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On the curvatures of the ruled surfaces of b-lift curves

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Abstract

In this study, we defined a new curve which is called the B-Lift curve, also obtained the Frenet vectors of the B-Lift curve. The ruled surfaces have been produced by taking base curves B-Lift curves. Moreover, the tangent, normal, and binormal surfaces of the B-Lift curve are introduced. Besides, the Darboux frame of these ruled surfaces is created. The characterizations of these ruled surfaces are also given and the cases of base curves as the asymptotic curve, geodesic curve, principal line are examined. Finally, examples of these ruled surfaces are given and we illustrate them.

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1. Introduction

In differential geometry, surfaces have important places and concepts in many disciplines such as kinematics, physics, computer science, etc. One of the most important of these surfaces is the ruled surface. A ruled surface is defined by a straight line that moves along a curve [1]. Many geometers have attracted the attention of ruled surfaces. T. Mert and M. Atçeken studied the ruled surfaces in de-Sitter and Hyperbolic 3-space [2-4]. E. Karaca and M. Çalışkan also studied the ruled surfaces generated by the natural lift curves on the Pseudo-Sphere [5]. E. Ergün and M. Çalışkan created ruled surfaces by accepting the base curve as a natural lift curve and they studied the integral invariants of these ruled surfaces [6]. The definition of the natural lift was given for the first time in Thorpe's book [7]. According to the definition, the natural lift curve is formed by combining the endpoints of the unit tangent vectors of the main curve.

S. Izumiya and N. Takeuchi examined the curves in terms of the geometry on the ruled surfaces. The principal normal surface of a space curve is defined as a ruled surface whose rules are given by the principal normal curve. They said that principal normal surfaces are by definition related to Bertrand curves and studied the general helices and Bertrand curves as curves on the ruled surfaces. General helices are a generalization of circular helices. In other words, the concept of Bertrand curves is a generalization of the concept of circular helices [8]. A curve is called a general helix if the tangent vector of the curve makes a constant angle with a fixed straight line. If both $\kappa \neq 0$ and $\tau > 0$ are constant, a curve is called a circular helix. M. A. Lancert proved that a curve is a general helix if and only if the ratio of curvatures along the curve is constant [9].

The Darboux frame along surface curves in 3-dimensional Euclidean space is a well-known concept. French mathematician Jean Gaston Darboux gave this field its name. It's an expansion of the Frenet-Serret frame applied to the surface geometry. Darboux frame is a natural moving frame built on a surface. Using this frame, the characteristics of the curve as a geodesic curve, asymptotic curve, or principal line can be given [10].

In this article, based on Thorpe's definition, we defined a new curve which is called the B-Lift curve, also calculated the Frenet vectors of this curve. Besides, introduced the ruled surfaces which are called tangent, principal normal, binormal surfaces by accepting the base curve as a B-Lift curve. After that, the Darboux frame of these surfaces was created and the situation of the geodesic curve, asymptotic curve, principal line of the B-Lift curve was examined.

2. Preliminaries

In this section, basic definitions and theorems for understanding this article are given.

Let $\vec{A} = (a_1, a_2, a_3)$ be a vector in $\mathbb{R}^3$. The norm is defined as $||\vec{A}|| = \sqrt{{a_1}^2 + {a_2}^2 + {a_3}^2}$. If $||\vec{A}||$ is equal to 1, then $\vec{A}$ is called unit vector in $\mathbb{R}^3$. For the vectors

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$\vec{A} = (a_1, a_2, a_3)$ and $\vec{B} = (b_1, b_2, b_3)$ in $\mathbb{R}^3$, the inner product is defined as $< \vec{A}, \vec{B} >= a_1 b_1 + a_2 b_2 + a_3 b_3$. If $\gamma'(s) \neq 0$, $\gamma: I \to \mathbb{R}^3$ is called the regular curve, for all $s \in I$. If $\gamma'(s)$ is equal to 1 then the curve $\gamma: I \to \mathbb{R}^3$ is a unit speed curve [1].

