Cumhuriyet Science Journal

e-ISSN: 2587-246X
ISSN: 2587-2680

Cumhuriyet Sci. J., 42(4) (2021) 862-872
http://dx.doi.org/10.17776/csj.942629

Approximation by the new modification of Bernstein-Stancu operators

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Abstract

The current paper deals with the new modification of Bernstein-Stancu operators which preserve constant and Korovkin's other test functions in limit case. We study the uniform convergence of the newly defined operators. The rate of convergence is investigated by means of the modulus of continuity, by using functions of Lipschitz class and by the help of Peetre-$\mathcal{K}$ functionals. Then a Voronovskaya type asymptotic formula for the newly constructed Bernstein-Stancu operators is presented. Finally, some graphs are given to illustrate the convergence properties of operators to some functions.

Article info

History:
Received: 25.05.2021
Accepted: 09.09.2021

Keywords:
Modulus of continuity,
Peetre-$\mathcal{K}$ functionals,
Voronovskaya-type theorem.

1. Introduction

Approximation theory concerns with the approximation of functions by the help of simpler calculated functions. Linear positive operators play a crucial role in this area. The well-known linear and positive operators are Bernstein operators [1], which are defined as

$$B_s(v;x) = \sum_{m=0}^{s} \binom{s}{m} x^s (1-x)^{s-m} v\left(\frac{m}{s}\right), \quad s \geq 1$$

for $v \in C[0,1]$ and $x \in [0,1]$. In 1983, Stancu [2] studied a generalization of Bernstein operators

$$S_s^{\alpha,\beta}(v;x) = \sum_{m=0}^{s} \binom{s}{m} x^m (1-x)^{s-m} v\left(\frac{m+\alpha}{s+\beta}\right)$$

with the condition $0 \leq \alpha \leq \beta$. After that, various versions of Bernstein operators are investigated by researchers such as [3-8]. Recently, a new modification of Bernstein operators is introduced by Usta [9] which unchanging constant test function and preserve Korovkin's other test functions t and t^2 in limit case by

$$B_s^*(v;x) = \frac{1}{s} \sum_{m=0}^{s} \binom{s}{m} (m-sx)^2 x^{m-1} (1-x)^{s-m-1} v\left(\frac{m}{s}\right).$$

And he proved the approximation properties of the $B_s^*(v;x)$ operators. Motivated by this work, we construct a modification of Bernstein-Stancu operators. The new modification of Bernstein-Stancu operators is presented for $v \in C[0,1]$ as follows:

$$B_s^{\alpha,\beta}(v;x) = \frac{1}{s} \sum_{m=0}^{s} \binom{s}{m} (m-sx)^2 x^{m-1} (1-x)^{s-m-1} v\left(\frac{m+\alpha}{s+\beta}\right), \tag{1}$$

where $0 \leq \alpha \leq \beta$, $x \in (0,1)$ and $s \geq 1$.

The aim of the current paper is to derive approximation properties for the operators (1) by working on Korovkin's theorem [10].

The rest of this paper is structured as follows: In Section 2, the certain elementary properties and uniform convergence of the newly constructed Bernstein-Stancu operators are investigated. In Section 3, the rate of convergence is studied, while Voronovskaya-type formula is given in Section 4. In Section 5, the

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numerical examples are illustrated to show the prosperousness of the theoretical results and the effectiveness of the defined operators. Finally, a brief conclusion about the paper is given in Section 6.

2. Approximation Properties of $B_s^{\alpha,\beta}$

In this part, we give some important auxiliary results which will be used in proving our main results of the following sections. First of all, we determine moments, central moments and some limit results for the operators (1).

