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A Study On the Kernels of Irreducible Characters of Finite Groups

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ABSTRACT

Let G be a finite group and $\chi \in Irr(G)$, where $Irr(G)$ denotes the set of all irreducible characters of G. The kernel of χ is defined by $ker(\chi) = \{ g \in G \mid \chi(g) = \chi(1)\}$, where $\chi(1)$ is the character degree of χ. The irreducible character χ of G is called as monolithic when the factor group $G/ker(\chi)$ has only one minimal normal subgroup. In this study, we have proven some results by concentrating on the kernels of nonlinear irreducible characters of G. First, we have provided an alternative proof for the classification of finite groups possessing two nonlinear irreducible characters by using their kernels. Also, we have presented the structure the solvable group G in which every nonlinear monolithic characters has same kernel

Keywords: Finite groups, Monolithic characters, Character degrees, Kernels of irreducible characters.

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Introduction

Let G be a finite group. Since past, many studies have been carried out to investigate the structure of G whose character degrees have some special properties. For example, some authors have found all finite groups in which nonlinear irreducible characters have same degree (see Chapter 12 of [1]). On the other hand, the structure of the group G which has certain conditions on the number of its irreducible characters has been considered in much of the studies. Then in some cases the structure of the group has been fully presented. For example, Seitz has shown in [2] that if G has one nonlinear irreducible character, then $G \cong ES(2^{2a+1})$, extraspecial 2-group, or $G \cong N \rtimes K$ is Frobenius group, where N is an elementary abelian group of order q^a and the complement K is a cyclic group of order $q^a - 1$ for some prime number q. Throughout this paper, a finite group having one nonlinear irreducible character is called as a Seitz group.

Materials and Methods

Before proving our main results, we should mention some definitions and notations in the character theory of finite groups for the convenience of the reader. Our notations are standard and taken mainly from [1]. From now on, all groups are considered as finite.

Definition 2.1. The $\mathbb{C}$-representation of the group G is a homomorphism $\psi: G \longrightarrow GL(n, \mathbb{C})$ for some integer n, where $\mathbb{C}$ is the field of complex numbers and $GL(n, \mathbb{C})$ is the multiplicative group of non-singular $n \times n$ matrices over $\mathbb{C}$.

Definition 2.2. If ψ is a $\mathbb{C}$-representation of a group G, then the $\mathbb{C}$-character χ of G is the function such that $\chi(g) = tr(\psi(g))$ for all $g \in G$,where $tr(\psi(g))$ is the sum of the diagonal entries of $\psi(g)$. A character corresponding to an irreducible representation of G is said to be irreducible.

The set of all irreducible characters and all nonlinear irreducible characters of G is denoted by $Irr(G)$ and $Irr_1(G)$, respectively. Also, $\chi(1)$ is called the degree of χ and χ is said to be a linear character when $\chi(1) = 1$. In the character theory, it is well-known that

$$|G| = |G: G'| + \sum_{\substack{\chi \epsilon Irr(G) \\ \chi(1)>1}} \chi(1)^2$$

and the number of linear irreducible characters of G is equal to $|G: G'|$, where G' is the commutator subgroup of G.

Definition 2.3. Let ψ be an irreducible character of the group G. The kernel of ψ is given by $ker(\psi) = \{ g \in G \mid \psi(g) = \psi(1) \}$ and if $ker(\psi) = 1$, then we say that ψ is faithful.

The restriction of a character χ to a subgroup H of G is a character of H, which is denoted by χ_H. Conversely, an irreducible character μ of H determines the character μ^G of G (see Definition 5.1 of [1]). If $\mu^G = \theta$ for some character θ of G, then we have that $|G: H|\mu(1) = \theta(1)$.

Definition 2.4. For a group G and $\chi \in Irr(G)$, if $G/ker(\chi)$ has a unique minimal normal subgroup, then χ is said to be a monolithic character.

