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A Study On Generalized Absolute Matrix Summability

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ABSTRACT
In the present paper, generalized absolute matrix summability method of infinite series has been studied. A known theorem on $|M|_\kappa$ summability method has been generalized using the $|M,p_\eta,\lambda;\mu|_\kappa$ summability method of infinite series. So a new theorem has been established and proved. Some results related to the new theorem also have been obtained.

Keywords: Summability factors, Absolute matrix summability, Infinite series, Hölder inequality, Minkowski inequality.

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Introduction

Summability theory is important for analysis, applied mathematics and engineering sciences. The purpose of this theory is to bring an appropriate value to the indefinite divergent series. Various summability methods have been defined by some researchers to find the value. Some of these methods are Cesàro [1], Abel [2], Nörlund [3], Riesz [4], matrix summability [5].

A significant increase began in studies on the summability theory in the second half of the 19th century. In 1890, Cesàro published a paper on the multiplication of series [1]. Das gave the definition of absolute summability [6]. Then Kishore and Hotta defined the summability factor [7]. The definition of $|M|_\kappa$ summability was given by Tanović-Miller [8]. Later Bor defined $|\bar{N},p_\eta|_\kappa$ and $|\bar{N},p_\eta;\mu|_\kappa$ summability of an infinite series [9, 10]. The definition of $|M,p_\eta;\mu|_\kappa$ summability of an infinite series was defined by Özarslan and Öğdük [11]. The definition of $|M,p_\eta,\lambda;\mu|_\kappa$ summability was given by Özarslan and Karakaş [12]. In this paper a theorem on absolute matrix summability is obtained using $|M,p_\eta,\lambda;\mu|_\kappa$ summability method. Now we give some definitions related to the summability which are used in this article.

Definition 1 [13]**.** Let (s_η) be partial sums of the infinite series $\sum m_\eta$. (p_η) is a sequence such that

$$p_\eta>0 \text{ and } P_\eta=\sum_{v=0}^{\eta}p_v\to\infty \text{ as } \eta\to\infty \; (P_{-j}=p_{-j}=0,\ j\geq 1). \tag{1}$$

(ϑ_η) is the $(\bar{N},p_\eta)$ means of the sequence (s_η) such that

$$\vartheta_\eta=\frac{1}{P_\eta}\sum_{v=0}^{\eta}p_v s_v. \tag{2}$$

Definition 2 [9]**.** The series $\sum m_\eta$ is called summable $|\bar{N},p_\eta|_\kappa$, $\kappa\geq 1$, if

$$\sum_{\eta=1}^{\infty}\left(\frac{P_\eta}{p_\eta}\right)^{\kappa-1}\left|\vartheta_\eta-\vartheta_{\eta-1}\right|^\kappa<\infty. \tag{3}$$

Definition 3 [8]**.** Let $M=(m_{\eta v})$ be a normal matrix, i.e, a lower triangular matrix of nonzero diagonal entries. By $M=(m_{\eta v})$, a transformation from sequence $s=(s_\eta)$ to $Ms=(M_\eta(s))$ can be constituted where

$$M_\eta(s)=\sum_{v=0}^{\eta}m_{\eta v}s_v,\ \eta=0,1,... \tag{4}$$

The series $\sum m_\eta$ is called summable $|M|_\kappa$, $\kappa\geq 1$, if

$$\sum_{\eta=1}^{\infty}\eta^{\kappa-1}\left|\bar{\Delta}M_\eta(s)\right|^\kappa<\infty, \tag{5}$$

where

$$\bar{\Delta}M_\eta(s)=M_\eta(s)-M_{\eta-1}(s). \tag{6}$$

Definition 4 [14]**.** The series $\sum m_\eta$ is called summable $|M,p_\eta|_\kappa$, $\kappa\geq 1$, if

$$\sum_{\eta=1}^{\infty}\left(\frac{P_\eta}{p_\eta}\right)^{\kappa-1}\left|\bar{\Delta}M_\eta(s)\right|^{\kappa}<\infty . \tag{7}$$

Definition 5 [12]**.** The series $\sum m_\eta$ is called summable $\left|M,p_\eta,\lambda;\mu\right|_\kappa$, $\kappa\geq 1$, $\mu\geq 0$ and λ is a real number if

$$\sum_{\eta=1}^{\infty}\left(\frac{P_\eta}{p_\eta}\right)^{\lambda(\mu\kappa+\kappa-1)}\left|\bar{\Delta}M_\eta(s)\right|^{\kappa}<\infty . \tag{8}$$

Here, if we choose $\lambda=1$ and $\mu=0$, $\left|M,p_\eta,\lambda;\mu\right|_\kappa$ summability reduces to $\left|M,p_\eta\right|_\kappa$ summability. Also, by taking $\lambda=1$, $\mu=0$ and $p_\eta=1$ for $\forall\,\eta\in\mathbb{N}$, $\left|M,p_\eta,\lambda;\mu\right|_\kappa$ summability reduces to $\left|M\right|_\kappa$ summability.