Lemma 2.1 For every $x \in (0,1)$ and $0 \leq \alpha \leq \beta$, we write

$$
\begin{aligned}
B_s^{\alpha,\beta}(e_0;x) &= 1,\\
B_s^{\alpha,\beta}(e_1;x) &= \frac{s-2}{s+\beta}x + \frac{\alpha+1}{s+\beta},\\
B_s^{\alpha,\beta}(e_2;x) &= \frac{s^2-7s+6}{(s+\beta)^2}x^2 + \frac{(5+2\alpha)s-6-4\alpha}{(s+\beta)^2}x + \frac{(\alpha+1)^2}{(s+\beta)^2},\\
B_s^{\alpha,\beta}(e_3;x) &= \frac{s^3-15s^2+38s-24}{(s+\beta)^3}x^3 + \frac{(12+3\alpha)s^2-(48+21\alpha)s+36+18\alpha}{(s+\beta)^3}x^2\\
&\quad + \frac{(13+15\alpha+3\alpha^2)s-14-18\alpha-6\alpha^2}{(s+\beta)^3}x + \frac{(\alpha+1)^3}{(s+\beta)^3},\\
B_s^{\alpha,\beta}(e_4;x) &= \frac{1}{(\beta+s)^4(x-1)}\big(-1+\alpha^4(x-1)+(10-8s)x+(-180+240s-61s^2)x^2+(390-615s\\
&\quad +247s^2-22s^3)x^3-(360-630s+317s^2-48s^3+s^4)x^4+(120-226s+131s^2\\
&\quad -26s^3+s^4)x^5+4\alpha^3(x-1)(1+(-2+s)x)\\
&\quad +6\alpha^2(x-1)(1+(-6+5s)x+(6-7s+s^2)x^2)+4\alpha(x-1)(1+(-14+13s)x\\
&\quad +12(3-4s+s^2)x^2+(-24+38s-15s^2+s^3)x^3)\big),
\end{aligned}
$$

where $e_m = t^m$ for $m = 0,1,2,3,4$.

Lemma 2.2 For every $x \in (0,1)$ and $0 \leq \alpha \leq \beta$, we have

$$
\begin{aligned}
B_s^{\alpha,\beta}(t-x;x) &= -\frac{2+\beta}{s+\beta}x + \frac{\alpha+1}{s+\beta},\\
B_s^{\alpha,\beta}((t-x)^2;x) &= \frac{-3s+6+4\beta+\beta^2}{(s+\beta)^2}x^2 + \frac{3s-(6+4\alpha+2(\alpha+1)\beta)}{(s+\beta)^2}x + \frac{(\alpha+1)^2}{(s+\beta)^2},\\
B_s^{\alpha,\beta}((t-x)^4;x) &= \frac{1}{(\beta+s)^4(x-1)}\big(-1+\alpha^4(x-1)+2(5+2\beta-2s)x-(6\beta^2+\beta(60-40s)\\
&\quad +15(12-12s+s^2))x^2+(4\beta^3-6\beta^2(-7+3s)-40\beta(-5+4s)+5(78-83s\\
&\quad +9s^2))x^3-(12\beta^3+\beta^4-36\beta^2(-2+s)-40\beta(-6+5s)+15(24-26s+3s^2))x^4\\
&\quad +(8\beta^3+\beta^4+\beta(96-80s)-18\beta^2(-2+s)+5(24-26s+3s^2))x^5\\
&\quad -4\alpha^3(x-1)(-1+(2+\beta)x)+6\alpha^2(x-1)(1+(-6-2\beta+3s)x\\
&\quad +(6+4\beta+\beta^2-3s)x^2)-4\alpha(x-1)(-1+(14+3\beta-10s)x-3(12+\beta^2-\\
&\quad 3\beta(-2+s)-10s)x^2+(24+6\beta^2+\beta^3-9\beta(-2+s)-20s)x^3)\big). \qquad (2)
\end{aligned}
$$

Proof.

With the help of the following equalities

$$B_s^{\alpha,\beta}(t-x;x) = B_s^{\alpha,\beta}(e_1;x) - xB_s^{\alpha,\beta}(1;x),$$

$B_s^{\alpha,\beta}((t-x)^2;x) = B_s^{\alpha,\beta}(e_2;x) - 2xB_s^{\alpha,\beta}(e_1;x) + x^2B_s^{\alpha,\beta}(1;x),$
$B_s^{\alpha,\beta}((t-x)^4;x) = B_s^{\alpha,\beta}(e_4;x) - 4xB_s^{\alpha,\beta}(e_3;x) + 6x^2B_s^{\alpha,\beta}(e_2;x) - 4x^3B_s^{\alpha,\beta}(e_1;x) + x^4B_s^{\alpha,\beta}(1;x),$

and with the help of Lemma 2.1, we get the desired results.

Lemma 2.3 We have following results

$$\lim_{s\to\infty} sB_s^{\alpha,\beta}(t-x;x) = -(2+\beta)x + \alpha + 1, \tag{3}$$

$$\lim_{s\to\infty} sB_s^{\alpha,\beta}((t-x)^2;x) = 3x(1-x), \tag{4}$$

$$\lim_{s\to\infty} s^2B_s^{\alpha,\beta}((t-x)^4;x) = 15(x-1)^2x^2. \tag{5}$$

Let the Banach space of all continuous functions v on $[0,1]$ is denoted by $C[0,1]$ endowed with the norm

$$\|v\| = \max_{x\in(0,1)} |v(x)|.$$

Theorem 2.1 For every $v \in C[0,1], x \in (0,1)$ and $0 \le \alpha \le \beta$

$$\left\|B_s^{\alpha,\beta}(v) - v\right\| \to 0, \tag{6}$$

uniformly as $s \to \infty$.