Assume that γ is a unit speed curve. $\{T(s), N(s), B(s)\}$ is called Frenet-Serret frame of the curve $\gamma(s)$. $T(s) = \gamma'(s)$ is the unit tangent vector of the curve $\gamma(s)$. The unit principal normal and binormal vectors are defined by $N(s) = \frac{\gamma''(s)}{||\gamma''(s)||}$ and $B(s) = T(s)xN(s)$, respectively. Frenet-Serret formulas of the unit speed curve $\gamma(s)$ are given as follows:

$$T'(s) = \kappa(s)N(s)$$

$$N'(s) = -\kappa(s)T(s) + \tau(s)B(s)$$

$$B'(s) = -\tau(s)N(s)\,,$$

where $\kappa(s) = \gamma''(s)$ and $\tau(s) = -< B'(s)\,, N(s) >$ are the curvature and torsion of the curve $\gamma(s)$, respectively [6].

Definition 1 [1] Let $\gamma: I \to M$ be a unit speed curve, where $M \subset \mathbb{R}^3$ is a hypersurface. We called an integral curve to the curve γ if

$$\gamma'(s) = \mathrm{X}\big(\gamma(s)\big),$$

where X is a differentiable vector field on M.

Definition 2 [6] Let $\gamma: I \to M$ be a unit speed curve. The natural lift curve $\overline{\gamma}: I \to TM$ of the curve γ is defined as follows:

$$\overline{\gamma}(s) = \big(\gamma(s), \gamma'(s)\big) = \gamma'(s)|_{\gamma(s)}.$$

Then, we can write

$$\frac{d\overline{\gamma}(s)}{ds} = \frac{d}{ds}(\gamma'(s))|_{\gamma(s)} = D_{\gamma'(s)}\gamma'(s),$$

where D is Levi-Civita connection in $\mathbb{R}^3$.

Definition 3 [1] Let γ be a regular curve and w be a unit direction of a straight line in $\mathbb{R}^3$, then the ruled surface

$$\varphi(s, v) = \gamma(s) + vw(s).$$

Definition 4 [6] Let γ be a regular curve in $\mathbb{R}^3$ and the set $\{T(s), N(s), B(s)\}$ be the Frenet vectors of the curve γ. Then the tangent, principal normal and binormal surfaces of the curve γ are given in the following equalities:

$$\varphi_T(s, v) = \gamma(s) + vT(s)$$

$$\varphi_N(s, v) = \gamma(s) + vN(s)$$

$$\varphi_B(s, v) = \gamma(s) + vB(s)$$

For the unit normal vector of the ruled surface φ, we have $\frac{\varphi_s \times \varphi_v}{||\varphi_s \times \varphi_v||}$.

Definition 5 [1] Let M be a surface in $\mathbb{R}^3$ and γ be a curve on M. Then the set $\{T, V, U\}$ defines a frame which is called Darboux frame, where $T = \gamma'$, U is a unit normal vector of M and $V = U \times T$. Frenet-Serret formulas for the Darboux frame follow as:

$$\begin{pmatrix} T' \\ V' \\ U' \end{pmatrix} = \begin{pmatrix} 0 & k_g & k_n \\ -k_g & 0 & \tau_g \\ -k_n & -\tau_g & 0 \end{pmatrix} \begin{pmatrix} T \\ V \\ U \end{pmatrix},$$

where $k_g =< U \times T, T' >$, $k_n =< \gamma'', U >$ and $\tau_g = < T,\ U \times U' >$ are the geodesic curvature, normal curvature and geodesic torsion, respectively.

Definition 6 [1] Let a regular curve γ lie on a surface M. Then the followings are provided:

(1) γ is a geodesic curve if and only if $k_g = 0$.

(2) γ is an asymptotic curve if and only if $k_n = 0$.

(3) γ is a principal line if and only if $\tau_g = 0$.

3. Curvatures of the Ruled Surfaces of B-Lift Curves

In this part, defined the B-Lift curve and the Frenet vectors of the B-Lift curve are calculated. Moreover, the tangent, principal normal, binormal surfaces are constructed and the Darboux frames of these surfaces are created. Besides, the geodesic curve, asymptotic curve, principal line of the B-Lift curve are examined.

Definition 7 For any unit speed curve $\gamma: I \to M$, $\gamma_B: I \to TM$ is called the B-Lift curve of γ which provides the following equation:

$$\gamma_B(s) = \big(\gamma(s), B(s)\big) = B(s)|_{\gamma(s)}, \tag{1}$$

where B is binormal vector of the curve γ.

