Cumhuriyet Sci. J., 43(4) (2022) 684-695
DOI: https://doi.org/10.17776/csj.1005486

Cumhuriyet Science Journal

| csj.cumhuriyet.edu.tr | Founded: 2002 ISSN: 2587-2680 e-ISSN: 2587-246X Publisher: Sivas Cumhuriyet University

Generalized Gamma, Beta and Hypergeometric Functions Defined by Wright Function and Applications to Fractional Differential Equations

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Research Article

History
Received: 06/10/2021
Accepted: 30/11/2022

ABSTRACT

When the literature is examined, it is seen that there are many studies on the generalizations of gamma, beta and hypergeometric functions. In this paper, new types of generalized gamma and beta functions are defined and examined using the Wright function. With the help of generalized beta function, new type of generalized Gauss and confluent hypergeometric functions are obtained. Furthermore, some properties of these functions such as integral representations, derivative formulas, Mellin transforms, Laplace transforms and transform formulas are determined. As examples, we obtained the solution of fractional differential equations involving the new generalized beta, Gauss hypergeometric and confluent hypergeometric functions. Finally, we presented their relationship with other generalized gamma, beta, Gauss hypergeometric and confluent hypergeometric functions, which can be found in the literature.

Keywords: Beta function, Wright function, Gauss hypergeometric function, Laplace transform, Fractional differential equation

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Introduction

Gamma and beta functions are very useful special functions in many sciences such as mathematics, physics, chemistry, biology, medicine and engineering. These functions have been the focus of attention of researchers due to their popularity. When the literature is examined, it is seen that the generalized gamma and beta functions are mostly obtained by using appropriate kernel functions in the integral representations of the original functions. For instance, generalized gamma and beta functions are defined by using exponential, confluent hypergeometric and Mittag-Leffler functions etc. as kernel functions in their integral representations. Gamma and beta functions can also be written using the Pochhammer symbol. Series representations of hypergeometric functions are also associated with the Pochammer symbol. Many researchers have defined various generalizations for hypergeometric functions by making use of these relations. Historically, these generalizations began in 1994 and 1997 when Chaudhry et al., [1,2] selected exponential functions as the kernel of integral representations of original gamma and beta functions. Many researchers defined new generalizations of these functions inspired by the work of Chaudhry et al., (see for example [1-24] and reference therein).

All the studies mentioned above motivated us to describe a new generalization of gamma and beta functions. For this, we used the Wright function, which has a more general form than many special functions. We also defined the generalized Gauss and confluent hypergeometric functions with the help of generalized beta function. Then we presented some properties of these new functions. As examples, we obtained the solution of fractional differential equations involving the new generalized beta, Gauss hypergeometric and confluent hypergeometric functions.

Preliminaries

In this section, we gave some preliminary information that is needed throughout this paper. Then we mentioned the generalized gamma, beta, Gauss hypergeometric, and confluent hypergeometric functions defined by Chaudhry et al.. Firstly we gave the Mellin, inverse Mellin, Laplace, inverse Laplace integral transformations and the Caputo fractional derivative operator below.The Mellin and inverse Mellin transforms [25] for $s \in \mathbb{C}$ respectively are defined by

$$\mathfrak{M}\{f(p)\} = F(s) = \int_0^\infty p^{s-1} f(p)dp,$$

and

$$\mathfrak{M}^{-1}\{F(s)\} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} p^{-s} F(s)ds, \quad (c > 0).$$

The Laplace and inverse Laplace transforms [25] for $Re(s) > 0$ respectively are given by

$$\mathfrak{L}\{f(p)\} = F(s) = \int_0^\infty exp(-sp)f(p)dp,$$

and

$$\mathfrak{L}^{-1}\{F(s)\} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} exp(sp)F(s)ds, \quad (c > 0).$$

The Caputo fractional derivative operator [26] of order $\varepsilon \in \mathbb{C}$ for $m \in N$, $m-1 < Re(\varepsilon) < m$ is given by

$^{c}D_p^{\varepsilon}[f(p)] = \frac{1}{\Gamma(m-\varepsilon)}\int_0^p (p-t)^{m-\varepsilon-1} f^{(m)}(t)dt, \quad (Re(\varepsilon) > 0; p > 0).$

Also, Laplace transform of Caputo fractional derivative for $m \in N$, $m-1 < Re(\varepsilon) \le m$ as follows [26]:

$$\mathfrak{L}\{ {}^{c}D_p^{\varepsilon}[f(p)]\} = s^{\varepsilon}F(s) - \sum_{k=0}^{m-1} s^{\varepsilon-k-1} f^{(k)}(0). \tag{1}$$

We presented the generalized gamma, beta, Gauss hypergeometric, and confluent hypergeometric functions defined by Chaudhry et al. chronologically below.

