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A New Decomposition Method for Integro-Differential Equations

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ABSTRACT

This present study developed a new Modified Adomian Decomposition Method (MADM) for integro-differential equations. The modification was carried out by decomposing the source term function into series. The terms in the series were then selected in pairs to form the initials for the prevailing approximation. The newly modified Adomian decomposition method (MADM) accelerates the convergence of the solution faster than the Standard Adomian Decomposition Method (SADM). This study recommends the use of the MADM for solving integro-differential equations

Keywords: Adomian polynomials, Integro-differential equations, Taylor series.

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Introduction

Integro-differential equations have been investigated in many fields, including biology, physics, and engineering. Integro-differential equations, on the other hand, are widely used in science and engineering to simulate a variety of physical phenomena. As a result, scientists and applied mathematicians have focused their efforts on finding exact and approximate solutions to integro-differential equations [1-7].

The fractional calculus is a powerful tool in applied mathematics for studying a variety of problems from various fields of science and engineering, with many breakthrough results in mathematical physics, finance, hydrology, biophysics, thermodynamics, control theory, statistical mechanics, astrophysics, cosmology, and bioengineering [8]. Since the fractional calculus piqued the interest of mathematicians and other scientists, the solutions of fractional integro-differential equations have been studied frequently in recent years [9-19], other approcahes of the least squares with shifted Chebyshev polynomials [20], least-squares method using Bernstein polynomials [21], fractional residual power series method [22], Taylor matrix method [23].

Laguerre polynomials are used to solve some integer order integro-differential equations. The Altarelli-Parisi equation [24], the Pantograph-type Volterra integro-differential equation [25] and the linear Fredholm integro-differential equation are examples of these. In addition, Laguerre polynomials are used to solve fractional integro-differential equations [20]. Algebraic equations, differential equations, integral equations, and other functional equations are frequently the result of mathematical modeling of real-life problems [26].

In many domains of science and engineering, differential and integral equations are often used. However, research into these areas has uncovered novel subtopics in which both differential and integral operators appear in the same equation. This new type of equation is known as integro-differential equation.

Integro-differential equations are equations that are known to emerge in both the derivatives and anti-derivatives of a function [27]. It is an equation in which the unknown function u(x) appears under the integral sign and has yet to be identified [28]. To solve polynomial issues, various types of analytical methods have been applied. Hirota's bilinear approach, Darboux transformation, symmetry method, inverse scatting transformation, variational iteration method used by [29-30]. The Adomian Decomposition Method (ADM) is a dependable and practical method for dealing with various equations, both linear and non-linear.

Differential equations, such as Boundary Value Problems (BVPs), have also been solved using this method in other sectors of science and engineering. For nonlinear operators, the method relies on the calculation of Adomian polynomials. The usage of the Adomian decomposition approach has various drawbacks that can develop due to the nature of the issues being considered, such as a relatively poor convergence rate and a huge functional evaluation for non-linear problems. [30] solved certain linear and nonlinear integral equations using a modified version of this ADM. Using Adomian Polynomials, this paper proposes a new version of the Adomian Decomposition Method for integro-differential equations.

This new decomposition modification introduces a change in the formulation of Adomian polynomials, which is superior to the usual Adomian technique. The novel modified Adomian Decomposition Method (MADM) improves the accuracy, speed of convergence, and reduces the number of functional calculations.

Methodology

Assuming that the nonlinear function is $F\big(y(x)\big)$ therefore, below are few of Adomian polynomials.

$$A_0 = F(y_0), \tag{1}$$

$$A_1 = y_1 F'(y_0), \tag{2}$$

$$A_2 = y_2 F'(y_0) + \tfrac{1}{2!} y_1^2 F''(y_0), \tag{3}$$

$$A_3 = y_3 F'(y_0) + y_1 y_2 F''(y_0) + \tfrac{1}{3!} y_1^3 F'''(y_0), \tag{4}$$

$$A_4 = y_4 F'(y_0) + \left(\tfrac{1}{2!} y_2^2 + y_1 y_3\right) F''(y_0) + \tfrac{1}{2!} y_1^2 y_2 F'''(y_0) + \tfrac{1}{4} y_1^4 F^{(iv)}(y_0), \tag{5}$$

Two important observations can be made here. First, A_0 depends only on y_0, A_1 depends only on y_0 and y_1 , A_2 depends only on y_0 , y_1 and y_2 , and so on.
Secondly, substituting these $A_j's$ in (3) gives:

$$F(y) = A_0 + A_1 + A_2 + A_3 + \ldots$$

$$= F(y_0) + (y_1 + y_2 + y_3 + \ \ldots) F'(y_0) + \tfrac{1}{2!}(y_1^2 + 2y_1 y_2 + 2y_1 y_3 + y_2^2) F''(y_0)$$
$$+ \frac{1}{3!}(y_1^3 + 3y_1^2 y_3 + 6y_1 y_2 y_3 + \ \ldots) F'''(y_0) + \ \ldots$$
$$= F(y_0) + (y - y_0) F'(y_0) + \frac{1}{2!}(y - y_0)^2 F''(y_0) + \ \ldots$$

In the following, we will calculate Adomian polynomials for several linear terms that may arise in nonlinear integral equations.