Known Results

The following lemmas and theorem on $\left|M\right|_\kappa$ summability of the series $\sum m_\eta\lambda_\eta X_\eta$ have been proved by Sulaiman in [15].

Lemma 1. If $\sum \eta^{-1}\lambda_\eta$ is convergent, then (λ_η) is non-negative and decreasing, $\lambda_\eta\log\eta=O(1)$, and $\eta\Delta\lambda_\eta=O(1/(\log\eta)^2)$.

Lemma 2. If $\sum \eta^{-1}\lambda_\eta X_\eta$ is convergent, such that

$$\eta\Delta\lambda_\eta=O(\lambda_\eta) \text{ as } \eta\to\infty, \tag{9}$$

$$\sum_{v=1}^{\eta}\lambda_v=O(\eta\lambda_\eta) \text{ as } \eta\to\infty, \tag{10}$$

then

$$\eta\lambda_\eta\Delta X_\eta=O(1), \tag{11}$$

$$\sum_{\eta=1}^{q}\lambda_\eta\Delta X_\eta=O(1) \text{ as } q\to\infty, \tag{12}$$

$$\sum_{\eta=1}^{q}\eta\lambda_\eta\Delta^2X_\eta=O(1) \text{ as } q\to\infty. \tag{13}$$

Theorem 1. Let (λ_η), (X_η) be two sequences such that $\sum_{\eta=1}^{\infty}\eta^{-1}\lambda_\eta X_\eta$ is convergent, and the conditions (9), (10) are satisfied. Let $M=(m_{\eta v})$ be a normal matrix with non-negative entries satisfying

$$\bar{m}_{\eta 0}=1, \quad \eta=0,1,..., \tag{14}$$

$$m_{\eta-1,v}\geq m_{\eta v}, \text{ for } \eta\geq v+1, \tag{15}$$

$$\eta m_{\eta\eta}=O(1),\ 1=O(\eta m_{\eta\eta}), \tag{16}$$

$$\sum_{v=1}^{\eta-1}m_{vv}\,\widehat{m}_{\eta v}=O(m_{\eta\eta}). \tag{17}$$

If $u_v^\kappa=O(1)\,(C,1)$, where $u_v=\frac{1}{v+1}\sum_{i=1}^{v}i\,m_i$, then the series $\sum m_\eta\lambda_\eta X_\eta$ is summable $\left|M\right|_\kappa$, $\kappa\geq 1$.

Lemma 3. According to Theorem 1, we have

$$\sum_{v=0}^{\eta-1}\left|\Delta_v(\widehat{m}_{\eta v})\right|=m_{\eta\eta}, \tag{18}$$

$$\widehat{m}_{\eta,v+1}\geq 0, \tag{19}$$

$$\sum_{\eta=v+1}^{q+1}\widehat{m}_{\eta,v+1}=O(1). \tag{20}$$

Main Result

There are many studies on absolute matrix summability of infinite series [16-29]. This study provides a generalization of above mentioned theorem to $\left|M,p_\eta,\lambda;\mu\right|_\kappa$ summability under some suitable conditions. For the convenience of the reader, we give some further notations.

Let $M=(m_{\eta v})$ be a normal matrix. The definition of two lower semi-matrices $\bar{M}=(\bar{m}_{\eta v})$ and $\widehat{M}=(\widehat{m}_{\eta v})$ are as follows.

$$\bar{m}_{\eta v}=\sum_{i=v}^{\eta}m_{\eta i},\ \eta,v=0,1,... \tag{21}$$

and

$$\widehat{m}_{00}=\bar{m}_{00}=m_{00},\ \widehat{m}_{\eta v}=\bar{m}_{\eta v}-\bar{m}_{\eta-1,v},\ \eta=1,2,... \tag{22}$$

It is well-known that

$$M_\eta(s)=\sum_{v=0}^{\eta}m_{\eta v}s_v=\sum_{v=0}^{\eta}\bar{m}_{\eta v}m_v \tag{23}$$

and

$$\bar{\Delta}M_\eta(s)=\sum_{v=0}^{\eta}\widehat{m}_{\eta v}m_v . \tag{24}$$

Now, let's give the main theorem.