We use the notations $Irr_m(G)$ and $Irr_{1,m}(G)$ to denote all monolithic characters and all nonlinear monolithic characters of G, respectively. Monolithic characters contain some fundamental information in determining the structure of the group. For example, it is known that the group G is abelian if and only if all monolithic characters of G are linear. We want to note that if μ is an irreducible character of a p-group, then μ

must be monolithic. Also, $Irr_m(G/N) \subseteq Irr_m(G)$ for all $N \trianglelefteq G$.

Remark 2.5. For a solvable group G, we note by Lemma 2 of [5] that $D_m := \bigcap_{i=1}^{n} ker(\chi_i) \leq Z(G)$, where χ_i ranges over all nonlinear monolithic characters of G and $D_m \cap G' = 1$.

Remark 2.6. For a solvable group G, if $\varphi \in Irr_1(G)$ has a maximal kernel among the nonlinear irreducible characters of G, then φ is monolithic by Lemma 12.3 of [1]. Also, we may deduce that every kernel of nonlinear irreducible characters of G is a subgroup of the kernel of a nonlinear monolithic character of G. If $N \trianglelefteq G$ and G/N is nonabelian, then we have $N \leq ker(\chi)$ for some $\chi \in Irr_1(G)$.

Definition 2.7. Let G be a group. The regular character of G is given by

$$\rho(x) = \begin{cases} |G|, & x = 1 \\ 0, & 1 \neq x \in G \end{cases}.$$

By Lemma 2.11 of [1], we conclude that $ker(\rho) = 1$. Here, it is convenient to note the following lemma since we will frequently use it for proving our results.

Lemma 2.8. If G is a nonabelian group and $T = \bigcap \{ker(\varphi) \mid \varphi \in Irr_1(G)\}$, then T is trivial.

Proof. Suppose that $T > 1$. By considering regular character of G, we have that $T \cap G' = 1$. Then we may write

$$|G| = |G:G'| + \sum_{\substack{\chi \epsilon Irr(G) \\ \chi(1)>1}} \chi(1)^2$$

and

$$|G/T| = |G:TG'| + \sum_{\substack{\chi \epsilon Irr(G) \\ \chi(1)>1}} \chi(1)^2.$$

Subracting $|G/T|$ from $|G|$, we get that

$$|G|\left(\frac{|T|-1}{|T|}\right) = |G|\left(\frac{1}{|G'|} - \frac{1}{|G'T|}\right).$$

By using $|G'T| = \frac{|G'||T|}{|G' \cap T|}$ and simplifying above equation, we get the contradiction that $|G'|$=1. This contradiction shows that $T = 1$

Lemma 2.9. Let $G = H \times A$ be a direct product group, where A is abelian and $(|H|, |A|) = 1$. If $\chi \in Irr(H)$ and $1 \neq \xi \in Irr(A)$, then $ker_G(\chi\xi) = ker_H(\chi) \times ker_A(\xi)$.

Proof. By Theorem 4.21 of [1], we know that the character $\chi\xi$ is an irreducible character of G. Let $g \in ker_H(\chi) \times ker_A(\xi)$. Then we may write $g = ha$ for $h \in ker_H(\chi)$ and $a \in ker_A(\xi)$. This implies that

$$(\chi\xi)(g) = \chi(h)\xi(a) = \chi(1)\xi(1) = (\chi\xi)(1).$$

Thus, we obtain that $g \in ker_G(\chi\xi)$, and hence $ker_H(\chi) \times ker_A(\xi) \leq ker_G(\chi\xi)$.

Now, assume that $g \in ker_G(\chi\xi)$. Therefore, we have $(\chi\xi)(g) = \chi(h)\xi(a) = \chi(1)\xi(1)$, and so $|\chi(h)| = \chi(1)$ because ξ is a linear character of the abelian group A. Thus, $\chi(h) = u\,\chi(1)$ for a complex number u with $|u| = 1$. On the other hand, from Lemma 2.15 of [1], u is the $|H|$th root of the unity. This yields that

$$\chi(1) = \chi(h)\xi(a) = u\,\chi(1)\xi(a),$$

and hence $u\,\xi(a) = 1$. Since $\xi(a)$ is the $|A|$th root of the unity and $(|H|, |A|) = 1$, we obtain that $u = 1$ and $\xi(a) = 1$. This gives us $\chi(h) = \chi(1)$ and $\xi(a) = \xi(1)$, which shows that $g \in ker_H(\chi) \times ker_A(\xi)$. This completes the proof.