Proof.

It can be seen clearly from Lemma 2.1 that

$$\lim_{s\to\infty} B_s^{\alpha,\beta}(e_i;x) = t^i, \qquad i = 0,1,2.$$

Thus, we conclude the proof of the theorem thanks to the result of Korovkin's theorem [10].

3. Rate of Convergence

In this section, we study the rate of convergence with the help of modulus of continuity, by using functions that belong to Lipschitz class and with the help of Peetre-$\mathcal{K}$ functionals, respectively.
For $v \in C[0,1]$, the modulus of continuity is given by

$$\omega(v,\delta) := \sup_{|t-x|\le\delta} \sup_{x\in(0,1)} |v(t) - v(x)|, \quad \delta > 0.$$

Modulus of continuity of function v has the property as follows:

$$|v(t) - v(x)| \le \left(1 + \frac{(t-x)^2}{\delta^2}\right) \omega(v,\delta).$$

Theorem 3.1 For every $x \in (0,1)$, $v \in C[0,1]$ and $0 \le \alpha \le \beta$,

$$|B_s^{\alpha,\beta}(v;x) - v(x)| \le 2\omega(v,\delta_s(x)). \tag{7}$$

Here,

$$\delta_s(x) = \frac{\sqrt{\alpha^2 + 2\alpha + 1 + (3s - (6 + 4\alpha + 2(\alpha+1)\beta))x + (-3s + 6 + 4\beta + \beta^2)x^2}}{s+\beta}.$$

Proof.

$$\left|B_s^{\alpha,\beta}(v;x)-v(x)\right| = \left|\frac{1}{s}\sum_{m=0}^{s}\binom{s}{m}(m-sx)^2x^{m-1}(1-x)^{s-m-1}v\left(\frac{m+\alpha}{s+\beta}\right)-v(x)\right|$$

$$\leq \frac{1}{s}\sum_{m=0}^{s}\binom{s}{m}(m-sx)^2x^{m-1}(1-x)^{s-m-1}\left|v\left(\frac{m+\alpha}{s+\beta}\right)-v(x)\right|$$

$$\leq \frac{1}{s}\sum_{m=0}^{s}\binom{s}{m}(m-sx)^2x^{m-1}(1-x)^{s-m-1}\left(1+\frac{1}{\delta^2}\frac{(m+\alpha-x(s+\beta))^2}{(s+\beta)^2}\right)\omega(v,\delta)$$

$$=\left(1+\frac{1}{\delta^2}\frac{\alpha^2+2\alpha+1+(3s-(6+4\alpha+2(\alpha+1)\beta))x+(-3s+6+4\beta+\beta^2)x^2}{(s+\beta)^2}\right)\omega(v,\delta).$$

If we choose

$$\delta=\delta_s(\mathrm{x})=\frac{\sqrt{\alpha^2+2\alpha+1+(3s-(6+4\alpha+2(\alpha+1)\beta))x+(-3s+6+4\beta+\beta^2)x^2}}{s+\beta},$$

then we achieve that

$$|B_s^{\alpha,\beta}(v;x)-v(x)|\leq 2\omega\left(v,\frac{\sqrt{\alpha^2+2\alpha+1+(3s-(6+4\alpha+2(\alpha+1)\beta))x+(-3s+6+4\beta+\beta^2)x^2}}{s+\beta}\right),$$

which is the desired result.

Just now, we investigate the rate of convergence of $B_s^{\alpha,\beta}(v;x)$ by using functions of Lipschitz class. Let's recall that a function $v\in Lip_K(c)$ on $(0,1)$ if the inequality

$$|v(t)-v(x)|\leq K|t-x|^c \ ;\ \forall t,x\in(0,1) \tag{8}$$

holds.

Theorem 3.2 Let $v\in Lip_K(c)$, $0<c\leq 1$, $0\leq\alpha\leq\beta$, then we have

$$\left|B_s^{\alpha,\beta}(v;x)-v(x)\right|\leq K[\delta_s(x)]^c,$$

where

$$\delta_s(x)=\frac{\sqrt{\alpha^2+2\alpha+1+(3s-(6+4\alpha+2(\alpha+1)\beta))x+(-3s+6+4\beta+\beta^2)x^2}}{s+\beta}.$$

Proof.

