numerical simulation results are illustrated to confirm our analytical results and display the irregular dynamical behaviors of system (2).

The Existence and Stability of Fixed Points

In this section, the existence of fixed points is studied and the stability properties for system (2) is investigated. With the help of a simple calculation, it can be shown that the following system

$$\delta x(1-x) - xy\frac{x}{x+m} + s = x \tag{3}$$

$$y(1-\alpha) + \beta xy = y$$

has three fixed points:

$$P_1 = \left(\frac{\delta - 1 + \sqrt{(\delta-1)^2 + 4\delta s}}{2\delta}, 0\right)$$

$$P_2 = \left(-\frac{-\delta + 1 + \sqrt{(\delta-1)^2 + 4\delta s}}{2\delta}, 0\right)$$

and

$$P_3 = (x^*, y^*) = \left(\frac{\alpha}{\beta}, \frac{(\alpha^2\beta + \alpha m\beta^2 - \alpha^3 - \alpha^2 m\beta)\delta}{\alpha^2\beta} + \frac{s\alpha\beta^2 + sm\beta^3 - \alpha^2\beta - \alpha m\beta^2}{\alpha^2\beta}\right) \tag{4}$$

P_3 is the unique positive coexistence fixed point of the system (2) where the parameters $\alpha, \beta, \delta, m, s$ are all positive, $\beta - \alpha < 0$ and $(s\beta - \alpha) > 0$. We focus on the coexistence fixed point P_3 when studying stability analysis of the system (2).

It is well known that the local stability of the discrete-time system (2) is determined by calculating the eigenvalues of the Jacobian matrix which is evaluated at the coexistence fixed point P_3. The Jacobian matrix of system (2) at P_3 is given as follows:

$$J(P_3) = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \tag{5}$$

where

$$J_{11} = -\frac{\delta\alpha^3 - \alpha^2\beta + (\delta m - 2m + s)\beta^2\alpha + 2sm\beta^3}{\beta(\alpha + m\beta)\alpha}$$

$$J_{12} = -\frac{\alpha^2}{\beta(\alpha + m\beta)}$$

$$J_{21} = \frac{\big(\alpha(\alpha + m\beta)(\beta - \alpha)\big)\delta + \beta(s\beta - \alpha)(\alpha + m\beta)}{\alpha^2}$$

$$J_{22} = 1$$

The matrix $J(P_3)$ yields the characteristic equation:

$$F(\lambda) = \lambda^2 - tr\big(J(P_3)\big)\lambda + det\big(J(P_3)\big) \tag{6}$$

where

$$tr\big(J(P_3)\big) = \frac{\delta\alpha^3 - 2\alpha^2\beta + (s - 3m + \delta m)\beta^2\alpha}{\beta(\alpha + m\beta)\alpha} + \frac{2sm\beta^3}{\beta(\alpha + m\beta)\alpha} \tag{7}$$

and

$$det\big(J(P_3)\big) = \frac{-\delta\alpha^3 + (-\beta - \delta + \delta\beta - \delta m\beta)\alpha^2}{\beta(\alpha + m\beta)} + \frac{(\beta - m\beta^2 + s\beta^2 + \delta m\beta^2)\alpha}{\beta(\alpha + m\beta)} + \frac{(-\delta m\beta^2 + 2m\beta^2 - s\beta^2 + sm\beta^3)\alpha - 2sm\beta^3}{\beta(\alpha + m\beta)\alpha} \tag{8}$$

Definition 2.1. Let λ_1 and λ_2 are the roots of the characteristic polynomial $F(\lambda) = \lambda^2 + B\lambda + C,\ B, C \in \mathbb{R}$. Then the fixed point P_3 of the system (3) is called

i) sink if $|\lambda_1| < 1$ and $|\lambda_2| < 1$,
ii) source if $|\lambda_1| > 1$ and $|\lambda_2| > 1$,
iii) saddle if $|\lambda_1| < 1$ and $|\lambda_2| > 1$ or $|\lambda_1| > 1$ and $|\lambda_2| < 1$.
iv) non-hyperbolic if $|\lambda_1| = 1$ or $|\lambda_2| = 1$.

Definition 2.2. A fixed point is locally asymptotically stable if $|\lambda_1| < 1$ and $|\lambda_2| < 1$.
With the help of the following lemma, the stability of the coexistence fixed point of the system (2) is investigated.

Lemma 2.1. [28] Assume $F(\lambda) = \lambda^2 + B\lambda + C$, where B and C are two real constants and let $F(1) > 0$. Suppose that λ_1 and λ_2 are two roots of $F(\lambda) = 0$. Then the following statements hold:

i) $|\lambda_1| < 1$ and $|\lambda_2| < 1$ if and only if $F(-1) > 0$ and $C < 1$,
ii) $|\lambda_1| > 1$ and $|\lambda_2| > 1$ if and only if $F(-1) > 0$ and $C > 1$,
iii) $|\lambda_1| > 1$ and $|\lambda_2| < 1$ if and only if $F(-1) < 0$,
iv) λ_1 and λ_2 are a pair of conjugate complex roots and $|\lambda_1| = |\lambda_2| = 1$ if and only if $B^2 - 4C < 0$ and $C = 1$.