Theorem 2. Let (λ_η) and (X_η) be two sequences such that $\sum_{\eta=1}^{\infty}\eta^{-1}\lambda_\eta X_\eta$ is convergent. The conditions (9), (10), (14)-(17) and

$$m_{\eta\eta}=O\left(\frac{p_\eta}{P_\eta}\right), \tag{25}$$

$$\sum_{\eta=v+1}^{q+1}\left(\frac{P_\eta}{p_\eta}\right)^{\lambda(\mu\kappa+\kappa-1)-\kappa+1}\widehat{m}_{\eta,v+1}=O(1) \text{ as } q\to\infty, \tag{26}$$

$$\sum_{\eta=v+1}^{q+1}\left(\frac{P_\eta}{p_\eta}\right)^{\lambda(\mu\kappa+\kappa-1)-\kappa+1}\left|\Delta_v(\widehat{m}_{\eta v})\right|=O\left(m_{vv}\right) \text{ as } q\to\infty, \tag{27}$$

$$\sum_{v=1}^{\eta-1}m_{vv}\,\widehat{m}_{\eta,v+1}=O(m_{\eta\eta}) \tag{28}$$

are satisfied.

If $\left(\frac{P_v}{p_v}\right)^{\lambda(\mu\kappa+\kappa-1)-\kappa+1}u_v^\kappa=O(1)\,(C,1)$, where (u_v) as in Theorem 1, then the series $\sum m_\eta\lambda_\eta X_\eta$ is summable $\left|M,p_\eta,\lambda;\mu\right|_\kappa$, $\kappa\geq 1$, $\mu\geq 0$ and $-\lambda(\mu\kappa+\kappa-1)+\kappa>0$.

Proof of Theorem 2

Let $\phi_\eta=\lambda_\eta X_\eta$ and (W_η) be $M-$transform of the series $\sum m_\eta\phi_\eta$. By (23) and (24), we get

$$\bar{\Delta}W_\eta=\sum_{v=1}^{\eta}\widehat{m}_{\eta v}m_v\phi_v=\sum_{v=1}^{\eta}\frac{\widehat{m}_{\eta v}\phi_v}{v}m_v\,v\,.$$

Using Abel's transformation, we obtain the following.

$$\bar{\Delta}W_\eta=\sum_{v=1}^{\eta-1}\Delta_v\left(\frac{\widehat{m}_{\eta v}\phi_v}{v}\right)\sum_{i=1}^{v}i\,m_i+\frac{\widehat{m}_{\eta\eta}\phi_\eta}{\eta}\sum_{i=1}^{\eta}i\,m_i$$

$$=\sum_{v=1}^{\eta-1}\Delta_v\left(\frac{\widehat{m}_{\eta v}\phi_v}{v}\right)(v+1)u_v+\frac{m_{\eta\eta}\phi_\eta}{\eta}(\eta+1)u_\eta$$

$$=\sum_{v=1}^{\eta-1}\frac{1}{v}\widehat{m}_{\eta v}\,\phi_v\,u_v+\sum_{v=1}^{\eta-1}\Delta_v(\widehat{m}_{\eta v})\phi_v\,u_v+\sum_{v=1}^{\eta-1}\widehat{m}_{\eta,v+1}\,\Delta\phi_v\,u_v$$

$$+\frac{\eta+1}{\eta}m_{\eta\eta}\,\phi_\eta\,u_\eta$$

$$=W_{\eta,1}+W_{\eta,2}+W_{\eta,3}+W_{\eta,4}\,.$$

It is sufficient to prove

$$\sum_{\eta=1}^{\infty}\left(\frac{P_\eta}{p_\eta}\right)^{\lambda(\mu\kappa+\kappa-1)}\left|W_{\eta,j}\right|^\kappa<\infty,\ \text{ for } j=1,2,3,4.$$