Let $v\in Lip_K(c)$, $0<c\leq 1$ and $0\leq\alpha\leq\beta$. By using (8) and the linearity and monotonicity of the operators $B_s^{\alpha,\beta}$ we have

$$\left|B_s^{\alpha,\beta}(v;x)-v(x)\right|\leq B_s^{\alpha,\beta}(|v(t)-v(x)|;x)$$

$$\leq KB_s^{\alpha,\beta}(|t-x|^c;x).$$

By taking $p=\frac{2}{c}$, $q=\frac{2}{2-c}$ in the Hölder's inequality, we get

$$\left|B_s^{\alpha,\beta}(v;x)-v(x)\right|\leq K\left\{B_s^{\alpha,\beta}((t-x)^2;x)\right\}^{\frac{c}{2}}$$

$$\leq K[\delta_s(x)]^c$$

immediately. If we choose

$$\delta_s(x) = \frac{\sqrt{\alpha^2 + 2\alpha + 1 + (3s - (6 + 4\alpha + 2(\alpha + 1)\beta))x + (-3s + 6 + 4\beta + \beta^2)x^2}}{s + \beta}$$

the proof is completed.

Lastly, we mention the rate of convergence of our operator $B_s^{\alpha,\beta}(v;x)$ by means of Peetre-$\mathcal{K}$ functionals. First of all, we give the following lemma:

Lemma 3.1 For $x \in (0,1)$, $v \in C[0,1]$ and $0 \le \alpha \le \beta$, we have

$$|B_s^{\alpha,\beta}(v;x)| \le ||v||. \tag{9}$$

Proof.
From the definiton of $B_s^{\alpha,\beta}(v;x)$, we write

$$\begin{aligned}|B_s^{\alpha,\beta}(v;x)| &= \left|\frac{1}{s}\sum_{m=0}^{s}\binom{s}{m}(m - sx)^2 x^{m-1}(1-x)^{s-m-1} v\left(\frac{m+\alpha}{s+\beta}\right)\right| \\ &\le \frac{1}{s}\sum_{m=0}^{s}\binom{s}{m}(m - sx)^2 x^{m-1}(1-x)^{s-m-1}\left|v\left(\frac{m+\alpha}{s+\beta}\right)\right| \\ &\le ||v|| B_s^{\alpha,\beta}(1;x) \\ &= ||v||.\end{aligned}$$

Currently, we recall the properties of Peetre-$\mathcal{K}$ functionals. $C^2[0,1]$ is the space of the functions v, for which v, v' and v'' are continuous on $[0,1]$. We define classical Peetre-$\mathcal{K}$ functional as follows:

$$\mathcal{K}(v,\delta) := \inf_{u \in C^2[0,1]} \{\|v - u\|_{C[0,1]} + \delta\|u''\|_{C[0,1]}\}$$

and second modulus of smoothness of the function is defined by

$$\omega_2(v,\delta) := \sup_{0<h<\delta} \sup_{x, x+2h \in (0,1)} |v(x+2h) - 2v(x+h) + v(x)|$$

where $\delta > 0$. By [11], it is known that for $A > 0$

$$\mathcal{K}(v,\delta) \le A\omega_2(v,\sqrt{\delta}).$$

Theorem 3.3 Let $x \in (0,1)$, $v \in C[0,1]$ and $0 \le \alpha \le \beta$. Then we have for all $s \in \mathbb{N}$, there exists a positive constant A such that,

$$|B_s^{\alpha,\beta}(v;x) - v(x)| \le A\omega_2(v, \alpha_s(x)) + 2\omega(v, \beta_s(x)),$$

where

$$\alpha_s(x) = \sqrt{\frac{3 + 6\alpha + 3\alpha^2 - 6\alpha(2+\beta)x - 2(8 + 3\beta - 3s)x + (16 + 12\beta + 3\beta^2 - 6s)x^2}{2(s+\beta)^2}}$$

and

$$\beta_s(x) = \left|\frac{1 + \alpha - (2+\beta)x}{s + \beta}\right|.$$

Proof.

Define an auxiliary operator $B_s^*: C[0,1] \to C[0,1]$ by

$$B_s^*(u;x) = B_s^{\alpha,\beta}(u;x) - u\left(\frac{(s-2)x + \alpha + 1}{s+\beta}\right) + u(x). \tag{10}$$

From Lemma 2.1, we have

$$B_s^*(1;x) = 1$$

$$B_s^*(t-x;x) = B_s^{\alpha,\beta}((t-x);x) - \left(\frac{(s-2)x+\alpha+1}{s+\beta} - x\right) + x - x$$
$$= \left(-\frac{2+\beta}{s+\beta}x + \frac{\alpha+1}{s+\beta}\right) - \left(\frac{(s-2)x+\alpha+1}{s+\beta} - x\right) + x - x$$
$$= 0. \quad (11)$$