Case 1.
The first four Adomian polynomials for $F(y) = y^2$ are given by

$$A_0 = y_0^2$$

$$A_1 = 2y_0 y_1$$

$$A_2 = 2y_0 y_2 + y_1^2$$

$$A_3 = 2y_0 y_3 + 2y_1 y_2$$

Case 2.
The first four Adomian polynomials for $F(y) = y^3$ are given by

$$A_0 = y_0^3 ,$$

$$A_1 = 3y_0^2\, y_1 ,$$

$$A_2 = 3y_0^2 y_2 + 3y_0 y_1^2 ,$$

$$A_3 = 3y_0^2 y_3 + 6y_0 y_1 y_2 + y_1^3$$

Case 3.
The first four Adomian polynomials for $F(y) = y^4$ are given by

$$A_0 = y_0^4 ,$$

$$A_1 = 4y_0^3 y_1 ,$$

$$A_2 = 4y_0^3 y_2 + 6y_0^2 y_1^2 ,$$

$$A_3 = 4y_0^3 y_3 + 4y_1^3 y_0 + 12y_0^2 y_1 + y_2$$

Case 4.
The first four Adomian polynomials for $F(y) = \sin y$ are given by

$$A_0 = \sin y_0 ,$$

$$A_1 = y_1 \cos y_0 ,$$

$$A_2 = y_2 \cos y_0 - \tfrac{1}{2!} y_1^2 \sin y_0 ,$$

$$A_3 = y_3 \cos y_0 - y_1 y_2 \sin y_0 - \tfrac{1}{3!} y_1^3 \cos y_0$$

Case 5.
The first four Adomian polynomials for $F(y) = \cos y$ are given by

$$A_0 = \cos y_0 ,$$

$$A_1 = -y_1 \sin y_0 ,$$

$$A_2 = -y_2 \sin y_0 - \tfrac{1}{2!} y_1^2 \cos y_0 ,$$

$$A_3 = -y_3 \sin y_0 - y_1 y_2 \cos y_0 + \tfrac{1}{3!} y_1^3 \sin y_0 ,$$

Case 6.
The first four Adomian polynomials for $F(y) = exp(y)$ are given by

$$A_0 = exp(y_0) ,$$

$$A_1 = y_1\, exp(y_0) ,$$

$$A_2 = \left(y_2 + \tfrac{1}{2!} y_1^2\right) \ exp(y_0) ,$$

$$A_3 = \left(y_3 + y_1 y_2 + \tfrac{1}{3!} y_1^3\right) \ exp(y_0) ,$$

The modification was carried out by decomposing the source term function into series of the form

$$g(x) = \sum_{j=0}^{+\infty} g_i(x)$$

and the new recursive relation was obtained as:

$$y_0(x) = g_0(x),$$

$$y_1(x) = g_1(x) + g_2(x) + \lambda \int_a^x k(x,t)\left(L\big(y_0(x)\big) + A_0\right)dt,$$

$$y_2(x) = g_3(x) + g_4(x) + \lambda \int_a^x k(x,t)\big(L\big(y_0(x) + y_1(x)\big) + A_1\big)\,dt,$$

.

$$y_{j+1}(x) = g_{2(j+1)}(x) + g_{2(j+1)-1}(x) + \lambda \int_a^x k(x,t)\left(L\left(y_j(x) + y_{j-1}(x)\right) + A_1\right)dt.$$

Numerical Examples

Example 1:

Consider the standard integro-differential equation;

$y'(x) = 1 - \frac{1}{3}x + \int_0^1 xty(t)dt\,;\; y(0) = 0,\; y(x) = x$

Let

$a_0 = 1$

$a_0 = \int_0^x adx$

$a_0 = x$

$y_0 = t$

$g_0 = -\frac{1}{6}x^2$

$a_1 = g_0 + \int_0^x x \int_0^1 ty_0 dtdx$

$a_1 = 0$

$y_1 = 0$

$g_1 = 0$

Then;

$y_n = y_0 + y_1 + y_2 + y_3 + y_4$

$y_n(t) = t$

$y_n(x) = x$

Example 2:

Consider the standard integro-differential equation;

