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Special helices on equiform differential geometry of timelike curves in $\mathbb{E}_1^4$

Fatma BULUT[1,*]

[1]*Bitlis Eren University, Faculty of Arts and Sciences, Department of Mathematics, Bitlis/TURKEY*

Abstract

In this paper, we introduce the moving Frenet frame along the timelike curve in $\mathbb{E}_1^4$ and then Frenet formulas with the equiform parameter in the equiform geometry of the Minkowski space-time. We obtain k-type helices for equiform differential geometry of timelike curves in Minkowski space-time $\mathbb{E}_1^4$, in terms of their curvature functions. We give some new characterizations for these helices and investigate the special helices in Minkowski space-time. Finally, we establish (k, m)-type slant helices for equiform differential geometry of timelike curves in $\mathbb{E}_1^4$.

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1. Introduction

Differential geometry is basically an area where the theory of curves and manifolds are studied. New theories are practically being built on it everyday applications. Especially, since the theory of curves finds application in many disciplines, it has become an important field for both mathematicians and biologists, physics and even engineers and medicine in some fields. However, the geometric structures built on the timelike and spacelike curves and the construction of Frenet vectors opened completely different doors and allowed to work on a very wide platform. Geometricians try to express and prove these and similar issues in various spaces, for example in the Lorentz-Minkowski space in the Euclidean space, and in the Semi-Euclidean space. In particular, the theory of curves in Lorentz-Minkowski and Semi-Euclidean space, and the differences arising from the classification of curves as spacelike, timelike and null have yielded very interesting results.

Recently, Izumiya and Takeuchi introduced the concept of slant helix in Euclidean space. For instance, in [1], the authors presented some necessary and sufficient conditions for a curve to be a slant helix in Euclidean n-space. In [2], the authors established equiform differential geometry of curves in Minkowski space-time. Geometricians [3-5] usually deal with the theoretical part and continue to work with spacelike, timelike curves, involute-evolute curves, helices, and various characterizations. M.Y. Yilmaz and M. Bektaş defined (k, m)-type slant helices in 4-dimensional Euclidean space in [6]. Furthermore, very important theories have been proved in the 4-dimensional Minkowski space, which contains the most interesting and most different curves [7-10] and similar subjects [11-13] have yielded quite remarkable results. Because equiform roofs are expressed in 4-dimensional Euclidean space, each of them has its own unique geometric structures, allowing the study of events in a broad perspective.Additionally, F. Bulut and M. Bektaş obtained helix types for equiform differential geometry of spacelike curves in $\mathbb{E}_1^4$ in [12].

In this paper, we examine the structures of (k, m)-type helices of the distinguished timelike curves and the timelike curves expressed by the s parameter. We present helix types which are called curves as k-type helices and (k, m)-type slant helices for equiform differential geometry of timelike curves in Minkowski space-time.

2. Geometric Preliminaries

The Minkowski space-time $\mathbb{E}_1^4$ is a Euclidean space provided with the indefinite flat metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2$$

*Corresponding author Fatma BULUT. *E-mail address: fbulut@beu.edu.tr*

where $x_1^1, x_1^2, x_1^3, x_1^4$ is a rectangular coordinate system of $\mathbb{E}_1^4$. Recall that an arbitrary vector $y \in \mathbb{E}_1^4 - \{0\}$ can be spacelike, timelike or null (lightlike vector), if holds $g\langle y, y\rangle\rangle 0$, $g\langle y, y\rangle\langle 0$ or $g\langle y, y\rangle = 0$ respectively. If y is a timelike vector, then $\parallel y', y' \parallel = \sqrt{-\langle y', y'\rangle}$. For an arbitrary the curve $\alpha(s)$ in $\mathbb{E}_1^4$ is named a spacelike, a timelike and a null (lightlike) curve, if all of its velocity vectors $\alpha'(s)$ are spacelike, timelike, and null (lightlike), respectively [13]. The normal vector on the spacelike or the timelike hypersurface is, respectively, a timelike or a spacelike vector.