For a given function $u \in C^2[0,1]$, we have by the Taylor expansion that

$$u(t) = u(x) + (t-x)u'(x) + \int_x^t (t-m)u''(m)dm, \;\; t \in (0,1). \quad (12)$$

Applying B_s^* operator to the both sides of the equation (12), we obtain

$$B_s^*(u;x) = B_s^*\left(u(x) + (t-x)u'(x) + \int_x^t (t-m)u''(m)dm; x\right)$$
$$= u(x) + B_s^*((t-x)u'(x);x) + B_s^*\left(\int_x^t (t-m)u''(m)dm; x\right).$$

So,

$$B_s^*(u;x) - u(x) = u'(x)B_s^*((t-x);x) + B_s^*\left(\int_x^t (t-m)u''(m)dm; x\right).$$

By using (10) and (11), we get

$$B_s^*(u;x) - u(x) = B_s^*\left(\int_x^t (t-m)u''(m)dm; x\right)$$
$$= B_s^{\alpha,\beta}\left(\int_x^t (t-m)u''(m)dm; x\right) - \int_x^{\frac{(s-2)x+\alpha+1}{s+\beta}} \left(\frac{(s-2)x+\alpha+1}{s+\beta} - m\right) u''(m)dm$$
$$+ \int_x^x \left(\frac{(s-2)x+\alpha+1}{s+\beta} - m\right) u''(m)dm. \quad (13)$$

Furthermore,

$$\left|\int_x^t (t-m)u''(m)dm\right| \le \int_x^t |t-m||u''(m)|dm \le ||u''|| \int_x^t |t-m|dm$$
$$\le (t-x)^2||u''||, \quad (14)$$

and

$$\left|\int_x^{\frac{(s-2)x+\alpha+1}{s+\beta}} \left(\frac{(s-2)x+\alpha+1}{s+\beta} - u\right) u''(m)dm\right| \le ||u''|| \int_x^{\frac{(s-2)x+\alpha+1}{s+\beta}} \left(\frac{(s-2)x+\alpha+1}{s+\beta} - m\right) dm$$
$$= \frac{||u''||}{2}\left(\left(\frac{(s-2)x+\alpha+1}{s+\beta}\right)^2 - 2\frac{(s-2)x+\alpha+1}{s+\beta}x + x^2\right)$$
$$= \frac{||u''||}{2}\left(\frac{(s-2)x+\alpha+1}{s+\beta} - x\right)^2. \quad (15)$$

Let's rewrite (14) and (15) in the absolute value of (13). Then we obtain

$$|B_s^*(u;x) - u(x)| \le ||u''||B_s^{\alpha,\beta}((t-x)^2;x) + \frac{||u''||}{2}\left(\frac{(s-2)x+\alpha+1}{s+\beta} - x\right)^2$$
$$= ||u''||\left(B_s^{\alpha,\beta}((t-x)^2;x) + \frac{1}{2}\left(\frac{(s-2)x+\alpha+1}{s+\beta} - x\right)^2\right)$$
$$= ||u''||\alpha_s^2(x),$$

where

$$\alpha_s(x) = \sqrt{B_s^{\alpha,\beta}((t-x)^2;x) + \frac{1}{2}\left(\frac{(s-2)x+\alpha+1}{s+\beta} - x\right)^2}$$

$$= \sqrt{\frac{3+6\alpha+3\alpha^2-6\alpha(2+\beta)x-2(8+3\beta-3s)x+(16+12\beta+3\beta^2-6s)x^2}{2(s+\beta)^2}}.$$

Now, we will find an upper bound for the auxiliary operator $B_s^*(u;x)$. In the light of the Lemma 3.1, we get

$$|B_s^*(u;x)| = |B_s^{\alpha,\beta}(u;x) - u\left(\frac{(s-2)x+\alpha+1}{s+\beta}\right) + u(x)|$$

$$\leq |B_s^{\alpha,\beta}(u;x)| + \left|u\left(\frac{(s-2)x+\alpha+1}{s+\beta}\right)\right| + |u(x)|$$

$$\leq 3||u||.$$

Accordingly,

$$|B_s^{\alpha,\beta}(v;x) - v(x)| = \left|B_s^*(v;x) - v(x) + v\left(\frac{(s-2)x+\alpha+1}{s+\beta}\right) - v(x) + u(x) - u(x)\right.$$
$$\left. + B_s^*(u;x) - B_s^*(u;x)\right|$$
$$\leq |B_s^*(v-u;x) - (v-u)(x)| + |B_s^*(u;x) - u(x)| + \left|v\left(\tfrac{(s-2)x+\alpha+1}{s+\beta}\right) - v(x)\right|$$
$$\leq 4||v-u|| + ||u''||\alpha_s^2(x) + \omega(v,\beta_s(x))\left(1 + \frac{\left|\frac{(s-2)x+\alpha+1}{s+\beta} - x\right|}{\beta_s(x)}\right)$$
$$= 4||v-u|| + ||u''||\alpha_s^2(x) + 2\omega\left(v, \left|\tfrac{(s-2)x+\alpha+1}{s+\beta} - x\right|\right), \quad (16)$$