By using Lemma 2.1, we determine stability conditions for the fixed point P_3 of the system (2).

$$F(1) = \frac{s\beta^2 + \beta\delta\alpha - \beta\alpha - \delta\alpha^2}{\beta} > 0 \tag{9}$$

If conditions $\delta < \delta_3$, $\beta < \alpha$ and $s > \frac{\alpha}{\beta}$ hold where

$$\delta_3 = \frac{(s\beta - \alpha)\beta}{\alpha(-\beta + \alpha)}$$

$$F(-1) = \left(\frac{-\alpha^3 + (-2 + \beta - m\beta)\alpha^2}{\beta(\alpha + m\beta)} + \frac{m\beta(\alpha - 2)}{(\alpha + m\beta)}\right)\delta + \frac{4\alpha^2 + 6\alpha m\beta - 4sm\beta^2}{(\alpha + m\beta)\alpha} + \frac{-2s\alpha\beta + s\alpha^2\beta + s\alpha\beta^2 m - \alpha^2(\alpha + m\beta)}{(\alpha + m\beta)\alpha} \tag{10}$$

We define δ_1 as a root of $F(-1) = 0$ where

$$\delta_1 = \frac{(-\alpha^2 + (4 + s\beta - m\beta)\alpha)\beta}{(\alpha^3 + (2 - \beta + m\beta)\alpha^2 - \alpha m\beta^2 + 2m\beta^2)} + \frac{\left((6m\beta - 2s\beta + sm\beta^2)\alpha - 4sm\beta^2\right)\beta}{\alpha(\alpha^3 + (2 - \beta + m\beta)\alpha^2 - \alpha m\beta^2 + 2m\beta^2)} \tag{11}$$

Let K is the coefficient of δ in $F(-1)$ where

$$K = \frac{-\alpha^3 + (-2 + \beta - m\beta)\alpha^2}{\beta(\alpha + m\beta)} + \frac{m\beta(\alpha - 2)}{(\alpha + m\beta)}$$

$$F(0) = \left(\frac{-\alpha^3 + (-1 + \beta - m\beta)\alpha^2}{\beta(\alpha + m\beta)} + \frac{m\beta(\alpha - 1)}{(\alpha + m\beta)}\right)\delta + \frac{-2sm\beta^2 + 2\alpha m\beta - s\alpha\beta}{(\alpha + m\beta)\alpha} + \frac{\alpha + s\alpha\beta + s\beta^2 m - \alpha^2 - \alpha\beta m}{(\alpha + m\beta)} \tag{12}$$

Let us δ_2 is the root of $F(0) - 1 = 0$ where

$$\delta_2 = \frac{(-\alpha^2 + (s\beta - m\beta)\alpha)\beta}{(\alpha^3 + (1 - \beta + m\beta)\alpha^2 - \alpha m\beta^2 + m\beta^2)} + \frac{\left((m\beta - s\beta + sm\beta^2)\alpha - 2sm\beta^2\right)\beta}{\alpha(\alpha^3 + (1 - \beta + m\beta)\alpha^2 - \alpha m\beta^2 + m\beta^2)} \tag{13}$$

We assume that S is the coefficient of δ in $F(0) - 1$ where

$$S = \frac{-\alpha^3 + (-1 + \beta - m\beta)\alpha^2}{\beta(\alpha + m\beta)} + \frac{m(\alpha - 1)}{(\alpha + m\beta)} \tag{14}$$

When the sign table according to δ is examined, we conclude the following results:

Theorem 2.1. Assume that $\delta < \delta_3$, $\alpha > \beta$ and $s > \frac{\alpha}{\beta}$. Then for the coexistence fixed point P_3 of the system (2) the following hold true:

i) P_3 is a sink if the following condition holds
$K < 0, S < 0$ and $\delta_2 < \delta < min\{\delta_1, \delta_3\}$

ii) P_3 is a source if the following condition holds
$K < 0, S < 0$ and $\delta < min\{\delta_1, \delta_2, \delta_3\}$

iii) P_3 is a saddle if the following condition holds
$K < 0, S < 0$ and $\delta_1 < \delta < \delta_3$

iv) Assume that λ_1 and λ_2 are roots of $F(\lambda)$ then $\lambda_1 = -1$ and $|\lambda_2| \neq 1$ if and only if
$K < 0, S < 0, \delta = \delta_1$ and

$$\delta_2 \neq \frac{Q}{P}, \frac{Q}{P} + \frac{2\alpha\beta(m\beta + \alpha)}{P}$$

where

$$Q = (-2sm\beta - s\alpha\beta + s\alpha^2\beta + \alpha sm\beta^2 - \alpha^3 - \alpha^2 m\beta + \alpha m\beta)\beta$$

$$P = (m\beta^2 + \alpha^2(1 - \beta) - \alpha m\beta(\beta - \alpha) + \alpha^2(1 + \alpha m\beta))\alpha.$$

Example 2.1. Taking parameters $\alpha = 0.8, \beta = 0.6, m = 0.2, \delta = 0.3, s = 1.8$ and initial condition $(x_0, y_0) = (1.3,\ 1)$, the coexistence fixed point of the system (2) is obtained as $P_3 = (1.333333333, 0287500000)$. Using these parameter values, we can get below values:

$\delta_1 = 0.5230142566$, $\delta_2 = -0.7167597768$, $\delta_3 = 1.050000000$, K$= -2.846376811$<0, S$= -1.556521739 < 0$. Characteristic polynomial of the system (2) at fixed point P_3 is obtained as $F(\lambda) = \lambda^2 - 0.2173913044\lambda - 0.5826086952$ and the roots of the characteristic polynomial are $\lambda_1 = 0.8796842643$ and $\lambda_2 = -0.6622929599$ that verify $|\lambda_1| < 1$ and $|\lambda_2| < 1$. Also, the fixed point $P_3 = (1.333333333, 0287500000)$ of the system (2) is local asymptotically stable for $0 < \delta < 0.5230142566$ which shows the correctness of the Theorem 2.1. From Figure 1, (a)-(b) fixed point P_3 of the system (2) is local asymptotically stable that graphs represent x_t and y_t populations. If the parameter $\delta = 0.8$ is selected, the fixed point (1.333333333,0.09583333333) of the system (2) is unstable. For this situation, phase portrait of the prey and predator densities are exhibited in Figure 1 (c)-(d).