Let $\alpha: I \subset \mathbb{R} \to \mathbb{E}_1^4$ be a curve in Minkowski space-time. The curve α is said to be a timelike curve if $\langle \dot{\alpha}(t), \dot{\alpha}(t)\rangle < 0$ for each $t \in I$. The arclength of a timelike curve α measured from $\alpha(t_0)(t_0 \in I)$ is

$$s(t) = \int_{t_0}^{t} \parallel \dot{\alpha}(t) \parallel \mathrm{d}t.$$

α is said to be parameterized by the arc-length function s, if $\parallel \alpha'(s) \parallel = -1$, where $\alpha'(s) = d\alpha/ds$. Consequently, we say that α is a timelike curve, if $\parallel \alpha'(s) \parallel = -1$. For any $x, y, z \in \mathbb{E}_1^4$, we define a vector $x \times y \times z$ by

$$x \times y \times z = \begin{vmatrix} -e_1 & e_2 & e_3 & e_4 \\ x_1^1 & x_1^2 & x_1^3 & x_1^4 \\ x_2^1 & x_2^2 & x_2^3 & x_2^4 \\ x_3^1 & x_3^2 & x_3^3 & x_3^4 \end{vmatrix},$$

where $x_i = (x_i^1, x_i^2, x_i^3, x_i^4)$, $1 \leq i \leq 3$. Let $\alpha: I \longrightarrow \mathbb{E}_1^4$ be a timelike curve in $\mathbb{E}_1^4$. Let $\{\mathbf{t}(s), \mathbf{n}(s), \mathbf{b_1}(s), \mathbf{b_2}(s)\}$ is a pseudo-orthogonal frame which satisfies the following Frenet-Serret formulas of $\mathbb{E}_1^4$ along α.

$$\begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b_1} \\ \mathbf{b_2} \end{bmatrix}' = \begin{bmatrix} 0 & \overline{\kappa}_1 & 0 & 0 \\ \mu_1\overline{\kappa}_1 & 0 & \mu_2\overline{\kappa}_2 & 0 \\ 0 & \mu_3\overline{\kappa}_2 & 0 & \mu_4\overline{\kappa}_3 \\ 0 & 0 & \mu_5\overline{\kappa}_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b_1} \\ \mathbf{b_2} \end{bmatrix}, \tag{1}$$

where $\overline{\kappa}_1$, $\overline{\kappa}_2$ and $\overline{\kappa}_3$ are respectively, first, second and third curvature of the timelike curve α and we have

$$\overline{\kappa}_1(s) = \parallel \alpha''(s) \parallel,$$
$$\mathbf{n}(s) = \frac{\alpha''(s)}{\overline{\kappa}_1(s)},$$
$$\mathbf{b_1}(s) = \frac{\mathbf{n}'(s) + \mu_1\overline{\kappa}_1(s)\mathbf{t}(s)}{\parallel \mathbf{n}'(s) + \mu_1\overline{\kappa}_1(s)\mathbf{t}(s) \parallel},$$
$$\mathbf{b_2}(s) = \mathbf{t}(s) \times \mathbf{n}(s) \times \mathbf{b_1}(s).$$

Denote by $\{\mathbf{t}(s), \mathbf{n}(s), \mathbf{b_1}(s), \mathbf{b_2}(s)\}$ the moving Frenet frame along the timelike curve α [1-7]. So, $t(s)$ is a timelike tangent vector and the principal normal vector $\mathbf{n}(s)$, the first binormal vector $\mathbf{b_1}(s)$ and the second binormal vector $\mathbf{b_2}(s)$, then $\mu_i = \mp 1 (1 \leq i \leq 5)$ and we get $\mu_1 = \mu_2 = \mu_4 = 1$, $\mu_3 = \mu_5 = -1$.

Now, let γ be a timelike curve. Then $\mathbf{T}$ is timelike vector and following Frenet formulas is given

$$\begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B_1} \\ \mathbf{B_2} \end{bmatrix}' = \begin{bmatrix} 0 & \overline{\kappa}_1 & 0 & 0 \\ -\overline{\kappa}_1 & 0 & \overline{\kappa}_2 & 0 \\ 0 & -\overline{\kappa}_2 & 0 & \overline{\kappa}_3 \\ 0 & 0 & -\overline{\kappa}_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B_1} \\ \mathbf{B_2} \end{bmatrix},$$

where $\overline{\kappa}_1, \overline{\kappa}_2$ and $\overline{\kappa}_3$ denote the first, the second and the third curvature functions according to of γ, respectively. Here, $\{\mathbf{T}, \mathbf{N}, \mathbf{B_1}, \mathbf{B_2}\}$ satisfy the following equations

$$\langle \mathbf{N}, \mathbf{N}\rangle = \langle \mathbf{B_1}, \mathbf{B_1}\rangle = \langle \mathbf{B_2}, \mathbf{B_2}\rangle = 1, \qquad \langle \mathbf{T}, \mathbf{T}\rangle = -1.$$

3. Equiform Differential Geometry of Timelike Curves

Let $\alpha: I \longrightarrow \mathbb{E}_1^4$ be a timelike curve. We define the equiform parameter of $\alpha(s)$ by

$$\sigma = \int \frac{ds}{\rho} = \int \overline{\kappa}_1 \, ds$$

where $\rho = \frac{1}{\overline{\kappa}_1}$ is the radius of curvature of the curve and $\frac{ds}{d\sigma} = \rho$.