where

$$\beta_s(x) = \left|\frac{(s-2)x+\alpha+1}{s+\beta} - x\right|$$
$$= \left|\frac{1+\alpha-(2+\beta)x}{s+\beta}\right|.$$

Finally, for all $v \in C^2[0,1]$ take the infimum of the equation (16). We achieve

$$|B_s^{\alpha,\beta}(v;x) - v(x)| \leq 4\mathcal{K}(u,\alpha_s^2(x)) + 2\omega(v,\beta_s(x)). \quad (17)$$

As a result, using the property of Peetre-$\mathcal{K}$ functional, we obtain

$$|B_s^{\alpha,\beta}(v;x) - v(x)| \leq A\omega_2(v,\alpha_s(x)) + 2\omega(v,\beta_s(x)). \quad (18)$$

Thus the proof is completed.

4. Voronovskaya-Type Theorem

In 1932, Voronovskaya [12] obtain the convergence speed of the the $B_s(v;x)$ operators to the function v. In this section, we give a Voronovskaya-type asymptotic formula for $B_s^{\alpha,\beta}(v;x)$ operators.

Theorem 4.1 Let v be integrable on the interval $(0,1)$, also u' and u'' exist at a fixed point $x \in (0,1)$. Then we have

$$\lim_{s\to\infty} s\left(B_s^{\alpha,\beta}(v;x) - v(x)\right) = (-(2+\beta)x+\alpha+1)v'(x) + \frac{3}{2}x(1-x)v''(x). \quad (19)$$

Proof.

By using the well-known Taylor's formula, we write

$$v(t) = v(x) + (t-x)v'(x) + \frac{(t-x)^2}{2}v''(x) + \mathcal{R}(t,x)(t-x)^2, \qquad (20)$$

where $\mathcal{R}(t,x) := \frac{v''(\xi)-v''(x)}{2}$ is Peano form of the remainder term such that ξ lying between x and t. Also, $\lim_{t\to x}\mathcal{R}(t,x) = 0$. By applying $B_s^{\alpha,\beta}$ operators to (20), we get

$$B_s^{\alpha,\beta}(v;x) - v(x) = v'(x)B_s^{\alpha,\beta}((t-x);x) + \frac{v''(x)}{2}B_s^{\alpha,\beta}((t-x)^2;x)$$
$$+B_s^{\alpha,\beta}(\mathcal{R}(t,x)(t-x)^2;x). \qquad (21)$$

When we multiply (21) by s and take the limit as s goes to infinity, we achieve

$$\lim_{s\to\infty} s\left(B_s^{\alpha,\beta}(v;x) - v(x)\right) = \lim_{s\to\infty} sv'(x)B_s^{\alpha,\beta}((t-x);x) + \lim_{s\to\infty} s\frac{v''(x)}{2}B_s^{\alpha,\beta}((t-x)^2;x)$$
$$+\lim_{s\to\infty} sB_s^{\alpha,\beta}(\mathcal{R}(t,x)(t-x)^2;x).$$

Considering Lemma 2.3, Eqn. (3) and Eqn. (4) we obtain

$$\lim_{s\to\infty} sv'(x)B_s^{\alpha,\beta}((t-x);x) = v'(x)\lim_{s\to\infty} sB_s^{\alpha,\beta}((t-x);x)$$
$$= v'(x)(-(2+\beta)x+\alpha+1)$$

and

$$\lim_{s\to\infty} s\frac{v''(x)}{2}B_s^{\alpha,\beta}((t-x)^2;x) = \frac{v''(x)}{2}\lim_{s\to\infty} sB_s^{\alpha,\beta}((t-x)^2;x)$$
$$= \frac{v''(x)}{2}(3x-3x^2).$$

Therefore, we have

$$\lim_{s\to\infty} s\left(B_s^{\alpha,\beta}(v;x) - v(x)\right) = (-(2+\beta)x+\alpha+1)v'(x) + \frac{3}{2}x(1-x)v''(x)$$
$$+\lim_{s\to\infty} sB_s^{\alpha,\beta}(\mathcal{R}(t,x)(t-x)^2;x). \qquad (22)$$