In 1994, Chaudhry and Zubair [1] gave the extended gamma function for $Re(x) > 0$, $Re(p) > 0$ as follows:

$$\Gamma_p(x) = \int_0^{\infty} t^{x-1} \; exp\left(-t - \frac{p}{t}\right) dt.$$

In 1997, Chaudhry et al. [2] gave the extended beta function for $Re(x) > 0$, $Re(y) > 0$, $Re(p) > 0$ as follows:

$$B(x,y;p) = \int_0^1 t^{x-1}(1-t)^{y-1} exp\left(-\frac{p}{t(1-t)}\right) dt.$$

In 2004, Chaudhry et al. [3] gave the extended Gauss and confluent hypergeometric functions respectively as:

$$F_p(a,b;c;z) = \sum_{n=0}^{\infty} (a)_n \frac{B(b+n, c-b; p)}{B(b, c-b)} \frac{z^n}{n!},$$

$$(p \ge 0; |z| < 1; Re(c) > Re(b) > 0),$$

and

$$\Phi_p(b;c;z) = \sum_{n=0}^{\infty} \frac{B(b+n, c-b; p)}{B(b, c-b)} \frac{z^n}{n!},$$

$$(p \ge 0; Re(c) > Re(b) > 0).$$

Here, value expressed by $(\lambda)_n$ is the Pochhammer symbol [28] and is given as follows:

$$(\lambda)_0 \equiv 1 \; and \; (\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}, \quad (Re(\lambda) > -n; \; n \in N; \lambda \ne 0, -1, -2, \dots).$$

The integral representations of the above series are as follows, respectively:

$$F_p(a,b;c;z) = \frac{1}{B(b, c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} \; exp\left(-\frac{p}{t(1-t)}\right) dt,$$

$$(p > 0; p = 0 \; and \; |\, arg(1-z)| < \pi; Re(c) > Re(b) > 0),$$

and

$$\Phi_p(b;c;z) = \frac{1}{B(b, c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} \; exp\left(zt - \frac{p}{t(1-t)}\right) dt,$$

$$(p > 0; p = 0 \; and \; Re(c) > Re(b) > 0).$$

In this paper, we use Wright function to define new generalizations of gamma and beta functions, which defined in [28] as:

$${}_0\Psi_1(\alpha,\beta;z) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!},$$

where $\alpha, \beta \in \mathbb{C}$ and $Re(\alpha) > -1$.

New Generalized Gamma and Beta Functions

In this section, we introduced new generalized gamma and beta functions and presented some of their properties.

Definition 1. The new generalized gamma and beta functions for $Re(x) > 0$, $Re(y) > 0$, $Re(\alpha) > -1$, $Re(p) > 0$, respectively are defined by

$$^{\Psi}\Gamma_p^{(\alpha,\beta)}(x) = \int_0^\infty t^{x-1}{}_0\Psi_1\left(\alpha,\beta;-t-\frac{p}{t}\right)dt, \tag{2}$$

and

$$^{\Psi}B_p^{(\alpha,\beta)}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,{}_0\Psi_1\left(\alpha,\beta;-\frac{p}{t(1-t)}\right)dt. \tag{3}$$

We call the new generalizations of gamma and beta functions as Ψ-gamma and Ψ-beta functions, respectively.

Theorem 1. Let $Re(s) > 0$, $Re(x+s) > 0$, $Re(y+s) > 0$, $Re(p) > 0$, $Re(\alpha) > -1$. Then,

$$\mathfrak{M}\left\{{}^{\Psi}B_p^{(\alpha,\beta)}(x,y)\right\} = B(x+s,y+s)\,{}^{\Psi}\Gamma^{(\alpha,\beta)}(s).$$

Proof. Multiplying the equation (3) by p^{s-1} and integrating from $p=0$ to $p=\infty$, we have

$$\mathfrak{M}\left\{{}^{\Psi}B_p^{(\alpha,\beta)}(x,y)\right\} = \int_0^\infty p^{s-1}\int_0^1 t^{x-1}(1-t)^{y-1}\,{}_0\Psi_1\left(\alpha,\beta;-\frac{p}{t(1-t)}\right)dtdp.$$

By interchanging the integrals, we get

$$\mathfrak{M}\left\{{}^{\Psi}B_p^{(\alpha,\beta)}(x,y)\right\} = \int_0^1 t^{x-1}(1-t)^{y-1}\int_0^\infty p^{s-1}\,{}_0\Psi_1\left(\alpha,\beta;-\frac{p}{t(1-t)}\right)dpdt.$$

By substituting $v = \frac{p}{t(1-t)}$, we obtain

$$\begin{aligned}\mathfrak{M}\left\{{}^{\Psi}B_p^{(\alpha,\beta)}(x,y)\right\} &= \int_0^1 t^{x+s-1}(1-t)^{y+s-1}dt\int_0^\infty v^{s-1}{}_0\Psi_1(\alpha,\beta;-v)dv \\ &= B(x+s,y+s)\,{}^{\Psi}\Gamma^{(\alpha,\beta)}(s).\end{aligned}$$

Corollary 1. The inverse Mellin transform of the above equation is obtained as:

$$^{\Psi}B_p^{(\alpha,\beta)}(x,y) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} B(x+s,y+s)^{\Psi}\Gamma^{(\alpha,\beta)}(s)p^{-s}ds, \quad (c > 0).$$

Theorem 2. Let $Re(x) > 0$, $Re(y) > 0$, $Re(p) > 0$, $Re(s) > 0$, $Re(\alpha) > -1$. Then,

$$\mathfrak{L}\left\{{}^{\Psi}B_p^{(\alpha,\beta)}(x,y)\right\} = \frac{1}{s}\,{}^{\Psi}B_{\frac{1}{s}}\left[\begin{matrix}(1,1)_{1,1}\\(\beta,\alpha)_{1,1}\end{matrix}\middle|x,y\right].$$