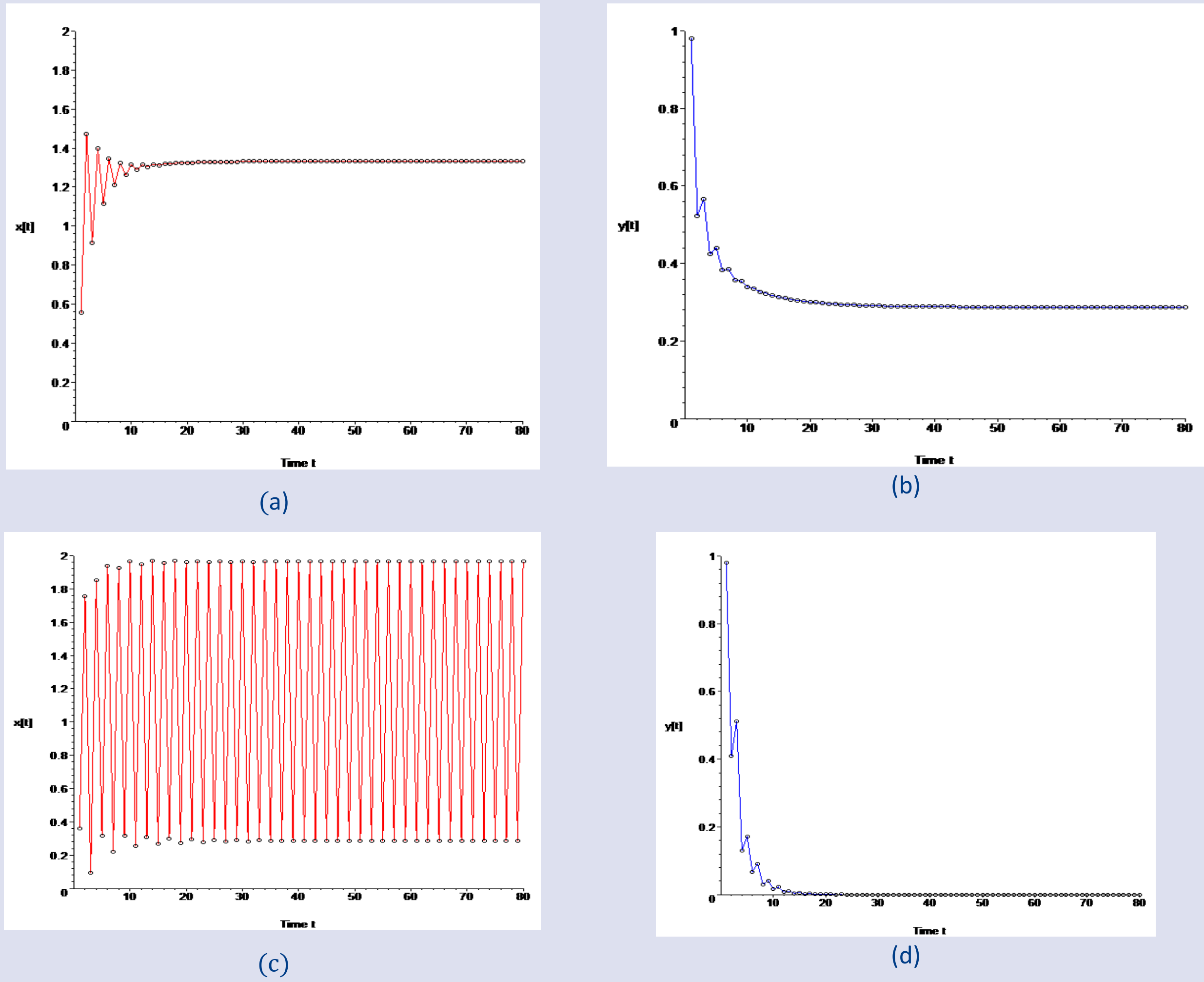

Figure 1. (a)-(b) A stable fixed point of the system (2) for $\alpha=0.8, \beta=0.6, m=0.2, \delta=0.3, s=1.8$ and initial condition $(x_0, y_0)=(1.3,\ 1)$. (c)-(d) An unstable fixed point (1.333333333,0.09583333333) of the system (2) for $\alpha=0.8, \beta=0.6, m=0.2, \delta=0.8, s=1.8$ and initial condition $(x_0, y_0)=(1.3,\ 1)$.

Period-Doubling Bifurcation

When it comes to case of dynamical systems, various types of bifurcation can occur as a result of changing stability of a fixed point, in other words, when a particular parameter exceeds its critical value. Depending on the bifurcation, various dynamical properties of the system under consideration can be studied. In this section, we investigate the parametric conditions for existence and directions of period-doubling bifurcation for the unique positive fixed point of system (2). When a discrete dynamical system goes through a period-doubling bifurcation, a small change in a parameter value in the system's equations causes a new behavior with twice the period of the original system undergoes.

In references [29-38], similar type of bifurcation analyses for discrete-time dynamical systems are studied.