Theorem 8 Let γ_B be the B-Lift curve of a regular curve γ. Then the following equations are provided:

$T_B(s) = -N(s),$

$$N_B(s) = \frac{\kappa(s)}{||W(s)||}T(s) - \frac{\tau(s)}{||W(s)||}B(s), \qquad (2)$$

$$B_B(s) = \frac{\tau(s)}{||W(s)||}T(s) + \frac{\kappa(s)}{||W(s)||}B(s),$$

where $\{T(s), N(s), B(s)\}$ and $\{T_B(s), N_B(s), B_B(s)\}$ are the Frenet vectors of the curve γ and γ_B, respectively. Furthermore, κ is the curvature, τ is the torsion and the Darboux vector $W = \tau T + \kappa B$ of the curve γ. (The torsion will be taken greater than zero.)

(i) Let γ_B be the B-Lift curve of the regular curve γ. Then the tangent surface of the B-Lift curve is

$$\varphi_{T_B}(s, v) = \gamma_B(s) + vT_B(s). \qquad (3)$$

From (1) and (2), we have

$$\varphi_{T_B}(s, v) = B(s) + v(-N(s)), \qquad (4)$$

$$(\varphi_{T_B})_s = -\tau N + v(\kappa T - \tau B), \qquad (5)$$

$$(\varphi_{T_B})_v = -N. \qquad (6)$$

The unit normal of the ruled surface φ_{T_B} is

$$U_{T_B} = \frac{(\varphi_{T_B})_s \times (\varphi_{T_B})_v}{||(\varphi_{T_B})_s \times (\varphi_{T_B})_v||} = \frac{(-v\tau, 0, -v\kappa)}{\sqrt{v^2\kappa^2 + v^2\tau^2}}. \qquad (7)$$

1) $k_g = < U_{T_B} \times T_B,\ {T_B}' >$

$= -v(\kappa^2 + \tau^2) \neq 0.$

2) $k_n = < {\gamma_B}'', U_{T_B} >$

$= -v(\kappa\kappa' + \tau\tau').$

3) $\tau_g =< T_B,\ U_{T_B} \times {U_{T_B}}' >$

$= v^2 (\frac{\tau}{\kappa})' \kappa^2.$

Corollary 9 For the regular curve γ, $(\kappa\kappa' + \tau\tau') = 0$ if and only if γ_B is the asymptotic curve of the ruled surface φ_{T_B}.

Corollary 10 γ is a general helix curve if and only if γ_B is a principal line of the ruled surface φ_{T_B}.

(ii) Let γ_B be the B-Lift curve of the regular curve γ. Then the principal normal surface of the B-Lift curve is

$$\varphi_{N_B}(s, v) = \gamma_B(s) + vN_B(s). \qquad (8)$$

From (1) and (2), the following equations are hold:

$$\varphi_{N_B}(s, v) = B(s) + v(\frac{\kappa(s)}{||W(s)||}T(s) - \frac{\tau(s)}{||W(s)||}B(s) \qquad (9)$$

$$(\varphi_{N_B})_s = -\tau N + v(\frac{\kappa'}{||W||}T + \frac{\kappa^2 + \tau^2}{||W||}N - \frac{\tau'}{||W||}B), \qquad (10)$$

$$(\varphi_{N_B})_v = \frac{\kappa}{||W||}T - \frac{\tau}{||W||}B. \qquad (11)$$

$$(\varphi_{N_B})_s \times (\varphi_{N_B})_v = (-\tau + \frac{\tau^2}{||W||}, \frac{v}{||W||^2}(\kappa'\tau - \kappa\tau'), -\kappa + \frac{\kappa\tau}{||W||}) \qquad (12)$$

The unit normal of the ruled surface φ_{N_B} is

$$U_{N_B} = \frac{(\varphi_{N_B})_s \times (\varphi_{N_B})_v}{||(\varphi_{N_B})_s \times (\varphi_{N_B})_v||}. \qquad (13)$$

1) $k_g = < U_{N_B} \times T_B,\ {T_B}' >$

$= \frac{(\kappa^2 + \tau^2)(\tau - 1)}{||W||}.$

2) $k_n = < {\gamma_B}'', U_{N_B} >$

$= -v \frac{\tau'(\kappa'\tau - \kappa\tau')}{||W||^2}.$

3) $\tau_g =< T_B,\ U_{N_B} \times {U_{N_B}}' >$

$= (\tau'\kappa - \kappa'\tau).[1 + \frac{\tau^2}{||W||^2} - \frac{2\tau}{||W||}].$

Corollary 11 For the regular curve γ, $\tau = 1$ if and only if γ_B is the geodesic curve of the ruled surface φ_{N_B}.