Proof. Applying the Laplace transform to the Ψ-beta function, we have

$$\begin{aligned}\mathfrak{L}\left\{{}^{\Psi}B_p^{(\alpha,\beta)}(x,y)\right\} &= \int_0^\infty exp(-sp)\,{}^{\Psi}B_p^{(\alpha,\beta)}(x,y)dp \\ &= \int_0^1 t^{x-1}(1-t)^{y-1}\int_0^\infty exp(-sp)\,{}_0\Psi_1\left(\alpha,\beta;-\frac{p}{t(1-t)}\right)dpdt \\ &= \int_0^1 t^{x-1}(1-t)^{y-1}\int_0^\infty exp(-sp)\sum_{n=0}^\infty \frac{1}{\Gamma(\alpha n+\beta)}\frac{\left(-\frac{p}{t(1-t)}\right)^n}{n!}dpdt\end{aligned}$$

$$=\frac{1}{s}\int_0^1 t^{x-1}(1-t)^{y-1}\sum_{n=0}^{\infty}\frac{\left(-\frac{\frac{1}{s}}{t(1-t)}\right)^n}{\Gamma(\alpha n+\beta)}dt$$

$$=\frac{1}{s}\,{}^{\Psi}B_{\frac{1}{s}}\left[\begin{matrix}(1,1)_{1,1}\\(\beta,\alpha)_{1,1}\end{matrix}\middle|x,y\right].$$

Corollary 2. The inverse Laplace transform of the above equation is obtained as:

$${}^{\Psi}B_p^{(\alpha,\beta)}(x,y)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}exp(sp)\frac{1}{s}\,{}^{\Psi}B_{\frac{1}{s}}\left[\begin{matrix}(1,1)_{1,1}\\(\beta,\alpha)_{1,1}\end{matrix}\middle|x,y\right]ds,\quad (c>0).$$

Remark 1. Note that, ${}^{\Psi}B_{\frac{1}{s}}\left[\begin{matrix}(1,1)_{1,1}\\(\beta,\alpha)_{1,1}\end{matrix}\middle|x,y\right]$ is special case of the generalized beta function defined by Ata and Kıymaz [4].

Theorem 3. Let $Re(x)>0$, $Re(y)>0$, $Re(p)>0$, $Re(\alpha)>-1$. Then,

$${}^{\Psi}B_p^{(\alpha,\beta)}(x,y)=2\int_0^{\frac{\pi}{2}}cos^{2x-1}(\theta)sin^{2y-1}(\theta)\,{}_0\Psi_1\big(\alpha,\beta;-psec^2(\theta)csc^2(\theta)\big)d\theta,$$

$${}^{\Psi}B_p^{(\alpha,\beta)}(x,y)=\int_0^{\infty}\frac{u^{x-1}}{(1+u)^{x+y}}\,{}_0\Psi_1\left(\alpha,\beta;-2p-p\left(u+\frac{1}{u}\right)\right)du.$$

Proof. The desired results are obtained by putting $t=cos^2(\theta)$ and $t=\frac{u}{1+u}$ in equation (3), respectively.

Theorem 4. Let $Re(x)>0$, $Re(y)>0$, $Re(p)>0$, $Re(s)>0$, $Re(\alpha)>-1$. Then,

$${}^{\Psi}B_p^{(\alpha,\beta)}(x,y+1)+{}^{\Psi}B_p^{(\alpha,\beta)}(x+1,y)={}^{\Psi}B_p^{(\alpha,\beta)}(x,y).$$

Proof. By making the necessary calculations, we get

$$\begin{aligned}{}^{\Psi}B_p^{(\alpha,\beta)}(x,y+1)+{}^{\Psi}B_p^{(\alpha,\beta)}(x+1,y)\\
&=\int_0^1 t^{x-1}(1-t)^y\,{}_0\Psi_1\left(\alpha,\beta;-\frac{p}{t(1-t)}\right)dt+\int_0^1 t^x(1-t)^{y-1}\,{}_0\Psi_1\left(\alpha,\beta;-\frac{p}{t(1-t)}\right)dt\\
&=\int_0^1\left(t^{x-1}(1-t)^y+t^x(1-t)^{y-1}\right){}_0\Psi_1\left(\alpha,\beta;-\frac{p}{t(1-t)}\right)dt\\
&=\int_0^1 t^{x-1}(1-t)^{y-1}\,{}_0\Psi_1\left(\alpha,\beta;-\frac{p}{t(1-t)}\right)dt\\
&={}^{\Psi}B_p^{(\alpha,\beta)}(x,y).\end{aligned}$$

Theorem 5. Let $Re(x)>0$, $Re(y)>0$, $Re(p)>0$, $Re(\alpha)>-1$. Then,

$$\begin{aligned}{}^{\Psi}\Gamma_p^{(\alpha,\beta)}(x)\,{}^{\Psi}\Gamma_p^{(\alpha,\beta)}(y)=4\int_0^{\frac{\pi}{2}}\int_0^{\infty}&r^{2(x+y)-1}cos^{2x-1}(\theta)sin^{2y-1}(\theta)\\
&\times\,{}_0\Psi_1\left(\alpha,\beta;-r^2cos^2(\theta)-\frac{p}{r^2cos^2(\theta)}\right)\\
&\times\,{}_0\Psi_1\left(\alpha,\beta;-r^2sin^2(\theta)-\frac{p}{r^2sin^2(\theta)}\right)drd\theta.\end{aligned}$$

Proof. By writing $t=\eta^2$ in (2), we have

$$^{\Psi}\Gamma_p^{(\alpha,\beta)}(x) = 2\int_0^\infty \eta^{2x-1}\,{}_0\Psi_1\left(\alpha,\beta;-\eta^2-\frac{p}{\eta^2}\right)d\eta.$$

Therefore,

$$^{\Psi}\Gamma_p^{(\alpha,\beta)}(x)^{\Psi}\Gamma_p^{(\alpha,\beta)}(y) = 4\int_0^\infty\int_0^\infty \eta^{2x-1}\xi^{2y-1}\,{}_0\Psi_1\left(\alpha,\beta;-\eta^2-\frac{p}{\eta^2}\right)\,{}_0\Psi_1\left(\alpha,\beta;-\xi^2-\frac{p}{\xi^2}\right)d\eta d\xi.$$

Taking $\eta = rcos(\theta)$ and $\xi = rsin(\theta)$, the desired result is obtained.