In order to complete the proof, we need to prove that

$$\lim_{s\to\infty} sB_s^{\alpha,\beta}(\mathcal{R}(t,x)(t-x)^2;x).$$

Using the Cauchy-Schwarz inequality for the remainder term, we obtain

$$sB_s^{\alpha,\beta}(\mathcal{R}(t,x)(t-x)^2;x) \le \sqrt{B_s^{\alpha,\beta}(\mathcal{R}^2(t,x);x)}\sqrt{s^2B_s^{\alpha,\beta}((t-x)^4;x)}. \qquad (23)$$

We already know the term $B_s^{\alpha,\beta}((t-x)^4;x)$ from Eqn. (2). Since $\mathcal{R}^2(.,x)$ is continuous at $t \in (0,1)$ and $\lim_{t\to x}\mathcal{R}(t,x) = 0$, we observe that

$$\lim_{s\to\infty} B_s^{\alpha,\beta}(\mathcal{R}^2(t,x);x) = \mathcal{R}^2(x,x) = 0. \qquad (24)$$

Hence, by using (2), (23), (24) and positivity of the linear operators $B_s^{\alpha,\beta}$, we have

$$\lim_{s\to\infty} sB_s^{\alpha,\beta}(\mathcal{R}(t,x)(t-x)^2;x) = 0. \qquad (25)$$

Finally, by substituting (25) in (22) we deduce

$$\lim_{s\to\infty} s\left(B_s^{\alpha,\beta}(v;x) - v(x)\right) = (-(2+\beta)x+\alpha+1)v'(x) + \tfrac{3}{2}x(1-x)v''(x)$$

as desired.

5. Graphical Analysis

In this section, we show the convergence of the newly constructed operators $B_s^{\alpha,\beta}$ with function v.
Let the the function v be

$$v(x) = \tan\left(\frac{x}{16}\right)\left(\frac{x}{8}\right)^2\left(1-\frac{x}{4}\right)^3.$$

We know that, the operators $B_s^{\alpha,\beta}$ have been given for the interval $(0,1)$. For this reason, we define the interval $[0+\epsilon, 1-\epsilon]$, where $\epsilon = 0.0001$.
Then for $\alpha = 1.01$ and $\beta = 1.02$, we have plotted the convergence of the B_s^* Bernstein operators [7] and newly constructed $B_s^{\alpha,\beta}$ Bernstein-Stancu operators to the function v in Fig. 1 for different s values.

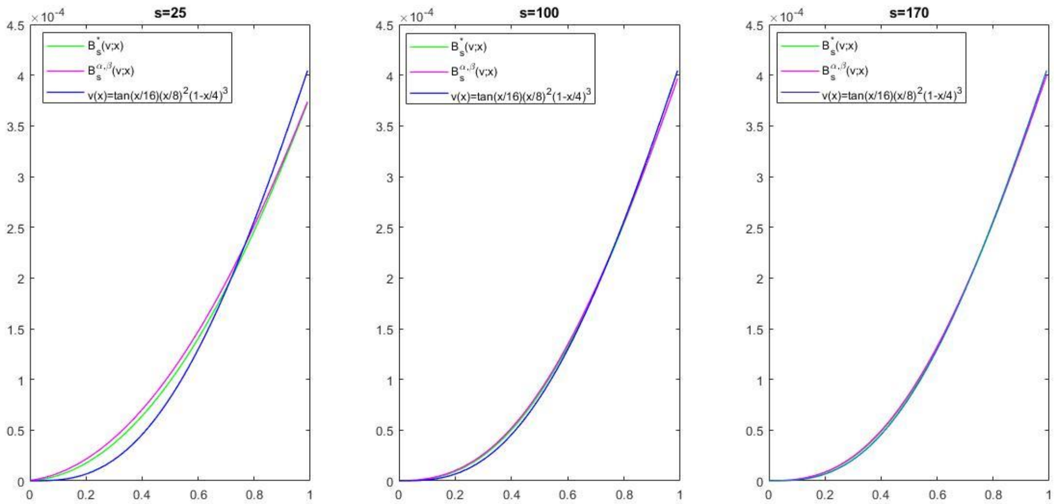

Figure 1. Convergence of $B_s^{\alpha,\beta}(v;x)$ and $B_s^*(v;x)$ to the function v(x) for different values of s with fixed $\alpha = 1.01$, $\beta = 1.02$.

The maximum error estimation for operators $B_s^{\alpha,\beta}$ and B_s^* to the function $v(x) = \tan\left(\frac{x}{16}\right)\left(\frac{x}{8}\right)^2\left(1-\frac{x}{4}\right)^3$ is presented in Table 1 for the interval $x \in [0+\epsilon, 1-\epsilon]$.