$y''(x) = \frac{1}{2}e^x + \frac{1}{2}\int_0^1 e^{x-2t}y^2(t)dt\,;\quad y(0) = 1,$
$y'(0) = 1$

Applying two fold integral linear operator defined by:

$L^{-1} = \int_0^x \int_0^x (.)dxdx$

The differential equation is transformed to:

$$y(x) = \frac{1}{2} + \frac{1}{2}x + \frac{1}{2}e^x + \frac{1}{2}L^{-1}\left[\int_0^1 e^{(x-2t)}y^2(t)dt\right]dxdx$$

Let

$r = \frac{1}{2} + \frac{1}{2}x + \frac{1}{2}e^x$

Using taylor (r,from x to 10)

$$1 + x + \frac{1}{4}x^2 + \frac{1}{12}x^3 + \frac{1}{48}x^4 + \frac{1}{120}x^5 + \frac{1}{1440}x^6 + \frac{1}{10080}x^8 + \frac{1}{725760}x^9 + 0(x^{10})$$

Then;

$a_0 = 1$

$y_0 = 1$

$g_0 = x + \frac{1}{4}x^2$

We have,

$a_1 = g_0 + \frac{1}{2}\int_0^x e^x \int_0^x \left[\int_0^1 e^{(-2t)}y_0^2 dt\right] dxdx$

$a_1 = x + \frac{1}{4}x^2 + 0.2161661792 + 0.2161661792e^x x - 0.2161661792e^x$

$y_1 = t + \frac{1}{4}t^2 + 0.2161661792 + 0.2161661792e^t t - 0.2161661792e^t$

Then,

$g_1 = \frac{1}{12}x^3 + \frac{1}{48}x^4$

$a_4 = g_3 + \frac{1}{2}\int_0^x e^x \int_0^x \left[\int_0^1 e^{(-2t)}a_{iv}dt\right] dxdx$

$a_4 = \frac{1}{1080}x^7 + \frac{1}{80640}x^8 + 0.02183314121 + 0.02183314121e^x x - 0.02183314121e^x$

$y_4 = \frac{1}{1080}t^7 + \frac{1}{80640}t^8 + 0.02183314121 + 0.02183314121e^t t - 0.02183314121e^t$

Then;

$$y_n = y_0 + y_1 + y_2 + y_3 + y_4$$

$$y_n(t) = 1.5044983400 + t + \frac{1}{4}t^2 + 0.5049834007e^t t - 0.5049834007e^t + \frac{1}{12}t^3 + \frac{1}{48}t^4 + \frac{1}{120}t^5 + \frac{1}{1440}t^6 + \frac{1}{1080}t^7 + \frac{1}{80640}t^8$$

$$y_n(x) = 1.5044983400 + x + \frac{1}{4}x^2 + 0.5049834007e^x x - 0.5049834007e^x + \frac{1}{12}x^3 + \frac{1}{48}x^4 + \frac{1}{120}x^5 + \frac{1}{1440}x^6 + \frac{1}{1080}x^7 + \frac{1}{80640}x^8$$

Table 1. Table of Absolute Errors for Example 2

X	Exact	NADM	Absolute Error
0.0	1.000.000.000	0.999514939	0.000485061
0.1	1.105.170.918	1.105.285.188	0.000114270
0.2	1.221.402.758	1.222.254.295	0.000851537
0.3	1.349.858.808	1.352.753.581	0.002894773
0.4	1.491.824.698	1.498.889.078	0.007064380
0.5	1.648.721.271	1.663.062.055	0.014340784
0.6	1.822.118.800	1.848.010.030	0.025891230
0.7	2.013.752.707	2.056.854.272	0.043101565
0.8	2.225.540.928	2.293.154.795	0.067613867
0.9	2.459.603.111	2.560.973.914	0.101370803
1.0	2.718.281.828	2.864.949.506	0.146667678

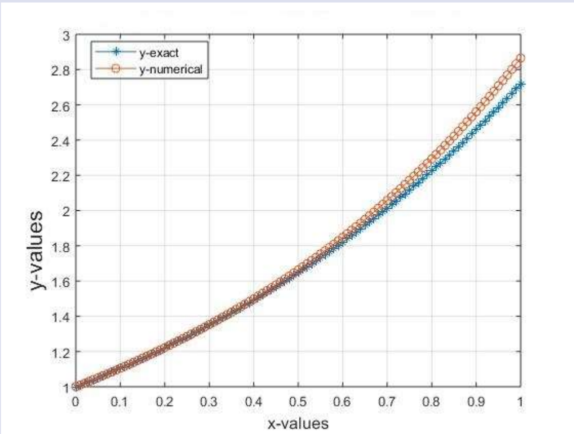

Figure 1. Graph of Comparison of the New ADM and the Exact for Example 2

Example 3:

$$y'(x) = e^x + \frac{1}{16}(3 + e^2)x + \frac{1}{4}\int_0^1 xt\big(1 + u(t) - y^2(t)\big)dt \ ;$$