Theorem 6. Let $Re(x) > 0$, $Re(y) < 1$, $Re(p) > 0$, $Re(\alpha) > -1$. Then,

$$^{\Psi}B_p^{(\alpha,\beta)}(x,1-y) = \sum_{n=0}^{\infty}\frac{(y)_n}{n!}\,{}^{\Psi}B_p^{(\alpha,\beta)}(x+n,1).$$

Proof. Using equation (3), we have

$$^{\Psi}B_p^{(\alpha,\beta)}(x,1-y) = \int_0^1 t^{x-1}(1-t)^{-y}\,{}_0\Psi_1\left(\alpha,\beta;-\frac{p}{t(1-t)}\right)dt.$$

The binomial series is as follows:

$$(1-t)^{-y} = \sum_{n=0}^{\infty}(y)_n\frac{t^n}{n!} \quad (|t| < 1).$$

Considering binomial series and interchanging summation and integration, we get

$$^{\Psi}B_p^{(\alpha,\beta)}(x,1-y) = \sum_{n=0}^{\infty}\frac{(y)_n}{n!}\int_0^1 t^{x+n-1}{}_0\Psi_1\left(\alpha,\beta;-\frac{p}{t(1-t)}\right)dt$$

$$= \sum_{n=0}^{\infty}\frac{(y)_n}{n!}\,{}^{\Psi}B_p^{(\alpha,\beta)}(x+n,1).$$

New Generalized Gauss and Confluent Hypergeometric Functions

In this section, we introduced new generalized Gauss and confluent hypergeometric functions and presented some of their properties.

Definition 2. The new generalized Gauss and confluent hypergeometric functions for $Re(c) > Re(b) > 0$, $Re(p) > 0$, $Re(\alpha) > -1$, respectively are defined by

$$^{\Psi}F_p^{(\alpha,\beta)}(a,b;c;z) = \sum_{n=0}^{\infty}(a)_n\frac{^{\Psi}B_p^{(\alpha,\beta)}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!}, \quad (|z| < 1),$$

and

$$^{\Psi}\Phi_p^{(\alpha,\beta)}(b;c;z) = \sum_{n=0}^{\infty}\frac{^{\Psi}B_p^{(\alpha,\beta)}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!}.$$

We call $^{\Psi}F_p^{(\alpha,\beta)}(a,b;c;z)$ as Ψ-Gauss hypergeometric function and $^{\Psi}\Phi_p^{(\alpha,\beta)}(b;c;z)$ as Ψ-confluent hypergeometric function.

Theorem 7. Let $Re(c) > Re(b) > 0$, $Re(p) > 0$, $Re(\alpha) > -1$. Then,

$$ {}^{\Psi}F_p^{(\alpha,\beta)}(a,b;c;z) = \frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}\,{}_0\Psi_1\left(\alpha,\beta;-\frac{p}{t(1-t)}\right)(1-zt)^{-a}dt, \tag{4} $$

$$ {}^{\Psi}F_p^{(\alpha,\beta)}(a,b;c;z) = \frac{1}{B(b,c-b)}\int_0^\infty u^{b-1}(1+u)^{a-c}(1+u(1-z))^{-a}\,{}_0\Psi_1\left(\alpha,\beta;-2p-p\left(u+\frac{1}{u}\right)\right)du, \tag{5} $$

$$ \begin{aligned} {}^{\Psi}F_p^{(\alpha,\beta)}(a,b;c;z) = \frac{2}{B(b,c-b)}\int_0^{\frac{\pi}{2}} & sin^{2b-1}(\theta)cos^{2c-2b-1}(\theta)\left(1-zsin^2(\theta)\right)^{-a} \\ & \times\,{}_0\Psi_1\left(\alpha,\beta;-\frac{p}{sin^2(\theta)cos^2(\theta)}\right)d\theta. \end{aligned} \tag{6} $$

Proof. By making the necessary calculations for equation (4), we have

$$ \begin{aligned} {}^{\Psi}F_p^{(\alpha,\beta)}(a,b;c;z) &= \sum_{n=0}^{\infty}(a)_n\frac{{}^{\Psi}B_p^{(\alpha,\beta)}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!} \\ &= \frac{1}{B(b,c-b)}\sum_{n=0}^{\infty}(a)_n\int_0^1 t^{b+n-1}(1-t)^{c-b-1}\,{}_0\Psi_1\left(\alpha,\beta;-\frac{p}{t(1-t)}\right)\frac{z^n}{n!}dt \\ &= \frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}\,{}_0\Psi_1\left(\alpha,\beta;-\frac{p}{t(1-t)}\right)\sum_{n=0}^{\infty}(a)_n\frac{(zt)^n}{n!}dt \\ &= \frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}\,{}_0\Psi_1\left(\alpha,\beta;-\frac{p}{t(1-t)}\right)(1-zt)^{-a}dt. \end{aligned} $$

By putting $u=\frac{t}{1-t}$ and $t=sin^2(\theta)$ in the last equation, equations (5) and (6) are obtained.