We first apply Hölder's inequality to obtain

$$\sum_{\eta=2}^{q+1}\left(\frac{P_\eta}{p_\eta}\right)^{\lambda(\mu\kappa+\kappa-1)}\left|W_{\eta,1}\right|^\kappa=\sum_{\eta=2}^{q+1}\left(\frac{P_\eta}{p_\eta}\right)^{\lambda(\mu\kappa+\kappa-1)}\left|\sum_{v=1}^{\eta-1}\frac{1}{v}\widehat{m}_{\eta v}\phi_v u_v\right|^\kappa$$

$$=O(1)\sum_{\eta=2}^{q+1}\left(\frac{P_\eta}{p_\eta}\right)^{\lambda(\mu\kappa+\kappa-1)}\left(\sum_{v=1}^{\eta-1}\frac{1}{v^\kappa}u_v^\kappa m_{vv}^{1-\kappa}\widehat{m}_{\eta v}\phi_v^\kappa\right)\left(\sum_{v=1}^{\eta-1}m_{vv}\widehat{m}_{\eta v}\right)^{\kappa-1}$$

By using (17), (25), (16) and (26), we get

$$\sum_{\eta=2}^{q+1}\left(\frac{P_\eta}{p_\eta}\right)^{\lambda(\mu\kappa+\kappa-1)}\left|W_{\eta,1}\right|^\kappa=O(1)\sum_{\eta=2}^{q+1}\left(\frac{P_\eta}{p_\eta}\right)^{\lambda(\mu\kappa+\kappa-1)}m_{\eta\eta}^{\kappa-1}\left(\sum_{v=1}^{\eta-1}\frac{1}{v^\kappa}u_v^\kappa m_{vv}^{1-\kappa}\widehat{m}_{\eta v}\phi_v^\kappa\right)$$

$$=O(1)\sum_{\eta=2}^{q+1}\left(\frac{P_\eta}{p_\eta}\right)^{\lambda(\mu\kappa+\kappa-1)-\kappa+1}\left(\sum_{v=1}^{\eta-1}u_v^\kappa m_{vv}\widehat{m}_{\eta v}\phi_v^\kappa\right)$$

$$=O(1)\sum_{v=1}^{q}m_{vv}u_v^\kappa\phi_v^\kappa\sum_{\eta=v+1}^{q+1}\left(\frac{P_\eta}{p_\eta}\right)^{\lambda(\mu\kappa+\kappa-1)-\kappa+1}\widehat{m}_{\eta v}$$

$$=O(1)\sum_{v=1}^{q}m_{vv}u_v^\kappa\phi_v^\kappa$$

$$=O(1)\sum_{v=1}^{q}\frac{1}{v}u_v^\kappa\phi_v\phi_v^{\kappa-1}\,.$$

Using $\phi_v^{\kappa-1}=O(1)$, we have

$$\sum_{\eta=2}^{q+1}\left(\frac{P_\eta}{p_\eta}\right)^{\lambda(\mu\kappa+\kappa-1)}\left|W_{\eta,1}\right|^\kappa=O(1)\sum_{v=1}^{q}\frac{\phi_v}{v}u_v^\kappa\,.$$

Here, applying Abel's transformation, we have

$$\sum_{\eta=2}^{q+1}\left(\frac{P_\eta}{p_\eta}\right)^{\lambda(\mu\kappa+\kappa-1)}\left|W_{\eta,1}\right|^\kappa=O(1)\sum_{v=1}^{q-1}\Delta\left(\frac{\phi_v}{v}\right)\sum_{i=1}^{v}u_i^\kappa+O(1)\frac{\phi_q}{q}\sum_{v=1}^{q}u_v^\kappa$$

$$=O(1)\sum_{v=1}^{q-1}v\,\Delta\left(\frac{\phi_v}{v}\right)+O(1)\phi_q\,.$$

Since

$$\Delta\left(\frac{\phi_v}{v}\right)=\frac{\phi_v}{v}-\frac{\phi_{v+1}}{v+1}<\frac{\phi_v}{v^2}+\frac{\Delta\phi_v}{v+1},$$

we have

$$\sum_{\eta=2}^{q+1}\left(\frac{P_\eta}{p_\eta}\right)^{\lambda(\mu\kappa+\kappa-1)}\left|W_{\eta,1}\right|^\kappa=O(1)\sum_{v=1}^{q-1}v\left(\frac{\phi_v}{v^2}+\frac{\Delta\phi_v}{v+1}\right)+O(1)\phi_q$$

$$=O(1)\sum_{v=1}^{q-1}\frac{\phi_v}{v}+O(1)\sum_{v=1}^{q-1}\Delta\phi_v+O(1)\phi_q\,.$$