Which subject to the initial condition?

$$u(0) = 2$$

Applying a one-fold integral linear operator defined by:

$$L^{-1} = \int_0^x (.)dx$$

The differential equation is transformed to

$$y(x) = 1 + exp(x) + \frac{1}{32}(3 + exp(2))x + \frac{1}{4}L^{-1}\left[\int_0^1 xt\big(1 + y(t) - y^2(t)\big)dt\right]dx$$

By using Taylor Series

$$(1 + \frac{973}{2997}x^2 + exp(x) \text{ from } x \text{ to } 10)$$

We have;

$$2 + x + \frac{4943}{5994}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^5 + \frac{1}{720}x^6 + \frac{1}{5040}x^7 + \frac{1}{40320}x^8 + \frac{1}{362880}x^9 + (0)x^{10}$$

Then;

$$a_0 = 2$$

$$y_0 = 2$$

and

$$a_i \ ; \ a_{ii} \ ; \ a_{iii} \ ; \ a_{iv} \ \ represent \ \ y_0^2 \ ; \ y_1^2 \ ; \ y_2^2 \ ; y_3^2$$

Then;
Integrate y_0;

$$a_1 = -0.3827304872x^2$$

$$y_1 = -0.3827304872t^2$$

$$a_{ii} = 2y_0y_1$$

$$a_{ii} = -0.7654609744(1 + e^t + 0.3246580031t^2)t^2$$

$$a_2 = \frac{1}{4}\int_0^x x\int_0^1 t(1 + y_1 - a_{ii})dtdx$$

$$a_2 = 0.1335487491x^2$$

$$y_2 = 0.1335487491t^2$$

Then; the sum of y_0 to y_4;

$$y_n = y_0 + y_1 + y_2 + y_3 + y_4$$

We have

$$y_n(t) = 2 + t + \frac{2955595512151}{4120514592768}t^2 + \frac{1}{6}t^3 + \frac{1}{24}t^4 + \frac{1}{120}t^5 + \frac{1}{720}t^6 + \frac{1}{5040}t^7 + \frac{1}{40320}t^8$$

$$y_n(x) = 2 + x + \frac{2955595512151}{4120514592768}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6 + \frac{1}{5040}x^7$$

Table 2. Table of Absolute Errors for Example 2

X	Exact	NADM	Absolute Error
0.0	2.000.000.000	2.000.000.000	0.000000000
0.1	2.105.170.918	2.107.343.798	0.002172880
0.2	2.221.402.759	2.230.094.277	0.008691518
0.3	2.349.858.807	2.369.414.725	0.019555918
0.4	2.491.824.697	2.526.590.771	0.034766074
0.5	2.648.721.265	2.703.043.255	0.054321990
0.6	2.822.118.771	2.900.342.437	0.078223666
0.7	3.013.752.588	3.120.223.689	0.106471101
0.8	3.225.540.527	3.364.604.822	0.139064295
0.9	3.459.601.938	3.635.605.188	0.176003250
1.0	3.718.278.771	3.935.566.732	0.217287961

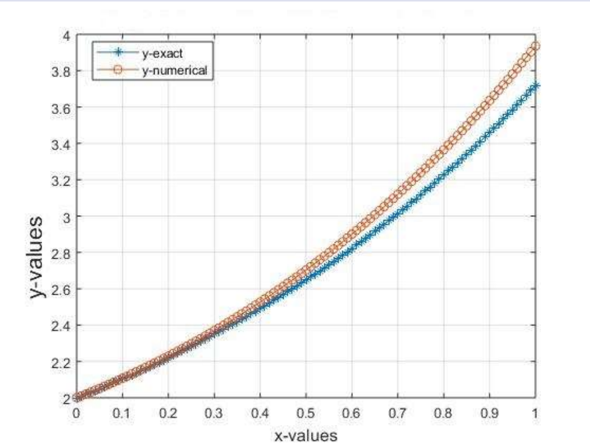

Figure 2. Graph of Comparison of the New ADM and the Exact for Example 3

Results and Discussion

The new Modified Adomian Decomposition Method (MADM) for integro-differential equations was introduced in this paper. This new approach converges faster, and it can be seen that the source term expansion should be as lengthy as feasible. The decomposed source term's convergence is improved by a little increase in the terms of decomposed source terms. The addition of more terms in the integral sign improves accuracy and, as a result, the Adomian polynomials.

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Conflicts of interest

With respect to this work, the authors state that there are no conflicts of interest.

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