Corollary 12 γ is a general helix curve if and only if γ_B is an asymptotic curve of the ruled surface φ_{N_B}.

Corollary 13 γ is a general helix curve if and only if γ_B is a principal line of the ruled surface φ_{N_B}.

(iii) Let γ_B be the B-Lift curve of the regular curve γ. Then the binormal surface of the B-Lift curve is

$$\varphi_{B_B}(s, v) = \gamma_B(s) + vB_B(s). \qquad (14)$$

From (1) and (2), we know

$$\varphi_{B_B}(s, v) = B(s) + v(\frac{\tau(s)}{||W(s)||}T(s) + \frac{\kappa(s)}{||W(s)||}B(s)) \qquad (15)$$

$$(\varphi_{B_B})_s = -\tau N + v(\frac{\tau'}{||W||}T + \frac{\kappa'}{||W||}B), \qquad (16)$$

$$(\varphi_{B_B})_v = \frac{\tau}{||W||}T + \frac{\kappa}{||W||}B. \qquad (17)$$

$$(\varphi_{B_B})_s \times (\varphi_{B_B})_v = \left(-\frac{\kappa\tau}{||W||}, -v\frac{(\kappa'\tau-\kappa\tau')}{||W||^2}, \frac{\tau^2}{||W||}\right). \quad (18)$$

The unit normal of the ruled surface φ_{B_B} is

$$U_{B_B} = \frac{(\varphi_{B_B})_s \times (\varphi_{B_B})_v}{||(\varphi_{B_B})_s \times (\varphi_{B_B})_v||}. \quad (13)$$

1) $k_g = < U_{B_B} \times T_B, {T_B}' >$

$= 0.$

2) $k_n = < {\gamma_B}'', U_{B_B} >$

$$= -\frac{\tau^2}{||W||}(\kappa^2+\tau^2) - v\frac{\tau'}{||W||^2}(\kappa'\tau-\kappa\tau').$$

3) $\tau_g =< T_B, U_{B_B} \times {U_{B_B}}' >$

$$= \frac{\tau^2(\kappa\tau'-\kappa'\tau)}{||W||^2}.$$

Corollary 14 γ_B is the geodesic curve of the ruled surface φ_{B_B}.

Corollary 15 γ is a general helix curve if and only if γ_B is a principal line of the ruled surface φ_{B_B}.

Example 16 Consider the general helix curve given by

$$\gamma(s) = \left(\frac{1}{3}s^{\frac{3}{2}}, \frac{1}{3}(1-s)^{\frac{3}{2}}, \frac{\sqrt{3}}{2}s\right).$$

Frenet vectors of the curve $\gamma(s)$ are given as follows:

$$T(s) = \left(\frac{1}{2}s^{\frac{1}{2}}, -\frac{1}{2}(1-s)^{\frac{1}{2}}, \frac{\sqrt{3}}{2}\right)$$

$$N(s) = \left((1-s)^{\frac{1}{2}}, s^{\frac{1}{2}}, 0\right)$$

$$B(s) = \left(-\frac{\sqrt{3}}{2}s^{\frac{1}{2}}, \frac{\sqrt{3}}{2}(1-s)^{\frac{1}{2}}, \frac{1}{2}\right).$$

Then the B-Lift curve of the curve $\gamma(s)$ is

$$\gamma_B(s) = \left(-\frac{\sqrt{3}}{2}s^{\frac{1}{2}}, \frac{\sqrt{3}}{2}(1-s)^{\frac{1}{2}}, \frac{1}{2}\right).$$

Frenet vectors of the curve γ_B are obtained as follows:

$$T_B(s) = \left(-(1-s)^{\frac{1}{2}}, -s^{\frac{1}{2}}, 0\right)$$

$$N_B(s) = \left(s^{\frac{1}{2}}, -(1-s)^{\frac{1}{2}}, 0\right)$$

$$B_B(s) = (0,0,1).$$

The tangent, principal normal and binormal surfaces of the curve γ_B are

$$\varphi_{T_B}(s,v) = \gamma_B(s) + vT_B(s)$$
$$= \left(-\frac{\sqrt{3}}{2}s^{\frac{1}{2}}, \frac{\sqrt{3}}{2}(1-s)^{\frac{1}{2}}, \frac{1}{2}\right) + v\left(-(1-s)^{\frac{1}{2}}, -s^{\frac{1}{2}}, 0\right)$$