We discuss period-doubling bifurcation of unique positive fixed point P_3 of the system (2) by using bifurcation theory and the center manifold theorem and taking δ as a bifurcation parameter. We suppose the condition

$$\begin{aligned}(-\delta\alpha^3+2\alpha^2\beta+(3m-s-\delta m)\beta^2\alpha-2sm\beta^3)^2 \\ > [-4\delta\alpha^4+(-4\beta-4\delta+4\delta\beta-4\delta m\beta)\alpha^3+(4\beta-4m\beta^2+4s\beta^2+4\delta m\beta^2)\alpha^2 \\ +(-4\delta m\beta^2+8m\beta^2-4s\beta^2+4sm\beta^3)\alpha-8sm\beta^3](\beta(\alpha+m\beta)\alpha)\end{aligned} \tag{15}$$

holds.

Then, λ_1 and λ_2 be distinct real roots of (6). Also, we assume that

$$\delta=\frac{(-\alpha^2+(4+s\beta-m\beta)\alpha)\beta}{(\alpha^3+(2-\beta+m\beta)\alpha^2-\alpha m\beta^2+2m\beta^2)}+\frac{((6m\beta-2s\beta+sm\beta^2)\alpha-4sm\beta^2)\beta}{\alpha(\alpha^3+(2-\beta+m\beta)\alpha^2-\alpha m\beta^2+2m\beta^2)} \tag{16}$$

Then, the roots of equation (6) are $\lambda_1 = -1$ and

$$\lambda_2 = -\frac{-4\alpha^3 + (-4m\beta + 2s\beta + 3\beta - 2)\alpha^2}{\alpha^3 + (2 - \beta + m\beta)\alpha^2 + (2 - \alpha)m\beta^2} \tag{17}$$

Furthermore, $|\lambda_2| \neq 1$ under the following conditions:

$$-\frac{-4\alpha^3 + (-4m\beta + 2s\beta + 3\beta - 2)\alpha^2}{\alpha^3 + (2 - \beta + m\beta)\alpha^2 + (2 - \alpha)m\beta^2} + \frac{(3m\beta^2 + 2sm\beta^2 - s\beta^2)\alpha - \beta^3 sm - 2m\beta^2}{\alpha^3 + (2 - \beta + m\beta)\alpha^2 + (2 - \alpha)m\beta^2} \neq \pm 1 \tag{18}$$

$$\alpha^3 + (2 - \beta + m\beta)\alpha^2 + (2 - \alpha)m\beta^2 \neq 0$$

Let us consider period-doubling set as follows

$$\Omega_{PDB} = \{(\alpha, \beta, \delta, s, m) \in \mathbb{R}_+^5 : K, S < 0\,, (16), (17) and (18)\ are\ satisfied\}.$$

For the aim of discussing the period-doubling bifurcation for the system (2) at its unique positive coexistence fixed point P_3, we take δ as bifurcation parameter. Then, variation of parameters α, β, δ, m and s in small neighborhood of Ω_{PDB} gives emergence of period-doubling bifurcation. Furthermore, we set

$$\delta_F = \frac{(-\alpha^2 + (4 + s\beta - m\beta)\alpha)\beta}{(\alpha^3 + (2 - \beta + m\beta)\alpha^2 - \alpha m\beta^2 + 2m\beta^2)} + \frac{\left((6m\beta - 2s\beta + sm\beta^2)\alpha - 4sm\beta^2\right)\beta}{\alpha(\alpha^3 + (2 - \beta + m\beta)\alpha^2 - \alpha m\beta^2 + 2m\beta^2)} \tag{19}$$

Then, for $(\alpha, \beta, \delta_F, m, s) \in \Omega_{PDB}$, system (2) can be expressed by the following two-dimensional map:

$$\begin{pmatrix} X \\ Y \end{pmatrix} \rightarrow \begin{pmatrix} \delta_F X(1 - X) - XY\dfrac{X}{X + m} + s \\ Y(1 - \alpha) + \beta XY \end{pmatrix} \tag{20}$$

Let us assume that $\bar{\delta}$ be a small bifurcation parameter such that $|\bar{\delta}| \ll 1$, then corresponding perturbed map for (20) is given by:

$$\begin{pmatrix} X \\ Y \end{pmatrix} \rightarrow \begin{pmatrix} (\delta_F + \bar{\delta})X(1 - X) - XY\dfrac{X}{X + m} + s \\ Y(1 - \alpha) + \beta XY \end{pmatrix} \tag{21}$$

Then, map (21) has unique fixed point

$$(\bar{X}, \bar{Y}) = \left(\frac{\alpha}{\beta}, \frac{(\alpha^2\beta + \alpha m\beta^2 - \alpha^3 - \alpha^2 m\beta)(\delta_F + \bar{\delta})}{\alpha^2\beta} + \frac{s\alpha\beta^2 + sm\beta^3 - \alpha^2\beta - \alpha m\beta^2}{\alpha^2\beta}\right)$$

For translating the fixed point to the origin, the transformations $x = X - \bar{X}$, $y = Y - \bar{Y}$ is done at point $(x, y) = (0,0)$, then we get the following map:

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} g_1(x, y, \bar{\delta}) \\ g_2(x, y, \bar{\delta}) \end{pmatrix} \tag{22}$$

where

$$g_1(x, y, \bar{\delta}) = a_{13}x^2 + a_{14}xy + b_1x^3 + b_2x^2y + d_1x\bar{\delta} + d_2x^2\bar{\delta} + O\left((|x| + |y| + |\bar{\delta}|)^4\right)$$

$$g_2(x, y, \bar{\delta}) = a_{24}xy + O\left((|x| + |y| + |\bar{\delta}|)^4\right)$$

where

$$a_{13} = -\frac{(\delta_F + \bar{\delta})\alpha^4 + (2m\delta_F + 2m\bar{\delta})\beta\alpha^3}{(\alpha + \beta m)^2\alpha^2} + + \frac{m^2\alpha^2\bar{\delta}\beta^2}{(\alpha + \beta m)^2\alpha^2} + \frac{(m^2\delta_F - m^2)\beta^3\alpha + m^2 s\beta^4}{(\alpha + \beta m)^2\alpha^2} \tag{23}$$

$$a_{14} = \frac{(\alpha + 2\beta m)\alpha}{(\alpha + \beta m)^2}$$

$$b_1 = \frac{m^2\beta^3(-\delta_F\alpha^2 + (-\beta + \beta\delta_F)\alpha + \beta^2 s)}{\alpha^2(\alpha + \beta m)^3}$$

$$b_2 = -\frac{\beta^3 m^2 y}{(\alpha + \beta m)^3}$$

$$d_1 = -\frac{(\beta\alpha m(-\beta + 2\alpha) - \alpha^2(\beta + 2\alpha))}{\beta\alpha(\alpha + \beta m)}$$

$$d_2 = -\frac{2\beta\alpha^3 m + \alpha^4 + \alpha^2\beta^2 m^2}{(\alpha + \beta m)^2\alpha^2}$$

$a_{24} = \beta$

For converting the coefficient matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

in map (22) into normal form, the following translation is used

$$\begin{pmatrix} x \\ y \end{pmatrix} = T\begin{pmatrix} u \\ v \end{pmatrix} \quad (24)$$

where

$$T = \begin{pmatrix} a_{12} & a_{12} \\ -1 - a_{11} & \lambda_2 - a_{11} \end{pmatrix} \quad (25)$$

be an invertible matrix. From (22) and (24), we obtain

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} g_3(x, y, \bar{\delta}) \\ g_4(x, y, \bar{\delta}) \end{pmatrix} \quad (26)$$

where

$$g_3(u, v, \bar{\delta}) = -\frac{(-\lambda_2 + a_{11})(a_{13} + d_2\bar{\delta})}{a_{12}(\lambda_2 + 1)}x^2 - \frac{(-\lambda_2 + a_{11})a_{14} + a_{12}a_{24}}{a_{12}(\lambda_2 + 1)}xy - \frac{(-\lambda_2 + a_{11})b_1}{a_{12}(\lambda_2 + 1)}x^3 - \frac{(-\lambda_2 + a_{11})b_2}{a_{12}(\lambda_2 + 1)}x^2 y - \frac{(-\lambda_2 + a_{11})d_1\bar{\delta}}{a_{12}(\lambda_2 + 1)}x + O\left((|u| + |v| + |\bar{\delta}|)^4\right)$$

$$g_4(u, v, \bar{\delta}) = \frac{(1 + a_{11})(a_{13} + d_2\bar{\delta})}{a_{12}(\lambda_2 + 1)}x^2 + \frac{(1 + a_{11})a_{14} - a_{12}a_{24}}{a_{12}(\lambda_2 + 1)}xy + \frac{(1 + a_{11})b_1}{a_{12}(\lambda_2 + 1)}x^3 + \frac{(1 + a_{11})b_2}{a_{12}(\lambda_2 + 1)}x^2 y + \frac{(1 + a_{11})d_1\bar{\delta}}{a_{12}(\lambda_2 + 1)}x + O\left((|u| + |v| + |\bar{\delta}|)^4\right)$$

$x = a_{12}(u + v),$

$y = -(1 + a_{11})u + (\lambda_2 - a_{11})v.$

In order to apply the center manifold theorem, we assume that $W^c(0,0,0)$ be the center manifold of (26) evaluated at (0,0) in a small neighborhood of $\bar{\delta} = 0$. We know

$W^c(0) = \{(x, y) \in R^c \times R^s | y = h(x), |x| < \delta, h(0) = 0, h'(0) = 0\}$

then $W^c(0,0,0)$ can be approximated as follows:

$W^c(0,0,0) = \{(\alpha, \beta, \bar{\delta}) \in \boldsymbol{R}^3 : v = h(u) = m_1 u^2 + m_2 u\bar{\delta} + m_3\bar{\delta}^2\},$

where

$$m_1 = \left(\frac{a_{12}a_{14}}{\lambda_2^2 - 1}\right)a_{11}^2 + a_{12}a_{11}\left(\frac{a_{12}(a_{24} - a_{13}) + 2a_{14}}{\lambda_2^2 - 1}\right) + a_{12}\left(\frac{a_{14} + a_{12}(a_{24} - a_{13})}{\lambda_2^2 - 1}\right)$$

$$m_2 = -\frac{(a_{11} + 1)d_1}{(\lambda_2 + 1)^2}$$

$$m_3 = 0$$

Therefore, the map is restricted to the center manifold $W^c(0,0,0)$ is given by

$$F: u \longrightarrow -u + k_1u^2 + k_2u\bar{\delta} + k_3u^2\bar{\delta}^2 + k_4u\bar{\delta}^2 + k_5u^3 + O\left(\left(|u| + |\bar{\delta}|\right)^4\right) \tag{27}$$

where

$$k_1 = -\frac{(a_{11} - \lambda_2)a_{12}a_{13}}{\lambda_2 + 1} + \left(-\frac{(a_{11} - \lambda_2)a_{14}}{a_{12}(\lambda_2 + 1)} - \frac{a_{24}}{\lambda_2 + 1}\right)(-1 - a_{11})a_{12}$$