Main Results

Finite groups having two nonlinear irreducible characters have been described by Berkovich in Theorem 6 of Chapter 31 of [4]. Berkovich's proof is based on the degrees of these irreducible characters. Here, we consider the relationship between the kernels of irreducible characters and the group structure. By investigating the kernels of these two nonlinear irreducible characters, we provide an alternate proof of Berkovich's theorem as follows:

Theorem 3.1. Let $Irr_1(G) = \{\chi, \theta\}$ for the group G. Then one of the following remains true:

(1) If $ker(\theta) = ker(\chi) = 1$, then one of the following holds:

(a) $|G| = 2^{2c}$ and cyclic center $Z(G) \geq G'$ with $|G'| = 2$ and $|Z(G)| = 4$.

(b) $G \cong ES(3^{2a+1})$.

(c) $G \cong G' \rtimes K$ is a Frobenius group, where K is the cyclic Frobenius complement with $2|K| = |G'| - 1$.

(2) If $ker(\theta) = 1$ and $ker(\chi) > 1$, then G satisfies one of the following :

(d) $G \cong (C_3 \times C_3) \rtimes Q_8$ is a Frobenius group with the Frobenius complement isomorphic to Q_8.

(e) $G/Z(G) \cong U \rtimes V$ is a Frobenius group possessing elementary abelian kernel U with a cyclic complement V. In this case, we also have $|Z(G)| = 2$, $|V| = |U| - 1$ and $Z(G) \cap G' = 1$.

(3) $ker(\theta) > 1$ and $ker(\chi) > 1$ if and only if $|G| = 2^{2\alpha+2}$, $Z(G) \cong V_4$, $G' \leq Z(G)$ and $|G'|$=2.

Proof. By Theorem 12.15 of [1], we know that G is solvable. Suppose that $ker(\theta) = ker(\chi)$. By Lemma 2.8, we have $ker(\theta) = ker(\chi) = 1$. Thus, G' becomes the unique minimal normal subgroup of G. It can be seen by Lemma 12.3 of [1] that $\chi(1) = \theta(1)$. Furthermore, G becomes a r-group and $G/Z(G)$ is elementary abelian of order $\theta(1)^2$ or a Frobenius group. Clearly, when G is a r-group, we deduce $|G'| = r$ since $G' \leq Z(G)$ and $Z(G)$ is cyclic. Consequently, we have the equation that

$$|G| = |G:G'| + \theta(1)^2 + \chi(1)^2 = \frac{|G|}{r} + 2r^{2a}$$

for some positive integer a. Computation yields that $(r-1)|G| = 2r^{2a+1}$ and this equality holds only when $r = 2$ or $r = 3$. Hence the cases (a) and (b) hold since the order of $G/Z(G)$ is r^{2a}. Now let G be a Frobenius group as in Lemma 12.3 of [1]. Thus, G' is the Frobenius kernel of G. On the other side, the Frobenius complement H becomes a cyclic group having the property that $2|H| = |G'| - 1$ since $|H| = \theta(1) = \chi(1)$. We see that G has the desired property in the case (c).

To prove the case (2), suppose that $ker(\theta) = 1$ and $ker(\chi) > 1$. Obviously, $G/ker(\chi)$ is a Seitz group. We also know from Lemma 2.8 that $ker(\chi)$ must be a minimal normal subgroup of G. Thus, we have that $|ker(\chi)| = r^a$ for some prime r and integer a. Now first, let us consider $G/ker(\chi)$ an extraspecial 2-group. Assume that $ker(\chi) \nleq G'$. Since $ker(\chi)$ is a minimal normal subgroup of G, we know that $ker(\chi) \cap G' = 1$. Thus G' is also a minimal normal subgroup of G. Hence $2 = |G'ker(\chi)/ker(\chi)| = |G'|$, and this implies $G' \leq Z(G)$. Therefore, G is nilpotent. It follows that r must be equal to 2. Otherwise, we obtain that $|Irr_1(G)| > 2$ by using the fact that $|Irr(G)| = |Irr(P)|.|Irr(ker(\chi))|$, where P is the Sylow 2-subgroup of G. Therefore, G is a 2-group and $|ker(\chi)| = 2$. We have a contradiction that $G' = ker(\chi)$ since $Z(G)$ is cyclic. This contradiction implies that $ker(\chi) \leq G'$ and also $ker(\chi)$ is the unique minimal normal subgroup of G. By using the equations

