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On Gaussian Jacobsthal-Padovan Numbers

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ABSTRACT

Gaussian Jacobsthal-Padovan numbers have been the central focus of this paper and firstly this number sequence has defined. Later, we have given the proof of the generating function of the Gaussian Jacobsthal-Padovan sequence. After that by using generating function, we have given the proof of the Binet formula for this number sequence. Additionally, we have investigated some properties such as Simson identity, summation formulas of this sequence. Finally, we have obtained some matrices whose elements are Gaussian Jacobsthal-Padovan numbers.

Keywords: Jacobsthal numbers, Jacobsthal-Padovan numbers, Gaussian Jacobsthal-Padovan numbers, Generating function, Binet formula

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Introduction

In recent years, there have been many studies on a variety of number sequences in the literature. Some of the famous examples are Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal and Jacobsthal-Lucas etc. Details can be found in [1-5]. In [4], Jacobsthal and Jacobsthal-Lucas number sequences are given respectively by

$$J_n = J_{n-1} + 2J_{n-2}, \quad J_0 = 0,\ J_1 = 1, \quad n \geq 2,$$
$$j_n = j_{n-1} + 2j_{n-2}, \quad j_0 = 2,\ j_1 = 1, \quad n \geq 2.$$

Gaussian forms of these sequences are also available in the literature. Gaussian forms of these sequences as well as their properties have been inquired into by many authors, see [6-10] for details. In [9], Jacobsthal and Jacobsthal-Lucas sequences' Gaussian forms are defined by the recursion formulas

$$GJ_n = GJ_{n-1} + 2GJ_{n-2}, \quad GJ_0 = \frac{i}{2},\ GJ_1 = 1, \quad n \geq 2,$$
$$Gj_n = Gj_{n-1} + 2Gj_{n-2},\ Gj_0 = 2 - \frac{i}{2} Gj_1 = 1 + 2i, n \geq 2,$$

respectively.

Also, Cerda-Morales [11] defined Gauss third-order Jacobsthal numbers and investigated some properties of this sequence.

Additionally, the Padovan (sequence A000931 in [12]), Pell-Padovan (sequence A066983 in [12]) and Jacobsthal-Padovan (sequence A159284 in [12]) sequences are given, respectively, by the third-order recurrence relations

$$P_n = P_{n-2} + P_{n-3}, \quad P_0 = P_1 = P_2 = 1, \quad n \geq 3,$$
$$R_n = 2R_{n-2} + R_{n-3}, \quad R_0 = R_1 = R_2 = 1, \quad n \geq 3,$$
$$JP_n = JP_{n-2} + 2JP_{n-3}, \quad JP_0 = JP_1 = JP_2 = 1, \quad n \geq 3.$$

Also, Taşçı [13] defined Gaussian Padovan and Gaussian Pell-Padovan sequences as follows:

$$GP_n = GP_{n-2} + GP_{n-3}, \quad GP_0 = 1, \quad GP_1 = 1 + i, \quad GP_2 = 1 + i, \quad n \geq 3,$$

$$GR_n = 2GR_{n-2} + GR_{n-3}, \quad GR_0 = 1 - i, \quad GR_1 = 1 + i,\ GR_2 = 1 + i, \quad n \geq 3,$$

respectively.

Moreover, in [13], some properties of these sequences are investigated. In addition, Yaşar Kartal [14] studied the Gaussian Padovan sequence.

In this study, we extended the Jacobsthal-Padovan sequence to Gaussian Jacobsthal-Padovan sequence. Then, we have derived generating function and the Binet formula for this sequence. Also, we have found numerous sums and various equalities for Gaussian Jacobsthal-Padovan sequence.

Main Results

First, we give the definition of Gaussian Jacobsthal-Padovan number sequence based on the recurrence relation.

Definition 1. The sequence $\{GJP_n\}_{n=0}^{\infty}$ of Gaussian Jacobsthal-Padovan numbers satisfies the following third-order recurrence relation:

$$GJP_n = GJP_{n-2} + 2GJP_{n-3}$$

with initial conditions $GJP_0 = 1,\ GJP_1 = 1 + i,\ GJP_2 = 1 + i$ and $n \geq 3$.

Then we get the Gaussian Jacobsthal-Padovan sequence

$$\{GJP_n\} = \{1, 1+i, 1+i, 3+i, 3+3i, 5+3i, 9+5i, \ldots\}.$$

Also, note that for $n \geq 0$

$$GJP_n = JP_n + iJP_{n-1}$$

where JP_n is the *n-th* Jacobsthal-Padovan numbers.

Theorem 1. The sequence $\{GJP_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$GJP_{-n} = -\frac{1}{2}GJP_{-(n-1)} + \frac{1}{2}GJP_{-(n-3)}$$

for $n \geq 1$.