Theorem 8. Let $\mathrm{Re(c)} > Re(\mathrm{b}) > 0$, $\mathrm{Re(p)} > 0$, $\mathrm{Re}(\alpha) > -1$. Then,

$$ {}^{\Psi}\Phi_p^{(\alpha,\beta)}(b;c;z) = \frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}exp(zt)\,{}_0\Psi_1\left(\alpha,\beta;-\frac{p}{t(1-t)}\right)dt, \tag{7} $$

$$ {}^{\Psi}\Phi_p^{(\alpha,\beta)}(b;c;z) = \frac{1}{B(b,c-b)}\int_0^1 u^{c-b-1}(1-u)^{b-1}exp(z(1-u))\,{}_0\Psi_1\left(\alpha,\beta;-\frac{p}{u(1-u)}\right)du. $$

Proof. Similarly, using equation (3) desired results are achieved.

Note 1. The beta function and the Pochammer symbol provide the following equations:

$$ B(b,c-b) = \frac{c}{b}B(b+1,c-b), $$

and

$$ (a)_{n+1} = a(a+1)_n. $$

These equations are used to calculate the theorems given below.

Theorem 9. Let $Re(c) > Re(b) > 0$, $Re(p) > 0$, $Re(\alpha) > -1$. Then,

$$ \frac{d^n}{dz^n}\left\{{}^{\Psi}F_p^{(\alpha,\beta)}(a,b;c;z)\right\} = \frac{(a)_n(b)_n}{(c)_n}\,{}^{\Psi}F_p^{(\alpha,\beta)}(a+n,b+n;c+n;z). $$

Proof. Differentiating the ${}^{\Psi}F_p^{(\alpha,\beta)}(a,b;c;z)$ function, we have

$$\frac{d}{dz}\left\{{}^{\Psi}F_p^{(\alpha,\beta)}(a,b;c;z)\right\}=\frac{d}{dz}\left\{\sum_{n=0}^{\infty}(a)_n\frac{{}^{\Psi}B_p^{(\alpha,\beta)}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!}\right\}$$

$$=\sum_{n=1}^{\infty}(a)_n\frac{{}^{\Psi}B_p^{(\alpha,\beta)}(b+n,c-b)}{B(b,c-b)}\frac{z^{n-1}}{(n-1)!}.$$

By writing $n\to n+1$, we get

$$\frac{d}{dz}\left\{{}^{\Psi}F_p^{(\alpha,\beta)}(a,b;c;z)\right\}=\frac{(a)(b)}{(c)}\sum_{n=0}^{\infty}(a+1)_n\frac{{}^{\Psi}B_p^{(\alpha,\beta)}(b+n+1,c-b)}{B(b+1,c-b)}\frac{z^n}{n!}$$

$$=\frac{(a)(b)}{(c)}{}^{\Psi}F_p^{(\alpha,\beta)}(a+1,b+1;c+1;z).$$

By the inductive method, the more general form is obtained as:

$$\frac{d^n}{dz^n}\left\{{}^{\Psi}F_p^{(\alpha,\beta)}(a,b;c;z)\right\}=\frac{(a)_n(b)_n}{(c)_n}{}^{\Psi}F_p^{(\alpha,\beta)}(a+n,b+n;c+n;z).$$

Theorem 10. Let $Re(c)>Re(b)>0,\ Re(p)>0,\ Re(\alpha)>-1$. Then,

$$\frac{d^n}{dz^n}\left\{{}^{\Psi}\Phi_p^{(\alpha,\beta)}(b;c;z)\right\}=\frac{(b)_n}{(c)_n}{}^{\Psi}\Phi_p^{(\alpha,\beta)}(b+n;c+n;z).$$

Proof. The desired result is obtained by performing similar operations as in the proof of Theorem 9.

Theorem 11. Let $Re(c)>Re(b)>0,\ Re(p)>0,\ Re(s)>0,\ Re(\alpha)>-1$. Then,

$$\mathfrak{M}\left\{{}^{\Psi}F_p^{(\alpha,\beta)}(a,b;c;z)\right\}=\frac{{}^{\Psi}\Gamma^{(\alpha,\beta)}(s)B(b+s,c+s-b)}{B(b,c-b)}\,{}_2F_1(a,b+s;c+2s;z).$$

Proof. Multiplying the equation (4) by p^{s-1} and integrating from $p=0$ to $p=\infty$, we have

$$\mathfrak{M}\{{}^{\Psi}F_p^{(\alpha,\beta)}(a,b;c;z)\}=\int_0^{\infty}p^{s-1}\,{}^{\Psi}F_p^{(\alpha,\beta)}(a,b;c;z)dp$$

$$=\int_0^{\infty}p^{s-1}\sum_{n=0}^{\infty}\frac{{}^{\Psi}B_p^{(\alpha,\beta)}(b+n,c-b)}{B(b,c-b)}(a)_n\frac{z^n}{n!}dp$$

$$=\frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\int_0^{\infty}p^{s-1}\,{}_0\Psi_1\left(\alpha,\beta;-\frac{p}{t(1-t)}\right)dpdt.$$

Putting $u=\frac{p}{t(1-t)}$ for the second integral to the right of the last equation, we get

$$\int_0^{\infty}p^{s-1}\,{}_0\Psi_1\left(\alpha,\beta;-\frac{p}{t(1-t)}\right)dp=t^s(1-t)^s\ {}^{\Psi}\Gamma^{(\alpha,\beta)}(s),$$

and then using this equation in the above equation, we obtain

$$\mathfrak{M}\left\{{}^{\Psi}F_p^{(\alpha,\beta)}(a,b;c;z)\right\}=\frac{{}^{\Psi}\Gamma^{(\alpha,\beta)}(s)B(b+s,c+s-b)}{B(b,c-b)}\,{}_2F_1(a,b+s;c+2s;z).$$