Let's indicate by $\{\mathbf{T}, \mathbf{N}, \mathbf{B_1}, \mathbf{B_2}\}$ the acting Frenet frame along the curve $\alpha(s)$ in the space $\mathbb{E}_1^4$ and so $\{\mathbf{T}, \mathbf{N}, \mathbf{B_1}, \mathbf{B_2}\}$ are, respectively, the unit tangent, the principal normal, the first binormal and the second binormal vector fields. We define the equiform parameter of $\alpha(s)$. Then, we can write

$$\mathbf{U_1} = \rho\mathbf{T},$$

$$\mathbf{U_2} = \rho\mathbf{N},$$

$$\mathbf{U_3} = \rho\mathbf{B_1},$$

$$\mathbf{U_4} = \rho\mathbf{B_2}.$$

Then, $\{\mathbf{U_1}, \mathbf{U_2}, \mathbf{U_3}, \mathbf{U_4}\}$ is an equiform invariant tetrahedron of the curve α [2]. σ is an equiform invariant parameter of α. The derivatives of these vectors with respect to s can be obtained by the following equations:

$$\mathbf{U_1'} = \frac{d}{d\sigma}(\mathbf{U_1}) = \rho \frac{d}{ds}(\rho\mathbf{T}) = \dot{\rho}\mathbf{U_1} + \mathbf{U_2},$$

$$\mathbf{U_2'} = \frac{d}{d\sigma}(\mathbf{U_2}) = \rho \frac{d}{ds}(\rho\mathbf{N}) = \mathbf{U_1} + \dot{\rho}\mathbf{U_2} + \left(\frac{\overline{\kappa}_2}{\overline{\kappa}_1}\right)\mathbf{U_3},$$

$$\mathbf{U_3'} = \frac{d}{d\sigma}(\mathbf{U_3}) = \rho \frac{d}{ds}(\rho\mathbf{B_1}) = -\left(\frac{\overline{\kappa}_2}{\overline{\kappa}_1}\right)\mathbf{U_2} + \dot{\rho}\mathbf{U_3} + \left(\frac{\overline{\kappa}_3}{\overline{\kappa}_1}\right)\mathbf{U_4},$$

$$\mathbf{U_4'} = \frac{d}{d\sigma}(\mathbf{U_4}) = \rho \frac{d}{ds}(\rho\mathbf{B_2}) = -\left(\frac{\overline{\kappa}_3}{\overline{\kappa}_1}\right)\mathbf{U_3} + \dot{\rho}\mathbf{U_4},$$

where the functions $\overline{\kappa}_1$, $\overline{\kappa}_2$ and $\overline{\kappa}_3$ are the curvatures of α [12]. Then, the Frenet formulas in the equiform geometry of the Minkowski space-time can be written as below:

$$\mathbf{U_1'} = \overline{K_1}\mathbf{U_1} + \mathbf{U_2},$$

$$\mathbf{U_2'} = \mathbf{U_1} + \overline{K_1}\mathbf{U_2} + \overline{K_2}\mathbf{U_3},$$

$$\mathbf{U_3'} = -\overline{K_2}\mathbf{U_2} + \overline{K_1}\mathbf{U_3} + \overline{K_3}\mathbf{U_4},$$

$$\mathbf{U_4'} = -\overline{K_3}\mathbf{U_3} + \overline{K_1}\mathbf{U_4}.$$

The functions $\overline{K_1}$, $\overline{K_2}$, $\overline{K_3}$ are the equiform curvatures of α.