$$|G| = |G:G'| + \theta(1)^2 + \chi(1)^2 \qquad (1)$$

and $|G/ker(\chi)| = |G:G'| + \chi(1)^2$, we get that

$$\theta(1)^2 = |G|\left(1 - \frac{1}{|ker(\chi)|}\right) = 2^{2b+1}(r^a - 1), \qquad (2)$$

where $|G/ker(\chi)| = 2^{2b+1}$ for some integer b. Since $ker(\chi)$ is abelian group, we know from Theorem 6.15 of [1] (Ito Theorem) that $\theta(1)$ divides $|G/ker(\chi)| = 2^{2b+1}$. Thus, $r^a - 1 = 2^\beta$ for some odd integer β. Thus, we get that $r \neq 2$. It follows that $Z(G) = 1$. Otherwise, G must be nilpotent since $ker(\chi) \leq Z(G)$. But this is a contradiction since we obtain that $|Irr_1(G)| > 2$ by using the equation $|Irr(G)| = |Irr(P)|.|Irr(ker(\chi))|$, where P is the Sylow 2-subgroup of G. Now we determine possible values of r. If $2 \mid a$, then we see that the only possibility is that $r^a - 1 = 8$ because $r^a - 1 = (r^{a/2} - 1)(r^{a/2} + 1) = 2^\beta$ and r is an odd prime. Thus, $ker(\chi) \cong C_3^2$ elementary abelian and also, $ker(\chi) = F(G)$ because $F(G)$ is a r-group. Let $K \in Syl_2(G)$. Since the centralizer $C_G(ker(\chi)) = ker(\chi)$, we conclude that K is isomorphic to a subgroup of the group $Aut(C_3 \times C_3)$ Therefore, we get $G \cong (C_3 \times C_3) \rtimes Q_8$. Now consider the case that $2 \nmid a$. If $a > 1$, then we see that there exists an odd prime which would have to divide

$$(r-1)(r^{a-1} + \ldots + r + 1) = r^a - 1 = 2^\beta,$$

and hence there clearly is no such prime r. Thus, $a = 1$ and we have that $r - 1 = 2^\beta$ for some odd number β. The only possibility is that $r = 3$ and we obtain that $ker(\chi)$ is a cyclic group of order 3. This is a contradiction because $Z(G) = 1$. Therefore, we need to consider that $G/ker(\chi)$ is a Seitz Frobenius group possessing $H/ker(\chi)$ Frobenius complement with the kernel $G'ker(\chi)/ker(\chi)$. Next, we claim that $ker(\chi) \nleq G'$. If not, $ker(\chi)$ becomes the unique minimal normal subgroup and so $|ker(\chi)| = r^a$ for some prime r and integer a. By using Equation (1) and $|G/ker(\chi)| = |G:G'| + \chi(1)^2$, we know that

$$\theta(1)^2 = q^c(q^c - 1)(r^a - 1), \qquad (3)$$

where $|G'/ker(\chi)| = q^c$ and $|H/ker(\chi)| = q^c - 1$. Now, we can consider the two cases that $F(G) = ker(\chi)$ or $F(G) = G'$ since $ker(\chi) \leq F(G) \leq G'$ and $G'/ker(\chi)$ is the unique minimal normal subgroup of $G/ker(\chi)$. Let $F(G) = ker(\chi)$. We now have $r \neq q$ and G' is not an abelian group. It follows that $G'' = ker(\chi)$ and so $ker(\chi) \nleq ker(\lambda)$ for every $\lambda \in Irr_1(G')$. Therefore, $ker(\chi) \nleq ker(\lambda^G)$ and by Clifford's Theorem (see Theorem 6.2 of [1]) we conclude that $\lambda^G = \theta$ since θ is the unique faithful irreducible character of G and G/G' is a cyclic group. Because the fact that $F(G') = ker(\chi)$ and $G'/F(G')$ is an abelian group, from Lemma 18.1 of [3], there must be an irreducible character φ of G' satisfying the property that $\varphi(1) = |G':F(G')| = q^c$. Thus, we obtain that