In that case, we obtain

$$\sum_{\eta=2}^{q+1}\left(\frac{P_\eta}{p_\eta}\right)^{\lambda(\mu\kappa+\kappa-1)}\left|W_{\eta,1}\right|^\kappa = O(1)\sum_{v=1}^{q-1}\frac{\lambda_v X_v}{v}+O(1)\sum_{v=1}^{q-1}\Delta(\lambda_v X_v)+O(1)\lambda_q X_q$$
$$=O(1) \text{ as } q\to\infty .$$

We now apply Hölder's inequality to obtain

$$\sum_{\eta=2}^{q+1}\left(\frac{P_\eta}{p_\eta}\right)^{\lambda(\mu\kappa+\kappa-1)}\left|W_{\eta,2}\right|^\kappa \le \sum_{\eta=2}^{q+1}\left(\frac{P_\eta}{p_\eta}\right)^{\lambda(\mu\kappa+\kappa-1)}\left(\sum_{v=1}^{\eta-1}\left|\Delta_v(\widehat{m}_{\eta v})\right|\phi_v u_v\right)^\kappa$$

$$\le \sum_{\eta=2}^{q+1}\left(\frac{P_\eta}{p_\eta}\right)^{\lambda(\mu\kappa+\kappa-1)}\left(\sum_{v=1}^{\eta-1}\left|\Delta_v(\widehat{m}_{\eta v})\right|\phi_v^\kappa u_v^\kappa\right)\left(\sum_{v=1}^{\eta-1}\left|\Delta_v(\widehat{m}_{\eta v})\right|\right)^{\kappa-1}.$$

By using (21) and (22), we get

$$\Delta_v(\widehat{m}_{\eta v}) = \widehat{m}_{\eta v}-\widehat{m}_{\eta,v+1} = \overline{m}_{\eta v}-\overline{m}_{\eta-1,v}-\overline{m}_{\eta,v+1}+\overline{m}_{\eta-1,v+1}$$
$$= m_{\eta v}-m_{\eta-1,v}. \tag{29}$$

Thus using (21), (14), (15) and (29), we obtain

$$\sum_{v=1}^{\eta-1}\left|\Delta_v(\widehat{m}_{\eta v})\right| = \sum_{v=1}^{\eta-1}(m_{\eta-1,v}-m_{\eta v})\le m_{\eta\eta}. \tag{30}$$

Then, we get

$$\sum_{\eta=2}^{q+1}\left(\frac{P_\eta}{p_\eta}\right)^{\lambda(\mu\kappa+\kappa-1)}\left|W_{\eta,2}\right|^\kappa = O(1)\sum_{\eta=2}^{q+1}\left(\frac{P_\eta}{p_\eta}\right)^{\lambda(\mu\kappa+\kappa-1)} m_{\eta\eta}^{\kappa-1}\left(\sum_{v=1}^{\eta-1}\left|\Delta_v(\widehat{m}_{\eta v})\right|\phi_v^\kappa u_v^\kappa\right)$$

By using (25) and (27), we have

$$\sum_{\eta=2}^{q+1}\left(\frac{P_\eta}{p_\eta}\right)^{\lambda(\mu\kappa+\kappa-1)}\left|W_{\eta,2}\right|^\kappa = O(1)\sum_{\eta=2}^{q+1}\left(\frac{P_\eta}{p_\eta}\right)^{\lambda(\mu\kappa+\kappa-1)-\kappa+1}\left(\sum_{v=1}^{\eta-1}\left|\Delta_v(\widehat{m}_{\eta v})\right|\phi_v^\kappa u_v^\kappa\right)$$

$$= O(1)\sum_{v=1}^{q}\phi_v^\kappa u_v^\kappa\sum_{\eta=v+1}^{q+1}\left(\frac{P_\eta}{p_\eta}\right)^{\lambda(\mu\kappa+\kappa-1)-\kappa+1}\left|\Delta_v(\widehat{m}_{\eta v})\right|$$
$$= O(1)\sum_{v=1}^{q} m_{vv}\, u_v^\kappa\phi_v^{\kappa-1}\phi_v$$
$$= O(1) \text{ as } q\to\infty ,$$

as in $W_{\eta,1}$.