Corollary 3. The inverse Mellin transform of the above equation is obtained as:

$$ {}^{\Psi}F_p^{(\alpha,\beta)}(a,b;c;z)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{{}^{\Psi}\Gamma^{(\alpha,\beta)}(s)B(b+s,c+s-b)}{B(b,c-b)}\,{}_2F_1(a,b+s;c+2s;z)p^{-s}ds,\quad (c>0). $$

Theorem 12. Let $Re(c)>Re(b)>0$, $Re(p)>0$, $Re(s)>0$, $Re(\alpha)>-1$. Then,

$$ \mathfrak{M}\left\{{}^{\Psi}\Phi_p^{(\alpha,\beta)}(b;c;z)\right\}=\frac{{}^{\Psi}\Gamma^{(\alpha,\beta)}(s)B(b+s,c+s-b)}{B(b,c-b)}\Phi(b+s;c+2s;z). $$

Proof. The desired result is obtained by performing similar operations as in the proof of Theorem 11.

Corollary 4. The inverse Mellin transform of the above equation is obtained as:

$$ {}^{\Psi}\Phi_p^{(\alpha,\beta)}(b;c;z)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{{}^{\Psi}\Gamma^{(\alpha,\beta)}(s)B(b+s,c+s-b)}{B(b,c-b)}\Phi(b+s;c+2s;z)p^{-s}ds,\quad (c>0). $$

Theorem 13. Let $Re(c)>Re(b)>0$, $Re(p)>0$, $Re(s)>0$, $Re(\alpha)>-1$. Then,

$$ \mathfrak{L}\left\{{}^{\Psi}F_p^{(\alpha,\beta)}(a,b;c;z)\right\}=\frac{1}{s}\,{}^{\Psi}F_{\frac{1}{s}}\left[\begin{matrix}(1,1)_{1,1}\\(\beta,\alpha)_{1,1}\end{matrix}\,|\,a,b;c;z\right]. $$

Proof. Using Laplace transform and making necessary calculation, we have

$$ \mathfrak{L}\{{}^{\Psi}F_p^{(\alpha,\beta)}(a,b;c;z)\}=\int_0^\infty exp(-sp)\,{}^{\Psi}F_p^{(\alpha,\beta)}(a,b;c;z)dp $$

$$ =\frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\int_0^\infty exp(-sp)\,{}_0\Psi_1\left(\alpha,\beta;-\frac{p}{t(1-t)}\right)dpdt $$

$$ =\frac{1}{s}\frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\sum_{n=0}^{\infty}\frac{\left(-\frac{\frac{1}{s}}{t(1-t)}\right)^n}{\Gamma(\alpha n+\beta)}dt $$

$$ =\frac{1}{s}\,{}^{\Psi}F_{\frac{1}{s}}\left[\begin{matrix}(1,1)_{1,1}\\(\beta,\alpha)_{1,1}\end{matrix}\,|\,a,b;c;z\right]. $$

Corollary 5. The inverse Laplace transform of the above equation is obtained as:

$$ {}^{\Psi}F_p^{(\alpha,\beta)}(a,b;c;z)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}exp(sp)\frac{1}{s}\,{}^{\Psi}F_{\frac{1}{s}}\left[\begin{matrix}(1,1)_{1,1}\\(\beta,\alpha)_{1,1}\end{matrix}\,|\,a,b;c;z\right]ds,\quad (c>0). $$

Remark 2. Note that, ${}^{\Psi}F_{\frac{1}{s}}\left[\begin{matrix}(1,1)_{1,1}\\(\beta,\alpha)_{1,1}\end{matrix}\,|\,a,b;c;z\right]$ is special case of the generalized Gauss hypergeometric function defined by Ata and Kıymaz [4].

Theorem 14. Let $Re(c)>Re(b)>0$, $Re(p)>0$, $Re(s)>0$, $Re(\alpha)>-1$. Then,

$$ \mathfrak{L}\left\{{}^{\Psi}\Phi_p^{(\alpha,\beta)}(b;c;z)\right\}=\frac{1}{s}\,{}^{\Psi}\Phi_{\frac{1}{s}}\left[\begin{matrix}(1,1)_{1,1}\\(\beta,\alpha)_{1,1}\end{matrix}\,|\,b;c;z\right]. $$

Proof. The desired result is obtained by performing similar operations as in the proof of Theorem 13.

Corollary 6. The inverse Laplace transform of the above equation is obtained as:

$$ {}^{\Psi}F_p^{(\alpha,\beta)}(a,b;c;z)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}exp(sp)\frac{1}{s}\,{}^{\Psi}\Phi_{\frac{1}{s}}\left[\begin{matrix}(1,1)_{1,1}\\(\beta,\alpha)_{1,1}\end{matrix}\,|\,b;c;z\right]ds,\quad (c>0). $$

Remark 3. Note that, ${}^{\Psi}\Phi_{\frac{1}{s}}\left[\begin{matrix}(1,1)_{1,1}\\(\beta,\alpha)_{1,1}\end{matrix}\Big| b;c;z\right]$ is special case of the generalized confluent hypergeometric function defined by Ata and Kıymaz [4].