$$\begin{bmatrix}\mathbf{U_1'}\\ \mathbf{U_2'}\\ \mathbf{U_3'}\\ \mathbf{U_4'}\end{bmatrix} = \begin{bmatrix}\overline{K_1} & 1 & 0 & 0\\ 1 & \overline{K_1} & \overline{K_2} & 0\\ 0 & -\overline{K_2} & \overline{K_1} & \overline{K_3}\\ 0 & 0 & -\overline{K_3} & \overline{K_1}\end{bmatrix}\begin{bmatrix}\mathbf{U_1}\\ \mathbf{U_2}\\ \mathbf{U_3}\\ \mathbf{U_4}\end{bmatrix}, \tag{2}$$

where

$$\overline{K_1} = \frac{1}{\rho^2}\langle \mathbf{U_j'}, \mathbf{U_j}\rangle; \ \ (j = 1,2,3,4),$$

$$\overline{K_2} = \frac{1}{\rho^2}\langle \mathbf{U_2'}, \mathbf{U_3}\rangle = -\frac{1}{\rho^2}\langle \mathbf{U_3'}, \mathbf{U_2}\rangle, \tag{3}$$

$$\overline{K_3} = \frac{1}{\rho^2}\langle \mathbf{U_3'}, \mathbf{U_4}\rangle = -\frac{1}{\rho^2}\langle \mathbf{U_4'}, \mathbf{U_3}\rangle.$$

4. k-Type Helices in $\mathbb{E}_1^4$

Definition 1. Let α be a timelike curve in $\mathbb{E}_1^4$ with equiform Frenet frame $\{\mathbf{U}_1, \mathbf{U}_2, \mathbf{U}_3, \mathbf{U}_4\}$. If there exists a non-zero constant vector field $\mathbf{U}$ in $\mathbb{E}_1^4$ such that $\langle \mathbf{U}_k, \mathbf{U}\rangle = \mathbf{c}_k$ is a constant for $1 \leq k \leq 4$, α is said to be a $k-$type slant helix and $\mathbf{U}$ is called the slope axis of α.

Theorem 1. Let α be a timelike curve with Frenet formulas in equiform geometry of the Minkowski space-time $\mathbb{E}_1^4$. Then, if the curve α is a 1-type helix (or general helix), then we have

$$\langle \mathbf{U}_2, \mathbf{U}\rangle = -\overline{K}_1 c,$$

where c is a constant.

Proof. Assume that α is a 1-type helix in $\mathbb{E}_1^4$, then for a constant field $\mathbf{U}$, we can write

$$\langle \mathbf{U}_1, \mathbf{U}\rangle = c \tag{4}$$

is a constant and differentiating (4) with respect to σ, we find as below:

$$\langle \mathbf{U}_1', \mathbf{U}\rangle = 0.$$

Using the equiform Frenet equations in equiform geometry, we have the following equation:

$$\langle \overline{K}_1 \mathbf{U}_1 + \mathbf{U}_2, \mathbf{U}\rangle = 0,$$

and it follows that

$$\overline{K}_1 \langle \mathbf{U}_1, \mathbf{U}\rangle + \langle \mathbf{U}_2, \mathbf{U}\rangle = 0.$$

Using (4), we obtain

$$\langle \mathbf{U}_2, \mathbf{U}\rangle = -\overline{K}_1 c.$$

The proof is completed.

Theorem 2. Let α be a timelike curve with Frenet formulas in equiform geometry of the Minkowski space-time $\mathbb{E}_1^4$. Then, if the curve α is a 2-type helix, then we have

$$\langle \mathbf{U}_1, \mathbf{U}\rangle + \overline{K}_2 \langle \mathbf{U}_3, \mathbf{U}\rangle = -\overline{K}_1 c_1,$$

where c_1 is a constant.

Proof. Let the curve α be a 2-type helix in $\mathbb{E}_1^4$, then for a constant field $\mathbf{U}$, in that case, the following equations can be obtained:

$$\langle \mathbf{U}_2, \mathbf{U}\rangle = c_1 \tag{5}$$

is a constant and differentiating this equation with respect to σ, we get

$$\langle \mathbf{U}_2', \mathbf{U}\rangle = 0$$

from the equiform Frenet equations in equiform geometry, we find

$$\langle \mathbf{U}_1 + \overline{K}_1 \mathbf{U}_2 + \overline{K}_2 \mathbf{U}_3, \mathbf{U}\rangle = 0,$$

and

$$\langle \mathbf{U}_1, \mathbf{U}\rangle + \overline{K}_1 \langle \mathbf{U}_2, \mathbf{U}\rangle + \overline{K}_2 \langle \mathbf{U}_3, \mathbf{U}\rangle = 0.$$

Using (5), we obtain the following equation:

$$\langle \mathbf{U}_1, \mathbf{U}\rangle + \overline{K}_2 \langle \mathbf{U}_3, \mathbf{U}\rangle = -\overline{K}_1 c_1.$$

The proof is completed.