We now apply Hölder's inequality again to obtain

$$\sum_{\eta=2}^{q+1}\left(\frac{P_\eta}{p_\eta}\right)^{\lambda(\mu\kappa+\kappa-1)}\left|W_{\eta,3}\right|^\kappa = O(1)\sum_{\eta=2}^{q+1}\left(\frac{P_\eta}{p_\eta}\right)^{\lambda(\mu\kappa+\kappa-1)}\left(\sum_{v=1}^{\eta-1}\widehat{m}_{\eta,v+1}\Delta\phi_v u_v\right)^\kappa$$

$$= O(1)\sum_{\eta=2}^{q+1}\left(\frac{P_\eta}{p_\eta}\right)^{\lambda(\mu\kappa+\kappa-1)}\left(\sum_{v=1}^{\eta-1}u_v^\kappa m_{vv}^{1-\kappa}\widehat{m}_{\eta,v+1}(\Delta\phi_v)^\kappa\right)\left(\sum_{v=1}^{\eta-1}m_{vv}\widehat{m}_{\eta,v+1}\right)^{\kappa-1}$$
.

Using (28), we have

$$\sum_{\eta=2}^{q+1}\left(\frac{P_\eta}{p_\eta}\right)^{\lambda(\mu\kappa+\kappa-1)}\left|W_{\eta,3}\right|^\kappa = O(1)\sum_{\eta=2}^{q+1}\left(\frac{P_\eta}{p_\eta}\right)^{\lambda(\mu\kappa+\kappa-1)} m_{\eta\eta}^{\kappa-1}\left(\sum_{v=1}^{\eta-1}u_v^\kappa m_{vv}^{1-\kappa}\widehat{m}_{\eta,v+1}(\Delta\phi_v)^\kappa\right)$$
.

By using (25), (26) and (16), we have

$$\sum_{\eta=2}^{q+1}\left(\frac{P_\eta}{p_\eta}\right)^{\lambda(\mu\kappa+\kappa-1)}\left|W_{\eta,3}\right|^\kappa = O(1)\sum_{\eta=2}^{q+1}\left(\frac{P_\eta}{p_\eta}\right)^{\lambda(\mu\kappa+\kappa-1)-\kappa+1}\left(\sum_{v=1}^{\eta-1}u_v^\kappa m_{vv}^{1-\kappa}\widehat{m}_{\eta,v+1}(\Delta\phi_v)^\kappa\right)$$

$$= O(1)\sum_{v=1}^{q}u_v^\kappa m_{vv}^{1-\kappa}(\Delta\phi_v)^\kappa\sum_{\eta=v+1}^{q+1}\left(\frac{P_\eta}{p_\eta}\right)^{\lambda(\mu\kappa+\kappa-1)-\kappa+1}\widehat{m}_{\eta,v+1}$$
$$= O(1)\sum_{v=1}^{q}u_v^\kappa v^{\kappa-1}(\Delta\phi_v)^\kappa .$$

Here $(v\Delta\phi_v)^{\kappa-1} = O(1)$, then we have

$$\sum_{\eta=2}^{q+1}\left(\frac{P_\eta}{p_\eta}\right)^{\lambda(\mu\kappa+\kappa-1)}\left|W_{\eta,3}\right|^\kappa = O(1)\sum_{v=1}^{q}u_v^\kappa\Delta\phi_v$$
$$= O(1)\sum_{v=1}^{q}u_v^\kappa\Delta\lambda_v X_v + O(1)\sum_{v=1}^{q}u_v^\kappa\lambda_{v+1}\Delta X_v$$
$$= Y_1+Y_2 .$$

The condition (9) and Abel's transformation enable us to write

$$Y_1 = O(1)\sum_{v=1}^{q}u_v^\kappa\frac{1}{v}\lambda_v X_v$$
$$= O(1)\sum_{v=1}^{q-1}\Delta\left(\frac{\lambda_v X_v}{v}\right)\sum_{i=1}^{v}u_i^\kappa + O(1)\frac{\lambda_q X_q}{q}\sum_{v=1}^{q}u_v^\kappa .$$