Theorem 15. Let $Re(c) > Re(b) > 0$, $Re(p) > 0$, $Re(\alpha) > -1$. Then,

$$^{\Psi}F_p^{(\alpha,\beta)}(a,b;c;z) = (1-z)^{-a}\ {}^{\Psi}F_p^{(\alpha,\beta)}\left(a,c-b;c;\frac{z}{z-1}\right).$$

Proof. Using equation

$$(1-z(1-t))^{-a} = (1-z)^{-a}\left(1+\frac{zt}{1-z}\right)^{-a}$$

and by writing $t \to 1-t$ in (4), we have

$$^{\Psi}F_p^{(\alpha,\beta)}(a,b;c;z) = \frac{(1-z)^{-a}}{B(b,c-b)}\int_0^1 t^{c-b-1}(1-t)^{b-1}\left(1-\frac{zt}{z-1}\right)^{-a}\ {}_0\Psi_1\left(\alpha,\beta;-\frac{p}{t(1-t)}\right)dt$$

$$= (1-z)^{-a}\ {}^{\Psi}F_p^{(\alpha,\beta)}\left(a,c-b;c;\frac{z}{z-1}\right).$$

Theorem 16. Let $Re(c) > Re(b) > 0$, $Re(p) > 0$, $Re(\alpha) > -1$. Then,

$$^{\Psi}\Phi_p^{(\alpha,\beta)}(b;c;z) = exp(z)\,{}^{\Psi}\Phi_p^{(\alpha,\beta)}(c-b;c;-z).$$

Proof. By writing $t \to 1-t$ in equation (7), the desired result is achieved.

Applications to Fractional Differential Equations

In this section, we obtained the solution of fractional differential equations involving the newly generalized beta, Gauss hypergeometric and confluent hypergeometric functions.

Example 1. Let $1 < Re(\varepsilon) \leq 2$, $Re(x) > 0$, $Re(y) > 0$, $Re(p) > 0$, $Re(\alpha) > -1$. Assume that the fractional differential equation

$$^{c}D_p^{\varepsilon}[f(p)] = p^{\beta-1}\ {}^{\Psi}B_{\varepsilon p^{\alpha}}^{(\alpha,\beta)}(x,y)$$

and initial conditions

$$f(0) = f'(0) = 0$$

are given. Considering equation (1) and applying Laplace transform to the fractional differential equation, we have

$$\mathfrak{L}\{{}^{c}D_p^{\varepsilon}[f(p)]\} = \mathfrak{L}\{p^{\beta-1}\ {}^{\Psi}B_{\varepsilon p^{\alpha}}^{(\alpha,\beta)}(x,y)\}$$

and then

$$s^{\varepsilon}F(s) - s^{\varepsilon-1}f(0) - s^{\varepsilon-2}f'(0) = s^{-\beta}\ B_{\frac{\varepsilon}{s^{\alpha}}}(x,y).$$

By using initial conditions, we get

$$F(s) = s^{-\varepsilon-\beta}\ B_{\frac{\varepsilon}{s^{\alpha}}}(x,y).$$

Applying inverse Laplace transform to both sides of the last equation, we obtain the solution function as:

$$f(p) = p^{\beta+\varepsilon-1}\ {}^{\Psi}B_{\varepsilon p^{\alpha}}^{(\alpha,\beta+\varepsilon)}(x,y).$$

Example 2. Let $1 < Re(\varepsilon) \leq 2$, $Re(c) > Re(b) > 0$, $Re(p) > 0$, $Re(\alpha) > -1$. Assume that the fractional differential equation

$$ {}^{c}D_{p}^{\varepsilon}[f(p)] = p^{\beta-1}\, {}^{\Psi}F_{\varepsilon p^{\alpha}}^{(\alpha,\beta)}(a,b;c;z) $$

and initial conditions

$$ f(0) = f'(0) = 0 $$

are given. Considering equation (1) and applying Laplace transform to the fractional differential equation, we have

$$ \mathfrak{L}\{ {}^{c}D_{p}^{\varepsilon}[f(p)]\} = \mathfrak{L}\{p^{\beta-1}\, {}^{\Psi}F_{\varepsilon p^{\alpha}}^{(\alpha,\beta)}(a,b;c;z)\} $$

and then

$$ s^{\varepsilon}F(s) - s^{\varepsilon-1}f(0) - s^{\varepsilon-2}f'(0) = s^{-\beta}\, F_{\frac{\varepsilon}{s^{\alpha}}}(a,b;c;z). $$

By using initial conditions, we get

$$ F(s) = s^{-\varepsilon-\beta}\, F_{\frac{\varepsilon}{s^{\alpha}}}(a,b;c;z). $$

Applying inverse Laplace transform to both sides of the last equation, we obtain the solution function as:

$$ f(p) = p^{\beta+\varepsilon-1}\, {}^{\Psi}F_{\varepsilon p^{\alpha}}^{(\alpha,\beta+\varepsilon)}(a,b;c;z). $$

Example 3. Let $1 < Re(\varepsilon) \leq 2$, $Re(c) > Re(b) > 0$, $Re(p) > 0$, $Re(\alpha) > -1$. Assume that the fractional differential equation

$$ {}^{c}D_{p}^{\varepsilon}[f(p)] = p^{\beta-1}\, {}^{\Psi}\Phi_{\varepsilon p^{\alpha}}^{(\alpha,\beta)}(b;c;z) $$

and initial conditions

$$ f(0) = f'(0) = 0 $$

are given. Considering equation (1) and applying Laplace transform to the fractional differential equation, we have

$$ \mathfrak{L}\{ {}^{c}D_{p}^{\varepsilon}[f(p)]\} = \mathfrak{L}\{p^{\beta-1}\, {}^{\Psi}\Phi_{\varepsilon p^{\alpha}}^{(\alpha,\beta)}(b;c;z)\} $$

and then

$$ s^{\varepsilon}F(s) - s^{\varepsilon-1}f(0) - s^{\varepsilon-2}f'(0) = s^{-\beta}\, \Phi_{\frac{\varepsilon}{s^{\alpha}}}(b;c;z). $$