$$\varphi^G(1) = |G:G'|\varphi(1) = (q^c - 1)q^c = \theta(1),$$

and hence by using Equation (3), we get $\theta(1) = q^c(q^c - 1) = (r^a - 1)$. By Clifford's Theorem, we have $\theta_{ker(\chi)} = \xi_1 + \xi_2 + \ldots + \xi_{r^a - 1}$, where $\xi_i \in Irr(ker(\chi))$ for $i \in \{1, \ldots, r^a - 1\}$. This leads by Clifford's Theorem to $I_G(\xi_i) = ker(\chi)$ for all nonprincipal irreducible characters ξ_i of $ker(\chi)$ because $|G:I_G(\xi_i)| = r^a - 1 = \theta(1)$. Thus, we have seen that there exists $K \leq G$ such that $G = ker(\chi) \rtimes K$ is a Frobenius group with $ker(\chi)$ being the Frobenius kernel. K has a unique involution since $|K| = q^c(q^c - 1)$ is even. This shows $Z(K) > 1$, which leads the contradiction as we know that $G/ker(\chi) \cong K$ is a Frobenius group.

Now, we may assume that $F(G) = G'$. We note that $r = q$ and $G' \in Syl_r(G)$ as $F(G)$ is a r-group. Thus we also know that G' is not an abelian group since $q \mid \theta(1)$. It follows that $ker(\chi) = Z(G') = G''$ since $G'/ker(\chi)$ is the unique minimal normal subgroup of $G/ker(\chi)$. By using the similar thought in the previous paragraph, we have $\psi^G = \theta$ for $\psi \in Irr_1(G')$. Therefore, $\theta(1) = r^t(r^c - 1)$, where $\psi(1) = r^t$ for some integer t. By using Equation (3), we get that

$$\theta(1)^2 = r^c(r^c - 1)(r^a - 1) = r^{2t}(r^c - 1)^2,$$

and by equating these expressions, we find that $a = c = 2t$. It follows that $|ker(\chi)| = r^c = |G'/ker(\chi)|$ and $\psi(1) = r^{c/2}$ for every nonlinear irreducible character ψ of G'. Since by Corollary 2.30 of [1] we have that $r^c = \psi(1)^2 \leq |G':Z(\psi)|$ and that $Z(G') \leq Z(\psi)$, we deduce that $Z(G') = Z(\psi)$. As $r^c = |G'/Z(G')|$, again by Corollary 2.30 of [1], for all $x \in G' - Z(G')$ we see that $\psi(x) = 0$.

Take $x \in G' - Z(G')$; then $\psi(x) = 0$ for all $\psi \in Irr_1(G')$. Then we have a contradiction that

$$\begin{aligned} r^c = |Z(G')| &< |C_{G'}(x)| \\ &= \textstyle\sum_{\psi \in Irr_1(G')} |\psi(x)|^2 + \sum_{\mu \in Lin(G')} |\mu(x)|^2 = r^c \end{aligned}$$

This proves our claim $ker(\chi) \nleq G'$. Thus, $ker(\chi)$ and G' are different minimal normal subgroups of G. By Ito's Theorem, $\theta(1) \mid q^c - 1$ since $G' \times ker(\chi) \trianglelefteq G$ is abelian, and hence it may be written the degree of θ as $\theta(1) = \frac{q^c - 1}{t}$ for some t. By using Equation (1),

$$q^c(q^c-1)r^a = |G| \\ = (q^c-1)r^a + (q^c-1)^2 + (q^c-1/t)^2$$

and this calculation shows that $r^a = 1 + \frac{1}{t^2}$, which is possible only if $r^a = 2$. This implies that $|ker(\chi)| = 2$. Actually, we have $ker(\chi) = Z(G)$, which yields that the case (b) holds.