Proof. From the recurrence relation of Gaussian Jacobsthal-Padovan sequence, we have

$$GJP_{n-3} = \frac{1}{2}GJP_n - \frac{1}{2}GJP_{n-2}.$$

Then, for $n = -n + 3$, we obtain

$$\begin{aligned} GJP_{-n} &= \frac{1}{2}GJP_{-n+3} - \frac{1}{2}GJP_{-n+1} \\ &= \frac{1}{2}GJP_{-(n-3)} - \frac{1}{2}GJP_{-(n-1)} \\ &= -\frac{1}{2}GJP_{-(n-1)} + \frac{1}{2}GJP_{-(n-3)} \end{aligned}$$

as required.
Presently, we deliver the generating function of Gaussian Jacobsthal-Padovan sequence with next theorem.

Theorem 2. The generating function of Gaussian Jacobsthal-Padovan sequence is obtained as

$$g(x) = \frac{1 + (1+i)x + ix^2}{1 - x^2 - 2x^3}.$$

Proof. Assume that $g(x)$ the generating function of $\{GJP_n\}_{n=0}^{\infty}$. By considering the recurrence relation of Gaussian Jacobsthal-Padovan sequence, and deriving $x^2\sum_{n=0}^{\infty} GJP_n x^n$ and $2x^3\sum_{n=0}^{\infty} GJP_n x^n$ from $\sum_{n=0}^{\infty} GJP_n x^n$ we get

$$\begin{aligned} (1 - x^2 - 2x^3)\sum_{n=0}^{\infty} GJP_n x^n &= \sum_{n=0}^{\infty} GJP_n x^n - x^2\sum_{n=0}^{\infty} GJP_n x^n - 2x^3\sum_{n=0}^{\infty} GJP_n x^n \\ &= \sum_{n=0}^{\infty} GJP_n x^n - \sum_{n=0}^{\infty} GJP_n x^{n+2} - 2\sum_{n=0}^{\infty} GJP_n x^{n+3} \\ &= \sum_{n=0}^{\infty} GJP_n x^n - \sum_{n=2}^{\infty} GJP_{n-2} x^n - 2\sum_{n=3}^{\infty} GJP_{n-3} x^n \\ &= (GJP_0 + GJP_1 x + GJP_2 x^2) - GJP_0 x^2 + \sum_{n=3}^{\infty}(GJP_n - GJP_{n-2} - 2GJP_{n-3})x^n \\ &= GJP_0 + GJP_1 x + (GJP_2 - GJP_0)x^2. \end{aligned}$$

Thus, by using the initial conditions, we obtain

$$\sum_{n=0}^{\infty} GJP_n x^n = \frac{1+(1+i)x+ix^2}{1-x^2-2x^3}$$

which is desired.

We now find the Binet formula for Gaussian Jacobsthal-Padovan sequence in the following theorem.

Theorem 3. n^{th} Gaussian Jacobsthal-Padovan number is

$$GJP_n = \frac{(x_1+1)(x_1+i)}{(x_1-x_2)(x_1-x_3)}x_1{}^n + \frac{(x_2+1)(x_2+i)}{(x_2-x_1)(x_2-x_3)}x_2{}^n + \frac{(x_3+1)(x_3+i)}{(x_3-x_1)(x_3-x_2)}x_3{}^n$$

where x_1, x_2 and x_3 are the different roots of the equation $x^3-x-2=0$ and whose roots are

$$x_1 = \sqrt[3]{1+\frac{\sqrt{78}}{9}} + \sqrt[3]{1-\frac{\sqrt{78}}{9}}, \quad x_2 = \omega\sqrt[3]{1+\frac{\sqrt{78}}{9}} + \omega^2\sqrt[3]{1-\frac{\sqrt{78}}{9}}, \quad x_3 = \omega^2\sqrt[3]{1+\frac{\sqrt{78}}{9}} + \omega\sqrt[3]{1-\frac{\sqrt{78}}{9}}$$

where $\omega = \frac{-1+i\sqrt{3}}{2}$.

Proof. Suppose that $g(x) = 1-x^2-2x^3$. Then using the roots x_1, x_2 and x_3 of the equation, we can write $g(x)$ as

$$g(x) = (1-x_1x)(1-x_2x)(1-x_3x),$$

namely,

$$1-x^2-2x^3 = (1-x_1x)(1-x_2x)(1-x_3x) \tag{1}$$

Thus, we find all roots of $g(x)$ as which $\frac{1}{x_1}$, $\frac{1}{x_2}$ and $\frac{1}{x_3}$.