$$k_2 = -\frac{(a_{11} - \lambda_2)d_1}{\lambda_2 + 1}$$

$$\begin{aligned}
k_3 = {} & \frac{\left(-\frac{(-\lambda_2 + a_{11})a_{14}}{a_{12}(\lambda_2 + 1)} - \frac{a_{24}}{\lambda_2 + 1}\right)(1 + a_{11})^2 d_1 a_{12}}{(\lambda_2 + 1)^2} \\
& - \left(\frac{\left(-\frac{(-\lambda_2 + a_{11})a_{14}}{a_{12}(\lambda_2 + 1)} - \frac{a_{24}}{\lambda_2 + 1}\right)}{(\lambda_2 + 1)^2}\right)\left(\frac{(\lambda_2 - a_{11})(1 + a_{11})d_1 a_{12}}{(\lambda_2 + 1)^2}\right) \\
& - \frac{(-\lambda_2 + a_{11})d_1\left(\frac{a_{12}a_{14}}{-1 + \lambda_2^2}\right)a_{11}^2}{\lambda_2 + 1} - \frac{(-\lambda_2 + a_{11})d_1 a_{12}^2\left(\frac{(a_{24} - a_{13}) + 2a_{14}}{-1 + \lambda_2^2}\right)a_{11}}{\lambda_2 + 1} \\
& - \frac{(-\lambda_2 + a_{11})d_1 a_{12}\left(\frac{a_{14} + a_{12}(a_{24} - a_{13})}{-1 + \lambda_2^2}\right)}{\lambda_2 + 1} + \frac{2(-\lambda_2 + a_{11})a_{12}a_{13}(1 + a_{11})d_1}{(\lambda_2 + 1)^3} \\
& - \frac{(-\lambda_2 + a_{11})a_{12}d_2}{\lambda_2 + 1}
\end{aligned}$$

$$k_4 = \frac{(-\lambda_2 + a_{11})d_1^2(1 + a_{11})}{(\lambda_2 + 1)^3}$$

$$\begin{aligned}
k_5 = {} & \left(-\frac{(-\lambda_2+a_{11})a_{14}}{a_{12}(\lambda_2+1)} - \frac{a_{24}}{\lambda_2+1}\right)(-1 - a_{11})a_{12}\left(\left(\frac{a_{12}a_{14}}{-1+\lambda_2^2}\right)a_{11}^2 + a_{12}\left(\frac{2a_{14}+a_{12}(a_{24}-a_{13})}{-1+\lambda_2^2}\right)a_{11} + \right. \\
& \left. a_{12}\left(\frac{a_{12}(a_{24}-a_{13})+a_{14}}{-1+\lambda_2^2}\right)\right) + \left(-\frac{(-\lambda_2+a_{11})a_{14}}{a_{12}(\lambda_2+1)} - \frac{a_{24}}{\lambda_2+1}\right)(\lambda_2 - a_{11})\left(a_{12}\left(\frac{a_{14}}{-1+\lambda_2^2}\right)a_{11}^2 + a_{12}\left(\frac{2a_{14}+a_{12}(a_{24}-a_{13})}{-1+\lambda_2^2}\right)a_{11} + \right. \\
& \left. a_{12}\left(\frac{a_{14}+a_{12}(a_{24}-a_{13})}{-1+\lambda_2^2}\right)\right)a_{12} - \frac{(-\lambda_2+a_{11})a_{12}b_2(-1-a_{11})}{\lambda_2+1} - \frac{(-\lambda_2+a_{11})a_{12}^2 b_1}{\lambda_2+1} - \frac{2(-\lambda_2+a_{11})a_{12}a_{13}\left(a_{12}\left(\frac{a_{14}}{-1+\lambda_2^2}\right)a_{11}^2\right)}{\lambda_2+1} - \\
& \frac{2(-\lambda_2+a_{11})a_{11}a_{12}^2a_{13}}{\lambda_2+1}\frac{\left(\frac{2a_{14}+a_{12}(a_{24}-a_{13})}{-1+\lambda_2^2}\right)}{\lambda_2+1} - \frac{a_{12}\left(\frac{a_{14}+a_{12}(a_{24}-a_{13})}{-1+\lambda_2^2}\right)}{\lambda_2+1}.
\end{aligned}$$

Next, the following two nonzero real numbers are defined:

$$n_1 = \left(\frac{\partial^2 g_3}{\partial u \partial \bar{\delta}} + \frac{1}{2}\frac{\partial F}{\partial \bar{\delta}}\frac{\partial^2 F}{\partial u^2}\right)_{(0,0)} = \frac{(1 + a_{11})d_1}{\lambda_2 + 1};$$

$$n_2 = \left(\frac{1}{6}\frac{\partial^3 F}{\partial u^3} + \left(\frac{1}{2}\frac{\partial^2 F}{\partial u^2}\right)^2\right)_{(0,0)} = k_5 + k_1^2 \neq 0$$

As a result of the above analysis, the following theorem gives the parametric conditions for existence and direction of period-doubling bifurcation for the system (2) at its positive coexistence fixed point P_3 [39].

Theorem 3.1. Suppose that $n_1 \neq 0$ and $n_2 \neq 0$ then system (2) goes through period-doubling bifurcation at the unique positive fixed point P_3 when parameter δ varies in small neighborhood of δ_F. Moreover, if $n_2 > 0$, then the period-two orbits that bifurcate from positive fixed point P_3 are stable, and if $n_2 < 0$, then these orbits are unstable.

Example 3.1. Taking parameters $\alpha = 0.9, \beta = 0.6, m = 0.2, s = 3$ the coexistence fixed point of the system (2) is $(x^*, y^*) = (1.5, 1.020300088)$. The critical value of period-doubling bifurcation point is obtained as $\delta_F = 0.1994704325$. By taking these parameters the characteristic polynomial of the system is obtained as $F(\lambda) = \lambda^2 + 0.405119152\,\lambda - 0.5948808467$ and the roots of the characteristic polynomial is $\lambda_1 = -1$ and $\lambda_2 = 0.5348808472$ that verifies the theoretical knowledge.