Let us consider the case (3), that is, $ker(\chi) > 1$ and $ker(\theta) > 1$. Take $N := ker(\chi) \cap ker(\theta)$, then we know from Lemma 2.8 that $|N| = 1$ and both $ker(\chi)$ and $ker(\theta)$ are minimal normal subgroups of G. Since $G/ker(\chi)ker(\theta)$ is abelian, we have that $G' \leq ker(\chi)ker(\theta)$. Suppose that $G = ker(\chi)ker(\theta)$. By using Equation (1), we get that

$$|G/ker(\theta)| - |G:G'| = \theta(1)^2 = |G| - |G/ker(\chi)|$$

since $ker(\chi)$ and $ker(\theta)$ are subgroups of G'. Thus, we have $|ker(\theta)|(|ker(\chi)| - 1) = |ker(\chi)| - 1$, which leads the contradiction that $|ker(\theta)| = 1$. Therefore, we see that $G' < ker(\chi)ker(\theta)$. Suppose that $ker(\chi) \leq G'$, then we get $G'ker(\theta) = ker(\chi)ker(\theta)$, and so $|G' \cap ker(\theta)| > 1$. Thus, we have $ker(\theta) \leq G'$, which contradicts with $G' < ker(\chi)ker(\theta)$. Then $ker(\chi) \nleq G'$. Similarly, we obtain that $ker(\theta) \nleq G'$. Thus G' is the another minimal normal subgroup. Now, we claim that $G'ker(\theta) = G'ker(\chi) = ker(\chi)ker(\theta)$. It is easy to see that $ker(\theta) = ker(\theta) \cap G'ker(\chi) \leq G'ker(\chi)$ and $ker(\chi) = ker(\chi) \cap G'ker(\theta) \leq G'ker(\theta)$, which yields that $G'ker(\theta) = G'ker(\chi) = ker(\chi)ker(\theta)$, as desired. We also note that $|G'| = |ker(\theta)| = |ker(\chi)|$.

First, we assume that $G/ker(\chi)$ is a Seitz Frobenius group. Since $|G'ker(\chi)/ker(\chi)| = |G'ker(\theta)/ker(\theta)|$, we see that $G/ker(\theta)$ cannot be an extraspecial 2-group. Thus, $G/ker(\theta)$ is also a Seitz Frobenius group whose order is $r^n(r^n-1)$ for some prime r and $\theta(1) = \chi(1) = r^n - 1$. Because $G' \cong G'ker(\chi)/ker(\chi) \cong ker(\theta)$, we conclude that $r^n = |G'| = |ker(\chi)|$, and hence we get $|G| = r^{2n}(r^n-1)$. By using Equation (1), we have

$$r^{2n}(r^n-1) = r^n(r^n-1) + 2(r^n-1)^2,$$

which gives a contradiction that $1 = r^n - 1 = \chi(1)$. Therefore we need to consider the case that $G/ker(\chi)$ is an extraspecial 2-group. Since we have $2 = |G'ker(\chi)/ker(\chi)| = |G'ker(\theta)/ker(\theta)|$, we deduce that $G/ker(\theta)$ is also an extraspecial 2-group, and so we get that G is a 2-group. Thus, minimal normal subgroups $ker(\chi)$, $ker(\theta)$ and G' are also subgroups of $Z(G)$ of order 2. In fact, $ker(\chi)\,ker(\theta) = Z(G)$ because $G/ker(\chi)$ is an extraspecial 2-group. Therefore, we obtain the case (3) that $Z(G) \cong V_4$, $|G| = 2^{2n+2}$ and $\chi(1) = \theta(1) = 2^n$, where V_4 is the Klein-4-group, and hence we are done.

Theorem 3.2. Let $Irr_{1,m}(G) = \{\chi_1, \chi_2, \ldots, \chi_n\}$ for a nonabelian solvable group G. Assume that $ker(\chi_i) = ker(\chi_j)$ for $i, j \in \{1, 2, \ldots, n\}$. Then $\chi_1(1) = \cdots = \chi_n(1) = d$ for some integer d and one of the following holds:

(i) $G = S \times T$, where $S \in Syl_p(G)$ and $T \trianglelefteq G$ is abelian. Also, $S/Z(S)$ is an elementary abelian p-group and $Z(S)$ is cyclic.