Theorem 3. Let α be a timelike curve with Frenet formulas in equiform geometry of the Minkowski space-time $\mathbb{E}_1^4$. In that case, if the curve α is a 3-type helix, then we have

$$-\overline{K}_2 \langle \mathbf{U}_2, \mathbf{U}\rangle + \overline{K}_3 \langle \mathbf{U}_4, \mathbf{U}\rangle = -\overline{K}_1 c_2$$

where c_2 is a constant.

Proof. Let the curve α be a 3-type helix. Thus, for a constant field $\mathbf{U}$ such that

$$\langle \mathbf{U}_3, \mathbf{U} \rangle = c_2 \tag{6}$$

is a constant. Differentiating (6) with respect to σ, we get

$$\langle \mathbf{U}_3', \mathbf{U} \rangle = 0,$$

and using equiform Frenet equations, we have

$$\langle -\overline{K}_2 \mathbf{U}_2 + \overline{K}_1 \mathbf{U}_3 + \overline{K}_3 \mathbf{U}_4, \mathbf{U} \rangle = 0,$$

and it follows that

$$-\overline{K}_2 \langle \mathbf{U}_2, \mathbf{U} \rangle + \overline{K}_1 \langle \mathbf{U}_3, \mathbf{U} \rangle + \overline{K}_3 \langle \mathbf{U}_4, \mathbf{U} \rangle = 0. \tag{7}$$

By setting (6) in (7), we can write

$$-\overline{K}_2 \langle \mathbf{U}_2, \mathbf{U} \rangle + \overline{K}_3 \langle \mathbf{U}_4, \mathbf{U} \rangle = -\overline{K}_1 c_2.$$

The proof is completed.

Theorem 4. Let α be a timelike curve with Frenet formulas in equiform geometry of the Minkowski space-time $\mathbb{E}_1^4$. If the curve α is a 4-type helix, then we have

$$\langle \mathbf{U}_3, \mathbf{U} \rangle = \frac{\overline{K}_1}{\overline{K}_3} c_3,$$

where c_3 is a constant.

Proof. Let the curve α be a 4-type helix in $\mathbb{E}_1^4$, then for a constant field $\mathbf{U}$, we can write the following equation:

$$\langle \mathbf{U}_4, \mathbf{U} \rangle = c_3 \tag{8}$$

is a constant. By differentiating (8) with respect to σ, we get

$$\langle \mathbf{U}_4', \mathbf{U} \rangle = 0$$

and using equiform Frenet equations, we find as below:

$$\langle -\overline{K}_3 \mathbf{U}_3 + \overline{K}_1 \mathbf{U}_4, \mathbf{U} \rangle = 0$$

and we can write

$$-\overline{K}_3 \langle \mathbf{U}_3, \mathbf{U} \rangle + \overline{K}_1 \langle \mathbf{U}_4, \mathbf{U} \rangle = 0.$$

By setting equation (8) in the last equation is written as follows:

$$\langle \mathbf{U}_3, \mathbf{U} \rangle = \frac{\overline{K}_1}{\overline{K}_3} c_3.$$

The proof is completed.

5. $(\boldsymbol{k}, \boldsymbol{m})$-Type Slant Helices for Equiform Differential Geometry in $\mathbb{E}_1^4$

In this section, we will define (k, m)-type slant helices for timelike curve with equiform Frenet frame in $\mathbb{E}_1^4$ such as [6].

Definition 2. Let α be a timelike curve in $\mathbb{E}_1^4$ with equiform Frenet frame $\{\mathbf{U}_1, \mathbf{U}_2, \mathbf{U}_3, \mathbf{U}_4\}$. We call α is a (k, m)-type slant helix if there exists a non-zero constant vector field $\mathbf{U} \in \mathbb{E}_1^4$ satisfies $\langle \mathbf{U}_k, \mathbf{U} \rangle = \mathbf{c}_k$, $\langle \mathbf{U}_m, \mathbf{U} \rangle = \mathbf{c}_m$ are constants for $1 \leq k, m \leq 4$, $k \neq m$. The constant vector $\mathbf{U}$ is an axis of (k, m)-type slant helix.

Theorem 5. If the curve α is a $(1,2)$-type slant helix in $\mathbb{E}_1^4$, then we have

$$\langle \mathbf{U}_3, \mathbf{U} \rangle = \frac{c_2^2 - c_1^2}{c_1} \frac{1}{\overline{K}_2},$$

where $\overline{K}_2 = -\frac{c_2}{c_1}$ is a constant.