Then, we have

$$Y_1 = O(1)\sum_{v=1}^{q-1}v\left(\frac{\lambda_v X_v}{v^2}+\frac{\Delta\lambda_v X_v}{v}+\frac{\lambda_{v+1}\Delta X_v}{v}\right)+O(1)\lambda_q X_q$$
$$= O(1)\sum_{v=1}^{q-1}\frac{\lambda_v X_v}{v}+O(1)\sum_{v=1}^{q-1}\Delta\lambda_v X_v+O(1)\sum_{v=1}^{q-1}\lambda_v\Delta X_v+O(1)\lambda_q X_q$$
$$= O(1) \text{ as } q\to\infty .$$

By Abel's transformation and (9), (12), (13), (11), we get

$$Y_2 = O(1)\sum_{v=1}^{q-1}\Delta(\lambda_v\Delta X_v)\sum_{i=1}^{v}u_i^\kappa + O(1)\lambda_q\Delta X_q\sum_{v=1}^{q}u_v^\kappa$$
$$= O(1)\sum_{v=1}^{q-1}v\Delta(\lambda_v\Delta X_v)+O(1)q\lambda_q\Delta X_q$$
$$= O(1)\sum_{v=1}^{q-1}v(\Delta\lambda_v\Delta X_v+\lambda_{v+1}\Delta^2 X_v)+O(1)q\lambda_q\Delta X_q$$

$$=O(1)\sum_{v=1}^{q-1}\lambda_v\Delta X_v+O(1)\sum_{v=1}^{q-1}v\lambda_v\Delta^2X_v+O(1)q\lambda_q\Delta X_q$$
$$=O(1) \text{ as } q\to\infty .$$

Since $Y_1=O(1)$ and $Y_2=O(1)$, we get

$$\sum_{\eta=2}^{q+1}\left(\frac{P_\eta}{p_\eta}\right)^{\lambda(\mu\kappa+\kappa-1)}\left|W_{\eta,3}\right|^\kappa=O(1) \text{ as } q\to\infty .$$

Finally, using (25), (16) and Abel's transformation, we obtain

$$\sum_{\eta=1}^{q}\left(\frac{P_\eta}{p_\eta}\right)^{\lambda(\mu\kappa+\kappa-1)}\left|W_{\eta,4}\right|^\kappa=O(1)\sum_{\eta=1}^{q}\left(\frac{P_\eta}{p_\eta}\right)^{\lambda(\mu\kappa+\kappa-1)}m_{\eta\eta}^{\kappa-1}m_{\eta\eta}\phi_\eta^\kappa u_\eta^\kappa$$
$$=O(1)\sum_{\eta=1}^{q}\left(\frac{P_\eta}{p_\eta}\right)^{\lambda(\mu\kappa+\kappa-1)-\kappa+1}m_{\eta\eta}\phi_\eta^{\kappa-1}\phi_\eta u_\eta^\kappa$$
$$=O(1)\sum_{\eta=1}^{q}\left(\frac{P_\eta}{p_\eta}\right)^{\lambda(\mu\kappa+\kappa-1)-\kappa+1}\frac{\phi_\eta}{\eta}u_\eta^\kappa$$
$$=O(1)\sum_{\eta=1}^{q-1}\Delta\left(\frac{\phi_\eta}{\eta}\right)\sum_{v=1}^{\eta}\left(\frac{P_v}{p_v}\right)^{\lambda(\mu\kappa+\kappa-1)-\kappa+1}u_v^\kappa+O(1)\frac{\phi_q}{q}\sum_{\eta=1}^{q}\left(\frac{P_\eta}{p_\eta}\right)^{\lambda(\mu\kappa+\kappa-1)-\kappa+1}u_\eta^\kappa$$
$$=O(1)\sum_{\eta=1}^{q-1}\eta\Delta\left(\frac{\phi_\eta}{\eta}\right)+O(1)\phi_q .$$

So, we have

$$\sum_{\eta=1}^{q}\left(\frac{P_\eta}{p_\eta}\right)^{\lambda(\mu\kappa+\kappa-1)}\left|W_{\eta,4}\right|^\kappa=O(1) \text{ as } q\to\infty ,$$

as in $W_{\eta,1}$.

Hence proof of the theorem is completed.

Conclusion

If we choose $\lambda=1$, $\mu=0$ and $p_\eta=1$ for $\forall\eta\in\mathbb{N}$, then we obtain Theorem 1. In that case, (25) reduces to $\eta m_{\eta\eta}=O(1)$ (first part of (16)). In addition, (26)-(28) are automatically satisfied.

Conflicts of interest

The author state that did not have conflict of interests.

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