(ii) $G/Z(G)$ is a Frobenius group possessing an abelian Frobenius complement whose order equals d, G' is an elementary abelian p-group and $G' \cap Z(G) = 1$. Also, for $R \in Syl_p(Z(G))$ we have $G = R \times K$. In fact, $K/Z(K)$ becomes a Frobenius group.

Proof. We first note that when every nonlinear monolithic character of G is faithful, then G' becomes the unique minimal normal subgroup of G from Remark 2.6, and hence we complete the proof by Lemma 12.3 of [1]. Therefore, we can suppose that $ker(\chi_i) \neq 1$ for $i \in \{1, 2, \ldots, n\}$. Since $1 < \bigcap_{i=1}^{n} ker(\chi_i) = ker(\chi_1)$, then by Remark 2.5, we have that $ker(\chi_1) \leq Z(G)$ and also $ker(\chi_1) \cap G' = 1$. Therefore, we conclude by Remark 2.6 that G' is a minimal normal subgroup of G. First let us consider $ker(\chi_1) < Z(G)$. Thus, we get that $G/Z(G)$ is abelian because it has no nonlinear monolithic character. So G becomes a nilpotent group. Because the fact that G is nonabelian and nilpotent, then there is a nonabelian Sylow p-subgroup $S \trianglelefteq G$ having the property with $G = S \times T$. Since $S' = G' \leq S$, then we observe that T is an abelian group. Furthermore, the factor group $G/ker(\chi_1)$ needs to be a nonabelian and p-group. This gives from Remark 2.6 that $T \leq ker(\chi_1)$. Since we know $Irr_{1,m}(G/T) = Irr_{1,m}(S)$, we obtain that S has exactly n nonlinear monolithic characters having same kernel. From Lemma 2.8, we deduce that every nonlinear irreducible characters of S needs to be faithful. Therefore, S is as in Lemma 12.3 (a) of [1], which is as desired result in (i).

From now on, we shall suppose that $ker(\chi_1) = Z(G)$. If the factor group $G/Z(G)$ is a p-group, G is nonabelian and nilpotent. Also, we note that $(|S|, |Z(G)|) \neq 1$, where S is a Sylow p-subgroup of G. Therefore, there exists $N < Z(G)$ subgroup with $G/N \cong S$. Since $Irr_{1,m}(S) \subseteq Irr_{1,m}(G)$ and $G/NZ(S) \cong S/Z(S)$, we have a contradiction that $n = |Irr_{1,m}(S/Z(S))| < |Irr_{1,m}(S)| \leq |Irr_{1,m}(G)| = n$.

Then $G/Z(G)$ becomes a Frobenius group as Lemma 12.3 (b) of [1], that is,

$$G/Z(G) \cong Z(G)G'/Z(G) \rtimes B/Z(G),$$

where $Z(G)G'/Z(G) \cong G'$ is an elementary abelian p-group. Let's pick $R \in Syl_p(Z(G))$. Because the fact that $Z(G) \cap G' = 1$, we get $G' \times R \in Syl_p(G)$. Since $G' \times R$ splits over the normal abelian group R, then we get by Gaschütz's Lemma that $G = R \times K$, where $K/Z(K)$ is a Frobenius group.

Conversely, the groups as in the theorem have nonlinear monolithic characters having equal kernel by Lemma 2.9, and hence we are done.

Conclusion

For many years, several authors have defined some new concepts and given theorems on the classifications of a finite group by using its irreducible characters. The aim of this paper is to consider the relation between the groups structure and their irreducible character kernels. Under some certain conditions related to irreducible character kernels, we have given a classification of finite groups. On above occassion, we want to emphasize that monolithic characters are important constituent of the set of irreducible characters. Therefore, this study may be considered as a pioneering work for classifying finite groups having more nonlinear monolithic character kernels.

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Conflicts of interest

The authors state that did not have conflict of interests.

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