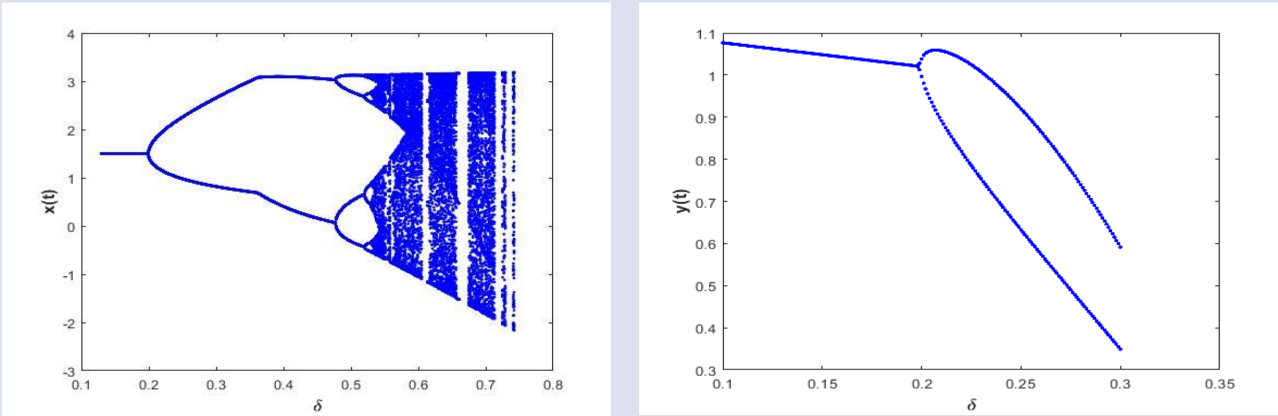

Figure 2. Bifurcation diagrams for x_t and y_t for the system (2) for values of $\alpha = 0.9, \beta = 0.6, m = 0.2, s = 3$ and initial condition $(x_0, y_0) = (1.45,\ 1.01)$

Conclusion

Allee effect and immigration have an important role in increasing the realism of the prey-predator model. So, we have considered a discrete-time prey-predator model with both Allee effect and immigration in this paper. We have investigated the complex dynamical behaviors of the system (2). Firstly, we have obtained existence conditions of the fixed points of the system (2). We have focused on coexistence fixed point due to biological meaning for showing complex dynamics of the system (2). We have analyzed topological classifications of the coexistence fixed point of the system (2). Later, we have obtained the required conditions on the parameters for period-doubling bifurcation of the system (2) by choosing δ as a bifurcation parameter. For period-doubling bifurcation analyses, we have used center manifold theorem and normal form theory [40]. Finally, we have given numerical simulations to support obtained theoretical finding. In Figure1, we have observed that the coexistence fixed point of the system (2) is local asymptotically stable on some conditions demonstrated in Theorem 2.1 (i). Also, in Figure 2, we have shown that the stability of the fixed point P_3 of the system (2) changes from stable to unstable when the bifurcation parameter δ, crosses a critical value δ_F. Thus, the period-doubling bifurcation arises from the fixed point P_3.

Conflicts of interest

No conflict of interest or common interest has been declared by the authors.

References

[1] Lotka A.J., Elements of physical biology, 1st ed. Baltimore: Williams and Wilkins Co., (1925).

[2] Volterra V., Variazioni e Fluttuazioni del Numero Dindividui in Spece Animali Conviventi, *Mem R Accad Naz dei Lincei*, 2 (6) (1926).

[3] Murray J.D., Mathematical biology. New York: Springer-Verlag, (1993).

[4] Walde S.J., Murdoch W.W., Spatial Density Dependence in Parasitoids, *Annu. Rev. of Entomol*., 33 (1988) 441-466.

[5] Kangalgil F., Flip Bifurcation and Stability in a Discrete-Time Prey-Predator Model with Allee Effect, Cumhuriyet Sci. J., 40 (2019) 141-149

[6] Beddington J.R., Free C.A., Lawton J.H., Dynamic Complexity in Predator-Prey Models Framed in Difference Equations*, Nature*, 255 (1975) 58-60.

[7] Blackmore D., Chen J., Perez J., Savescu M., Dynamical Properties of Discrete Lotka-Volterra Equations, *Chaos Solution. Fract.,* 12 (2001) 2553-2568.

[8] Danca M., Codreanu S., Bako B., Detailed Analysis of a Nonlinear Prey-Predator Model, *J. Biol. Phys*., 23 (1997) 11-20.

[9] Hadeler K.P., Gerstmann I., The Discrete Rosenzweig Model, *Math. Biosci.,* 98 (1) (1990) 49-72.

[10] Işık S., A Study of Stability and Bifurcation Analysis in Discrete-Time Predator-Prey System Involving the Allee Effect, Int. J. Biomath., 12 (01) (2019).

[11] Işık S., Kangalgil F., On the Analysis of Stability, Bifurcation, and Chaos Control of Discrete-Time Predator-Prey Model with Allee Effect on Predator, Hacet. J. Math. Stat., 51 (2) (2022) 404-420.

[12] Selvam A.G.M., Jacintha M., Dhineshbabu R., Bifurcation Analysis and Chaotic Behaviour in Discrete-Time Predator Prey System, *Int. J. Comput. Eng. Res*., 9 (4) (2019).