Proof. Let the curve α be a $(1,2)$-type slant helix in $\mathbb{E}_1^4$, then for a constant field $\mathbf{U}$, we can write following equations:

$$\langle \mathbf{U}_1, \mathbf{U} \rangle = c_1 \tag{9}$$

is a constant, and

$$\langle \mathbf{U}_2, \mathbf{U} \rangle = c_2 \tag{10}$$

is a constant. Differentiating (9) and (10) with respect to σ, we get

$$\langle \mathbf{U}_1', \mathbf{U} \rangle = 0$$

and

$$\langle \mathbf{U}_2', \mathbf{U} \rangle = 0.$$

Using equiform Frenet equations, we find the following equations:

$$\langle \overline{K}_1 \mathbf{U}_1 + \mathbf{U}_2, \mathbf{U} \rangle = 0$$

and it follows that

$$\langle \mathbf{U}_1 + \overline{K}_1 \mathbf{U}_2 + \overline{K}_2 \mathbf{U}_3, \mathbf{U} \rangle = 0.$$

In that case, we get

$$\overline{K}_1 \langle \mathbf{U}_1, \mathbf{U} \rangle + \langle \mathbf{U}_2, \mathbf{U} \rangle = 0, \tag{11}$$

$$\langle \mathbf{U}_1, \mathbf{U} \rangle + \overline{K}_1 \langle \mathbf{U}_2, \mathbf{U} \rangle + \overline{K}_2 \langle \mathbf{U}_3, \mathbf{U} \rangle = 0. \tag{12}$$

By setting (9) and (10) in (11), we find

$$\overline{K}_1 c_1 + c_2 = 0.$$

Substituting (9) and (10) to (12), we obtain as below:

$$c_1 + \overline{K}_1 c_2 + \overline{K}_2 \langle \mathbf{U}_3, \mathbf{U} \rangle = 0. \tag{13}$$

Finally, we get

$$\overline{K}_1 = -\frac{c_2}{c_1}, \tag{14}$$

and by setting (14) in (13), we get

$$\langle \mathbf{U}_3, \mathbf{U} \rangle = \frac{c_2^2 - c_1^2}{c_1} \frac{1}{\overline{K}_2}.$$

The proof is completed.

Theorem 6. If the curve α is a $(1,3)$-type slant helix in $\mathbb{E}_1^4$, then there exists a constant such that

$$\langle \mathbf{U}_4, \mathbf{U} \rangle = -\frac{\overline{K}_1 \overline{K}_2}{\overline{K}_3} c_1 - \frac{\overline{K}_1}{\overline{K}_3} c_3$$

where c_1, c_3 are constants.

Proof. Let the curve α be a $(1,3)$-type slant helix in $\mathbb{E}_1^4$, then for a constant field $\mathbf{U}$, we can write as below:

$$\langle \mathbf{U}_1, \mathbf{U} \rangle = c_1 \tag{15}$$

is a constant, and

$$\langle \mathbf{U}_3, \mathbf{U} \rangle = c_3 \tag{16}$$

is a constant. Differentiating (15) and (16) with respect to σ, we get

$\langle \mathbf{U}_1', \mathbf{U} \rangle = 0$

and

$\langle \mathbf{U}_3', \mathbf{U} \rangle = 0.$

Using equiform Frenet equations, we obtain the following equations:

$\langle \overline{K}_1 \mathbf{U}_1 + \mathbf{U}_2, \mathbf{U} \rangle = 0,$

and we have that

$\langle -\overline{K}_2 \mathbf{U}_2 + \overline{K}_1 \mathbf{U}_3 + \overline{K}_3 \mathbf{U}_4, \mathbf{U} \rangle = 0.$

(We know that $\mathbf{U}$ is a constant). Thus, we can write as below:

$$\overline{K}_1 \langle \mathbf{U}_1, \mathbf{U} \rangle + \langle \mathbf{U}_2, \mathbf{U} \rangle = 0, \tag{17}$$

$$-\overline{K}_2 \langle \mathbf{U}_2, \mathbf{U} \rangle + \overline{K}_1 \langle \mathbf{U}_3, \mathbf{U} \rangle + \overline{K}_3 \langle \mathbf{U}_4, \mathbf{U} \rangle = 0. \tag{18}$$

By setting equation (15) in equation (17), we get

$$\langle \mathbf{U}_2, \mathbf{U} \rangle = -\overline{K}_1 c_1. \tag{19}$$

Substituting (16) and (19) to (18), we find

$$\langle \mathbf{U}_4, \mathbf{U} \rangle = -\frac{\overline{K}_1 \overline{K}_2}{\overline{K}_3} c_1 - \frac{\overline{K}_1}{\overline{K}_3} c_3.$$

The proof is completed.