Now, we write the equation (1) and the generating function of $\{GJP_n\}_{n=0}^{\infty}$ as:

$$\begin{aligned}\sum_{n=0}^{\infty} GJP_n x^n &= \frac{1+(1+i)x+ix^2}{1-x^2-2x^3}\\ &= \frac{1+(1+i)x+ix^2}{(1-x_1x)(1-x_2x)(1-x_3x)}\\ &= \frac{A}{1-x_1x} + \frac{B}{1-x_2x} + \frac{C}{1-x_3x}\end{aligned} \tag{2}$$

Hence,

$$1+(1+i)x+ix^2 = A(1-x_2x)(1-x_3x) + B(1-x_1x)(1-x_3x) + C(1-x_1x)(1-x_2x).$$

Then, for $x = \frac{1}{x_1}$, we have $1+(1+i)\frac{1}{x_1}+i\frac{1}{x_1{}^2} = A\left(1-\frac{x_2}{x_1}\right)\left(1-\frac{x_3}{x_1}\right)$. From here, we find

$$A = \frac{x_1{}^2\left[1+(1+i)\frac{1}{x_1}+i\frac{1}{x_1{}^2}\right]}{(x_1-x_2)(x_1-x_3)} = \frac{x_1{}^2+(1+i)x_1+i}{(x_1-x_2)(x_1-x_3)} = \frac{(x_1+1)(x_1+i)}{(x_1-x_2)(x_1-x_3)}.$$

In a similar way, we obtain

$$B = \frac{(x_2+1)(x_2+i)}{(x_2-x_1)(x_2-x_3)}, \quad C = \frac{(x_3+1)(x_3+i)}{(x_3-x_1)(x_3-x_2)}.$$

Consequently, we can write the equation (2) as in the following way

$$\sum_{n=0}^{\infty} GJP_n x^n = A(1-x_1x)^{-1} + B(1-x_2x)^{-1} + C(1-x_3x)^{-1}$$

$$= A\sum_{n=0}^{\infty} x_1{}^n x^n + B\sum_{n=0}^{\infty} x_2{}^n x^n + C\sum_{n=0}^{\infty} x_3{}^n x^n$$

$$= \sum_{n=0}^{\infty} (Ax_1{}^n + Bx_2{}^n + Cx_3{}^n)x^n.$$

Thus, we obtain for all $n \geq 0$

$$GJP_n = Ax_1{}^n + Bx_2{}^n + Cx_3{}^n$$

$$= \frac{(x_1+1)(x_1+i)}{(x_1-x_2)(x_1-x_3)}x_1{}^n + \frac{(x_2+1)(x_2+i)}{(x_2-x_1)(x_2-x_3)}x_2{}^n + \frac{(x_3+1)(x_3+i)}{(x_3-x_1)(x_3-x_2)}x_3{}^n$$

as required.

The following theorem gives the Simson formula for Gaussian Jacobsthal-Padovan sequence.

Theorem 4. For $n \in \mathbb{Z}$, we have

$$\begin{vmatrix} GJP_{n+2} & GJP_{n+1} & GJP_n \\ GJP_{n+1} & GJP_n & GJP_{n-1} \\ GJP_n & GJP_{n-1} & GJP_{n-2} \end{vmatrix} = -2^n(1-i).$$

Proof. We show the proof of this theorem by induction over n. For $n = 1$, the statement is true. In fact

$$\begin{vmatrix} GJP_3 & GJP_2 & GJP_1 \\ GJP_2 & GJP_1 & GJP_0 \\ GJP_1 & GJP_0 & GJP_{-1} \end{vmatrix} = -2 + 2i = -2^1(1-i).$$

Now, let this statement be true for $n = k$. That is,

$$\begin{vmatrix} GJP_{k+2} & GJP_{k+1} & GJP_k \\ GJP_{k+1} & GJP_k & GJP_{k-1} \\ GJP_k & GJP_{k-1} & GJP_{k-2} \end{vmatrix} = -2^k(1-i).$$

Finally, we must show that the statement is correct for $n = k + 1$. We obtain from induction hypothesis and the properties of determinant function.

$$\begin{vmatrix} GJP_{k+3} & GJP_{k+2} & GJP_{k+1} \\ GJP_{k+2} & GJP_{k+1} & GJP_k \\ GJP_{k+1} & GJP_k & GJP_{k-1} \end{vmatrix} = \begin{vmatrix} GJP_{k+1} + 2GJP_k & GJP_{k+2} & GJP_{k+1} \\ GJP_k + 2GJP_{k-1} & GJP_{k+1} & GJP_k \\ GJP_{k-1} + 2GJP_{k-2} & GJP_k & GJP_{k-1} \end{vmatrix}$$

$$= \begin{vmatrix} GJP_{k+1} & GJP_{k+2} & GJP_{k+1} \\ GJP_k & GJP_{k+1} & GJP_k \\ GJP_{k-1} & GJP_k & GJP_{k-1} \end{vmatrix} + \begin{vmatrix} 2GJP_k & GJP_{k+2} & GJP_{k+1} \\ 2GJP_{k-1} & GJP_{k+1} & GJP_k \\ 2GJP_{k-2} & GJP_k & GJP_{k-1} \end{vmatrix}$$