[13] Zhu G., Wei J., Global Stability and Bifurcation Analysis of a Delayed Predator-Prey System with Prey Immigration, *Electron. J. Qual. Theory Differ. Equ.,* 13 (2016) 1-20.

[14] Sugie J., Saito Y., Uniqueness of Limit Cycles in a Rosenzweig-Macarthur Model with Prey Immigration, *SIAM J. Appl. Math*., 72 (1) (2012) 299-316.

[15] Stone L., Hart D., Effects of Immigration on Dynamics of Simple Population Models, *Theor. Popul. Biol*., 55 (3) (1999) 227-234.

[16] Ak Gümüş Ö., Kangalgil F., Dynamics of a Host-Parasite Model Connected with Immigration, *New Trend. Math. Sci*., 5 (3) (2017) 332-339.

[17] Misra J.C., Mitra A., Instabilities in Single-Species and Host-Parasite Systems: Period-Doubling Bifurcations and Chaos, *Comput. Math. with Appl*., 52 (3) (2006) 525-538.

[18] Holt R.D., Immigration and the Dynamics of Peripheral Populations, Advances in Herpetology and Evolutionary Biology (Rhodin and Miyata, Eds.), Museum of Comparative Zoology, Harvard University, Cambridge: M.A., (1983).

[19] McCallum H.I., Effects of Immigration on Chaotic Population Dynamics*, J. Theor. Biol*., 154 (1992) 277-284.

[20] Stone L., Hart D., Effects of Immigration on the Dynamics of Simple Population Models, *Theor. Popul. Biol.,* 55 (3) (1999) 227-234.

[21] Ruxton G.D., Low Levels of Immigration between Chaotic Populations can Reduce System Extinctions by Inducing Asynchronous Regular Cycles, *Proc. Royal Soc. B,* 256 (1994) 189-193.

[22] Rohani P., Miramontes O., Immigration and the Persistence of Chaos in Population Models, *J. Theor. Biol*., 175 (2) (1995) 203-206.

[23] Zhou S., Liu Y., Wang G., The Stability of Predator-Prey ssstems Subject to the Allee Effects, *Theor. Popul. Biol*., 67 (1) (2005) 23-31.

[24] Sen M., Banarjee M., Morozou A., Bifurcation Analysis of a Ratio-Dependent Prey-Predator Model with the Allee Effect, *Ecol. Complex*., 11 (2012) 12-27.

[25] Cheng L., Cao H., Bifurcation Analysis of a Discrete-Time Ratio-Dependent Prey-Predator Model with the Allee Effect, Commun, *Nonlinear Sci. Numer. Simul.*, 38 (2016) 288-302.

[26] Kangalgil F., Ak Gümüş Ö., Allee Effect in a New Population Model and Stability Analysis, *Gen. Math. Notes*, 35 (1) (2016) 54-65.

[27] Lin Q., Allee Effect Increasing the Final Density of the Species Subject to Allee Effect in a Lotka-Volterra Commensal Symbiosis, *Model, Adv. Differ. Equ*., 196 (2018).

[28] Kangalgil F., Işık S., Controlling Chaos and Neimark-Sacker Bifurcation in a Discrete-Time Predator-Prey System, *Hacettepe J. Mathematics and Statistics*, 49 (5) (2020) 1761-1776.

[29] Din Q., Global Stability of Beddington Model, *Qual. Theory Dyn. Syst.,* 16 (2017) 391-415.

[30] Din Q., Global Stability and Neimark-Sacker Bifurcation of a Host-Parasitoid Model, *Int. J. Syst. Sci.,* 48 (6) (2016) 1194-1202.

[31] Din Q., Neimark-Sacker Bifurcation and Chaos Control in Hassell-Varley Model, *J. Differ. Equ. Appl.,* 23 (4) (2017) 741-762.

[32] Din Q., Ak Gümüş Ö., Khalil H., Neimark-Sacker Bifurcation and Chaotic Behaviour of a Modified Host-Parasitoid Model, *Z. Naturforsch*., 72 (1) (2016) 25-37.

[33] Din Q., Complexity and chaos control in a discrete-time prey-predator model, *Commun. Nonlinear Sci. Numer. Simul.*, 49 (2017) 113-134.

[34] Elabbasy E.M., Elsadany A.A., Zhang Y., Bifurcation Analysis and Chaos in a Discrete Reduced Lorenz System, *Appl. Math. Comput.,* 228 (2014) 184-194.

[35] Din Q., Donchev T., Kolev D., Stability, Bifurcation Analysis and Chaos Control in Chlorine Dioxide-Iodine-Malonic Acid Reaction, *MATCH Commun. Math. Comput. Chem.,* 79 (2018) 577-606.

[36] Allen L.J.S., An introduction to mathematical biology, Texas Tech. University, (2007).

[37] Kılıç H., Topsakal N., Kangalgil F., Stability Analysis of a Discrete-Time Prey-Predator Population Model with Immigration*, Cumhuriyet Sci. J.,* 41 (4) (2020) 884-900.

[38] Sucu G., Bir Ayrık Av-Avcı Modelinin Kararlılık ve Çatallanma Analizi, Yüksek Lisans Tezi, TOBB Ekonomi ve Teknoloji Üniversitesi, Fen Bilimleri Enstitüsü, 2016.

[39] He Z., Lai X., Bifurcation and Chaotic Behavior of a Discrete-Time Prey-Predator System, *Nonlinear Anal. Real World Appl.*, 12 (2011) 403-417.

[40] Kuznetsov Y.A., Elements of applied bifurcation theory, 2nd ed. New York: Springer-Verlag, (1998).