Table 1. Error estimation table

s	$\max_{x\in[0+\epsilon,1-\epsilon]} \lvert B_s^*(v;x) - v(x)\rvert$	$\max_{x\in[0+\epsilon,1-\epsilon]} \lvert B_s^{\alpha,\beta}(v;x) - v(x)\rvert$
25	$3.17794.\ 10^{-5}$	$3.05525.\ 10^{-5}$
75	$1.06654.\ 10^{-5}$	$1.05226.\ 10^{-5}$
100	$8.00549.\ 10^{-6}$	$7.92454.\ 10^{-6}$
170	$4.71371.\ 10^{-6}$	$4.68533.\ 10^{-6}$

For a second example, let the the function v be

$$v(x) = \frac{x^3}{3} - \frac{x^2}{10} + \frac{3x}{10}$$

and $[0+\epsilon, 0.8-\epsilon]$ where $\epsilon = 0.01$. Then for $\alpha = 0.01$ and $\beta = 0.1$, we have plotted the convergence of the $B_s^{\alpha,\beta}$ Bernstein-Stancu operators to the function v in Fig. 2 for $s = 170$.

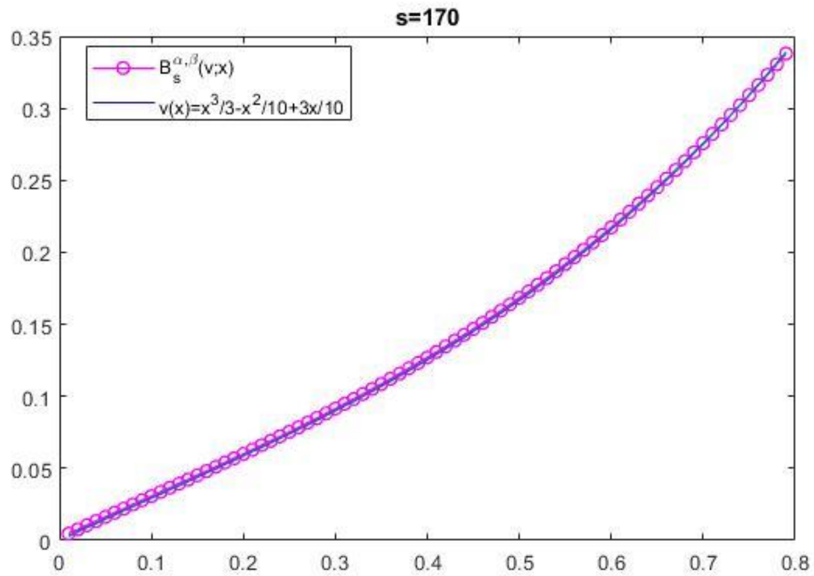

Figure 2. Convergence of $B_s^{\alpha,\beta}(v;x)$ for $s = 170$ with fixed $\alpha = 0.01$, $\beta = 0.1$.

The error estimation for operators $B_s^{\alpha,\beta}$ to the function $v(x) = \frac{x^3}{3} - \frac{x^2}{10} + \frac{3x}{10}$ is presented in Table 2 at the points x=0.1, 0.2, 0.4, 0.6 and x=0.79. Additionally, maximum error for Bernstein-Stancu operators to the function v is calculated for $x \in [0 + \epsilon, 0.8 - \epsilon]$, $\epsilon = 0.01$. .

Table 2. Error estimation table

s	Error for x=0.1	Error for x=0.2	Error for x=0.4	Error for x=0.6	Error for x=0.79	$\max_{x\in[0+\epsilon,0.8-\epsilon]} \lvert B_s^{\alpha,\beta}(v;x) - v(x)\rvert$
25	0.00962	0.00944	0.01114	0.00827	0.00663	0.01157
75	0.00313	0.00305	0.00373	0.00291	0.00210	0.00388
100	0.00234	0.00227	0.00280	0.00220	0.00156	0.00291
170	0.00137	0.00133	0.00164	0.00130	0.00091	0.00171

As we can see in Fig. 1 and Fig. 2, the newly defined Bernstein-Stancu operators present good approximation behaviour in function approximation.

6. Conclusion

In this paper, we investigated Stancu-type generalization of the new modification of Bernstein operators. We examined that the newly defined Bernstein-Stancu operators fix constant and preserve Korovkin's other test functions in limit case. Also, we improved many approximation properties such as the rate of approximation, Voronovskaya-type asymptotic formula. Furthermore, we demonstrated the theoretical results by using graphical representations.

Conflicts of interest

The authors declared there is no conflict of interest associated with this work.

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