By using initial conditions, we get

$$ F(s) = s^{-\varepsilon-\beta}\, \Phi_{\frac{\varepsilon}{s^{\alpha}}}(b;c;z). $$

Applying inverse Laplace transform to both sides of the last equation, we obtain the solution function as:

$$ f(p) = p^{\beta+\varepsilon-1}\, {}^{\Psi}\Phi_{\varepsilon p^{\alpha}}^{(\alpha,\beta+\varepsilon)}(b;c;z). $$

Conclusion

In this paper, we defined Ψ-gamma and Ψ-beta functions involving Wright function in the kernels and then we defined Ψ-Gauss and Ψ-confluent hypergeometric functions with the help of Ψ-beta function. Furthermore, we gave some properties of these functions and we presented their applications to fractional differential equations. In fact, most of the generalized gamma, beta, and hypergeometric functions in the literature seem to be special cases of the new generalized functions introduced in this article, such that: Chaudhry et al. [1,2,3],

$$^{\Psi}\Gamma_p^{(0,1)}(x) = \Gamma_p(x),$$

$$^{\Psi}B_p^{(0,1)}(x,y) = B(x,y;p),$$

$$^{\Psi}F_p^{(0,1)}(a,b;c;z) = F_p(a,b;c;z),$$

$$^{\Psi}\Phi_p^{(0,1)}(b;c;z) = \Phi_p(b;c;z).$$

Özergin et al. [5],

$$^{\Psi}\Gamma_p^{(0,1)}(x) = \Gamma_p^{(\alpha,\alpha)}(x),$$

$$^{\Psi}B_p^{(0,1)}(x,y) = B_p^{(\alpha,\alpha)}(x,y),$$

$$^{\Psi}F_p^{(0,1)}(a,b;c;z) = F_p^{(\alpha,\alpha)}(a,b;c;z),$$

$$^{\Psi}\Phi_p^{(0,1)}(b;c;z) = \Phi_p^{(\alpha,\alpha)}(b;c;z).$$

Lee et al. [6],

$$^{\Psi}B_p^{(0,1)}(x,y) = B(x,y;p;1),$$

$$^{\Psi}F_p^{(0,1)}(a,b;c;z) = F_p(a,b;c;z;1),$$

$$^{\Psi}\Phi_p^{(0,1)}(b;c;z) = \Phi_p(b;c;z;1).$$

Parmar [7],

$$^{\Psi}\Gamma_p^{(0,1)}(x) = \Gamma_p^{(\alpha,\alpha;1)}(x),$$

$$^{\Psi}B_p^{(0,1)}(x,y) = B_p^{(\alpha,\alpha;1)}(x,y),$$

$$^{\Psi}F_p^{(0,1)}(a,b;c;z) = F_p^{(\alpha,\alpha;1)}(a,b;c;z),$$

$$^{\Psi}\Phi_p^{(0,1)}(b;c;z) = \Phi_p^{(\alpha,\alpha;1)}(b;c;z).$$

Srivastava et al. [8],

$$^{\Psi}B_p^{(0,1)}(x,y) = B_p^{(\alpha,\alpha;1,1)}(x,y),$$

$$^{\Psi}F_p^{(0,1)}(a,b;c;z) = F_p^{(\alpha,\alpha;1,1)}(a,b;c;z).$$

Shadab et al. [9],

$$^{\Psi}B_p^{(0,1)}(x,y) = B_1^{p}(x,y),$$

$$^{\Psi}F_p^{(0,1)}(a,b;c;z) = F_{p,1}(a,b;c;z),$$

$$^{\Psi}\Phi_p^{(0,1)}(b;c;z) = \Phi_{p,1}(b;c;z).$$

Rahman et al. [10],

$$^{\Psi}B_p^{(0,1)}(x,y) = B_p^{1;1}(x,y),$$

$$^{\Psi}F_p^{(0,1)}(a,b;c;z) = F_p^{1;1}(a,b;c;z),$$

$$^{\Psi}\Phi_p^{(0,1)}(b;c;z) = \Phi_p^{1;1}(b;c;z).$$

Classic functions [27],

$$^{\Psi}\Gamma_0^{(0,1)}(x) = \Gamma(x),$$

$$^{\Psi}B_0^{(0,1)}(x,y) = B(x,y),$$

$$^{\Psi}F_0^{(0,1)}(a,b;c;z) = {}_2F_1(a,b;c;z),$$

$$^{\Psi}\Phi_0^{(0,1)}(b;c;z) = \Phi(b;c;z).$$

Acknowledgment

Note that, a part of this work was published without peer review at Cornell University arXiv.org on March 5, 2018 as preprint arXiv:1803.03121 [math.CA]. Then on March 12, 2021, it was revised to arXiv:1803.03121v3 [math.CA] [24]. The authors are thankful to the referees for their valuable remarks, comments and suggestions that led to the improvement of this research and to the editors for their interest.

Conflicts of interest

There are no conflicts of interest in this work.

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