$$= 2\begin{vmatrix} GJP_k & GJP_{k+2} & GJP_{k+1} \\ GJP_{k-1} & GJP_{k+1} & GJP_k \\ GJP_{k-2} & GJP_k & GJP_{k-1} \end{vmatrix}$$

$$= -2\begin{vmatrix} GJP_{k+2} & GJP_k & GJP_{k+1} \\ GJP_{k+1} & GJP_{k-1} & GJP_k \\ GJP_k & GJP_{k-2} & GJP_{k-1} \end{vmatrix}$$

$$= 2\begin{vmatrix} GJP_{k+2} & GJP_{k+1} & GJP_k \\ GJP_{k+1} & GJP_k & GJP_{k-1} \\ GJP_k & GJP_{k-1} & GJP_{k-2} \end{vmatrix}$$

$$= 2\left[-2^{k}(1-i)\right]$$
$$= -2^{k+1}(1-i).$$

Therefore, the statement is also correct for $n = k + 1$. □

In the next theorem, we give some summation formulas of Gaussian Jacobsthal-Padovan sequence.

Theorem 5. For $n \geq 1$, we have the following sums:

i. $\sum_{k=1}^{n} GJP_k = \frac{1}{2}(GJP_{n+2} + GJP_{n+3}) - (2 + i)$,

ii. $\sum_{k=1}^{n} GJP_{2k} = \frac{1}{2} GJP_{2n+3} - (3 + i)$,

iii. $\sum_{k=1}^{n} GJP_{2k-1} = \frac{1}{2} GJP_{2n+2} - (1 + i)$.

Proof (i). From the recursive relation of Gaussian Jacobsthal-Padovan sequence, we have

$$GJP_{n-3} = \tfrac{1}{2} GJP_n - \tfrac{1}{2} GJP_{n-2}. \qquad (3)$$

Thus, we have from the equation (3)

$$GJP_1 = \frac{1}{2} GJP_4 - \frac{1}{2} GJP_2$$
$$GJP_2 = \frac{1}{2} GJP_5 - \frac{1}{2} GJP_3$$
$$GJP_3 = \frac{1}{2} GJP_6 - \frac{1}{2} GJP_4$$
$$\vdots$$
$$GJP_n = \frac{1}{2} GJP_{n+3} - \frac{1}{2} GJP_{n+1}.$$

After performing necessary calculations, we obtain

$$\sum_{k=1}^{n} GJP_k = \frac{1}{2}(GJP_{n+2} + GJP_{n+3}) - \frac{1}{2}(GJP_2 + GJP_3)$$

$$= \frac{1}{2}(GJP_{n+2} + GJP_{n+3}) - (2 + i)$$

which is desired.

The proof of (ii) and (iii) can be done similarly to the proof of (i).

Theorem 6. For $n \in \mathbb{Z}^{+}$, we have

$$\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{n} . \begin{pmatrix} 1+i \\ 1+i \\ 1 \end{pmatrix} = \begin{pmatrix} GPJ_{n+2} \\ GPJ_{n+1} \\ GPJ_{n} \end{pmatrix}.$$

Proof. We can prove the theorem by induction on n. For $n = 1$, we get

$$\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{1} . \begin{pmatrix} 1+i \\ 1+i \\ 1 \end{pmatrix} = \begin{pmatrix} 3+i \\ 1+i \\ 1+i \end{pmatrix} = \begin{pmatrix} GPJ_{3} \\ GPJ_{2} \\ GPJ_{1} \end{pmatrix}.$$

Assume that the equality holds for $n = k$, namely,

$$\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{k} . \begin{pmatrix} 1+i \\ 1+i \\ 1 \end{pmatrix} = \begin{pmatrix} GPJ_{k+2} \\ GPJ_{k+1} \\ GPJ_{k} \end{pmatrix}.$$

Now, we need to show that it is true for $n = k + 1$. Hence, we obtain

$$\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{k+1} . \begin{pmatrix} 1+i \\ 1+i \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} . \left[\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{k} . \begin{pmatrix} 1+i \\ 1+i \\ 1 \end{pmatrix} \right]$$

$$= \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} . \begin{pmatrix} GPJ_{k+2} \\ GPJ_{k+1} \\ GPJ_{k} \end{pmatrix}$$

$$= \begin{pmatrix} GPJ_{k+1} + 2GJP_{k} \\ GPJ_{k+2} \\ GPJ_{k+1} \end{pmatrix}$$

$$= \begin{pmatrix} GPJ_{k+3} \\ GPJ_{k+2} \\ GPJ_{k+1} \end{pmatrix}.$$

The proof is completed.

Conflicts of interest

The authors state that did not have conflict of interests.

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