Theorem 7. If the curve α is a $(1,4)$-type slant helix in $\mathbb{E}_1^4$, then there exists a constant such that

$\langle \mathbf{U}_2, \mathbf{U} \rangle = -\overline{K}_1 c_1$

and

$$\langle \mathbf{U}_3, \mathbf{U} \rangle = \frac{\overline{K}_1}{\overline{K}_3} c_4$$

where c_1, c_4 are constants.

Proof. Let the curve α be a $(1,4)$-type slant helix in $\mathbb{E}_1^4$, then for a constant field $\mathbf{U}$, we can write the following equations:

$$\langle \mathbf{U}_1, \mathbf{U} \rangle = c_1 \tag{20}$$

is a constant and

$$\langle \mathbf{U}_4, \mathbf{U} \rangle = c_4 \tag{21}$$

is a constant. Differentiating (20) and (21) with respect to σ, we get

$\langle \mathbf{U}_1', \mathbf{U} \rangle = 0$

and

$\langle \mathbf{U}_4', \mathbf{U} \rangle = 0.$

Using equiform Frenet equations, we find

$\langle \overline{K}_1 \mathbf{U}_1 + \mathbf{U}_2, \mathbf{U} \rangle = 0$

and

$\langle -\overline{K}_3 \mathbf{U}_3 + \overline{K}_1 \mathbf{U}_4, \mathbf{U} \rangle = 0.$

So, the following equations can be obtained:

$$\overline{K}_1 \langle \mathbf{U}_1, \mathbf{U} \rangle + \langle \mathbf{U}_2, \mathbf{U} \rangle = 0, \tag{22}$$

$$-\overline{K}_3\langle \mathbf{U}_3, \mathbf{U}\rangle + \overline{K}_1\langle \mathbf{U}_4, \mathbf{U}\rangle = 0. \tag{23}$$

By setting (20) in (22), we have

$$\langle \mathbf{U}_2, \mathbf{U}\rangle = -\overline{K}_1 c_1.$$

Substituting (21) to (23), we get

$$\langle \mathbf{U}_3, \mathbf{U}\rangle = \frac{\overline{K}_1}{\overline{K}_3} c_4.$$

The proof is completed.

Theorem 8. If the curve α is a $(2,3)$-type slant helix in $\mathbb{E}_1^4$, then there exists a constant such as

$$\langle \mathbf{U}_1, \mathbf{U}\rangle = -\overline{K}_1 c_2 - \overline{K}_2 c_3$$

and

$$\langle \mathbf{U}_4, \mathbf{U}\rangle = \frac{\overline{K}_2}{\overline{K}_3} c_2 - \frac{\overline{K}_1}{\overline{K}_3} c_3.$$

Proof. Let the curve α be a $(2,3)$-type slant helix in $\mathbb{E}_1^4$, then for a constant field $\mathbf{U}$, we can write

$$\langle \mathbf{U}_2, \mathbf{U}\rangle = c_2 \tag{24}$$

is a constant and

$$\langle \mathbf{U}_3, \mathbf{U}\rangle = c_3 \tag{25}$$

is a constant. Differentiating (24) and (25) with respect to σ, we find

$$\langle \mathbf{U}_2', \mathbf{U}\rangle = 0$$

and

$$\langle \mathbf{U}_3', \mathbf{U}\rangle = 0.$$

Using equiform Frenet formulas, the following equations can be obtained:

$$\langle \mathbf{U}_1, \mathbf{U}\rangle + \overline{K}_1\langle \mathbf{U}_2, \mathbf{U}\rangle + \overline{K}_2\langle \mathbf{U}_3, \mathbf{U}\rangle = 0, \tag{26}$$

$$-\overline{K}_2\langle \mathbf{U}_2, \mathbf{U}\rangle + \overline{K}_1\langle \mathbf{U}_3, \mathbf{U}\rangle + \overline{K}_3\langle \mathbf{U}_4, \mathbf{U}\rangle = 0. \tag{27}$$

By setting (24) and (25) in (26), we get

$$\langle \mathbf{U}_1, \mathbf{U}\rangle = -\overline{K}_1 c_2 - \overline{K}_2 c_3,$$

and substituting (24) and (25) to (27), we have

$$\langle \mathbf{U}_4, \mathbf{U}\rangle = \frac{\overline{K}_2}{\overline{K}_3} c_2 - \frac{\overline{K}_1}{\overline{K}_3} c_3.$$

The proof is completed.