$$\varphi_{N_B}(s,v) = \gamma_B(s) + vN_B(s)$$
$$= \left(-\frac{\sqrt{3}}{2}s^{\frac{1}{2}}, \frac{\sqrt{3}}{2}(1-s)^{\frac{1}{2}}, \frac{1}{2}\right) + v\left(s^{\frac{1}{2}}, -(1-s)^{\frac{1}{2}}, 0\right)$$

$$\varphi_{B_B}(s,v) = \gamma_B(s) + vB_B(s)$$
$$= \left(-\frac{\sqrt{3}}{2}s^{\frac{1}{2}}, \frac{\sqrt{3}}{2}(1-s)^{\frac{1}{2}}, \frac{1}{2}\right) + v(0,0,1).$$

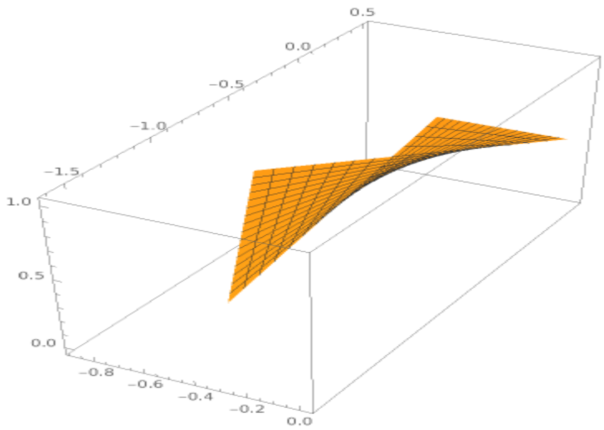

Figure 1. The tangent surface φ_{T_B} of the curve γ_B.

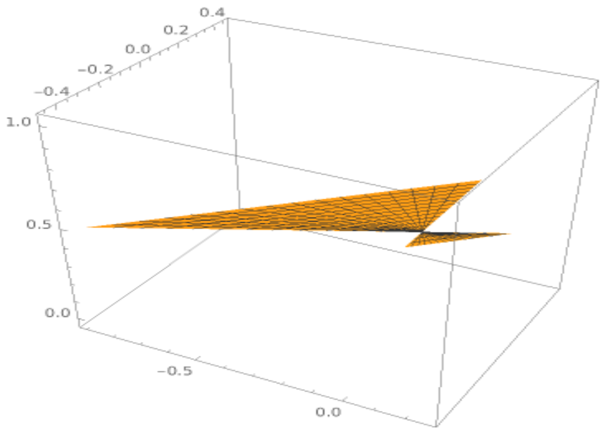

Figure 2. The principal normal surface φ_{N_B} of the curve γ_B.

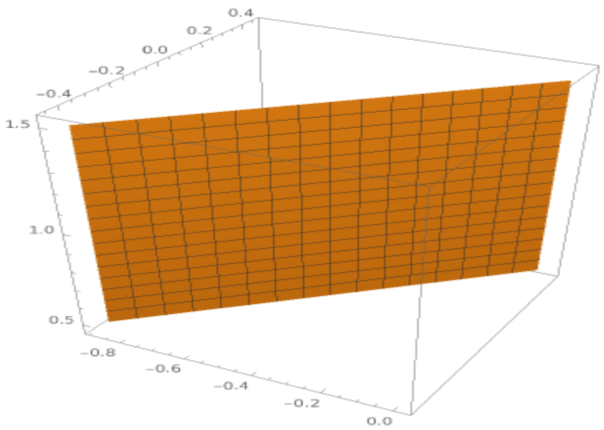

Figure 3. The binormal surface φ_{B_B} of the curve γ_B.

(i) For the ruled surface φ_{T_B}, since γ_B is the general helix, from Corollary 10, γ_B is a principal line of the ruled surface φ_{T_B}.

(ii) For the ruled surface φ_{N_B}, since γ_B is the general helix, from Corollary 12-13, γ_B are an asymptotic curve and principal line of the ruled surface φ_{N_B}.

(iii) For the ruled surface φ_{B_B}, from Corollary 14, γ_B is a geodesic curve of the ruled surface φ_{B_B} and since γ_B is the general helix, from Corollary 15, γ_B is a principal line of the ruled surface φ_{B_B}.

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Conflicts of interest

The authors state that did not have conflict of interests.

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