Theorem 9. If the curve α is a $(2,4)$-type slant helix in $\mathbb{E}_1^4$, then there exists a constant such as

$$\langle \mathbf{U}_1, \mathbf{U}\rangle = -\overline{K}_1 c_2 - \overline{K}_2 \frac{\overline{K}_1}{\overline{K}_3} c_4,$$

where $c_2,\ c_4$ are constants.

Proof. Let the curve α be a $(2,4)$-type slant helix in $\mathbb{E}_1^4$, then for a constant field $\mathbf{U}$, we can write the following equations:

$$\langle \mathbf{U}_2, \mathbf{U}\rangle = c_2 \tag{28}$$

and

$$\langle \mathbf{U}_4, \mathbf{U} \rangle = c_4 \tag{29}$$

are constants. By differentiating (28) and (29) with respect to σ, we get the following equations:

$$\langle \mathbf{U}_2', \mathbf{U} \rangle = 0$$

and

$$\langle \mathbf{U}_4', \mathbf{U} \rangle = 0.$$

Using equiform Frenet equations, we find

$$\langle \mathbf{U}_1, \mathbf{U} \rangle + \overline{K}_1 \langle \mathbf{U}_2, \mathbf{U} \rangle + \overline{K}_2 \langle \mathbf{U}_3, \mathbf{U} \rangle = 0, \tag{30}$$

$$-\overline{K}_3 \langle \mathbf{U}_3, \mathbf{U} \rangle + \overline{K}_1 \langle \mathbf{U}_4, \mathbf{U} \rangle = 0. \tag{31}$$

Substituting (28) to (30), we obtain as follows:

$$\langle \mathbf{U}_1, \mathbf{U} \rangle + \overline{K}_2 \langle \mathbf{U}_3, \mathbf{U} \rangle = -\overline{K}_1 c_2. \tag{32}$$

By setting (29) in (31), we have the following equation:

$$\langle \mathbf{U}_3, \mathbf{U} \rangle = \frac{\overline{K}_1}{\overline{K}_3} c_4 \tag{33}$$

and by setting (33) in (32), we obtain

$$\langle \mathbf{U}_1, \mathbf{U} \rangle = -\overline{K}_1 c_2 - \overline{K}_2 \frac{\overline{K}_1}{\overline{K}_3} c_4.$$

The proof is completed.

Theorem 10. If the curve α is a $(3,4)$-type slant helix in $\mathbb{E}_1^4$, then there exists a constant such as

$$\langle \mathbf{U}_2, \mathbf{U} \rangle = \frac{\overline{K}_3}{\overline{K}_2} \left(\frac{c_3^2}{c_4} + c_4 \right)$$

where c_3, c_4 are constants.

Proof. Let the curve α be a $(3,4)$-type slant helix in $\mathbb{E}_1^4$, then for a constant field $\mathbf{U}$, we can write as follows:

$$\langle \mathbf{U}_3, \mathbf{U} \rangle = c_3 \tag{34}$$

is a constant and

$$\langle \mathbf{U}_4, \mathbf{U} \rangle = c_4 \tag{35}$$

is a constant. By differentiating (34) and (35) with respect to σ, we have the following equations:

$$\langle \mathbf{U}_3', \mathbf{U} \rangle = 0$$

and

$$\langle \mathbf{U}_4', \mathbf{U} \rangle = 0.$$

Using equiform Frenet formulas, we find as below:

$$-\overline{K}_2 \langle \mathbf{U}_2, \mathbf{U} \rangle + \overline{K}_1 \langle \mathbf{U}_3, \mathbf{U} \rangle + \overline{K}_3 \langle \mathbf{U}_4, \mathbf{U} \rangle = 0, \tag{36}$$

and

$$-\overline{K}_3 \langle \mathbf{U}_3, \mathbf{U} \rangle + \overline{K}_1 \langle \mathbf{U}_4, \mathbf{U} \rangle = 0. \tag{37}$$

Substituting (34) and (35) to (37), we can write

$$\overline{K}_1 = \overline{K}_3 \frac{c_3}{c_4}, \tag{38}$$

and by setting (34), (35) and (38) in (36), we obtain

$$\langle \mathbf{U}_2, \mathbf{U} \rangle = \frac{\overline{K}_3}{\overline{K}_2}\frac{c_3^2}{c_4} + \frac{\overline{K}_3}{\overline{K}_2}c_4.$$

The proof is completed.

6. Conclusion

In this study we investigate equiform differential geometry of timelike curves and $k-$ and (k,m) $-$type slant helices for equiform differential geometry of timelike curves in the Minkowski space-time.

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Conflicts of interest

The authors stated that did not have conflict of interests.

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