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On the Associated Curves of a Frenet Curve in R_1^4

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ABSTRACT

In the present work, we have dealt with the properties of associated curves of a Frenet curve in R_1^4. In addition to this, we define principal direction curve, B_1 −direction curve, B_2 − direction curve of a given Frenet curve by using integral curves of 4-dimensional Minkowski space. Then we introduce some characterizations for general helix and slant helix. Finally, some new associated curves and theorems obtained for space-like curves and time- like curves in R_1^4. Also, an example is given.

Keywords: Frenet curve, Associated curve, Principal -direction curve, B_1 −direction curve, B_2 − direction curve.

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Introduction

One of the important and productive area of differential geometry is curve theory for many researchers. The special defined curves such as helices, slant helices, rectifying curves , Bertrand and Mannheim curve pairs are characterized by their curvatures , in many ways. Also, among these special defined curves an interesting one is associated curves obtained by the integral curves of the Frenet elements.

In general, we denominate these curves by the name of Frenet elements, ea. principal direction curve, binormal direction. The researchers focus on the subject from the different point of view and most of them based on the different dimension and different spaces because of the variety of the Frenet equations.

In [1] Babaarslan ,Tandoğan and Yaylı defined Bertrand curves and constant slope surfaces according to Darboux frame. Moreover, Bektaş, Ergüt and Öğrenmiş in [2] , mentioned a special curves of 4D Galilean space. Also, in [3] author introduced special helices on equiform differential geometry of time-like curves in E_1^4. In [4-5], authors defined associated curves on different spaces and researched their applications. Following the studies above the geometers introduce associated directional curves in various spaces in [6-9]. According to these studies, in [10] Sahiner obtained characterizations for quaternionic direction curve and some special dual direction in R^3. Due to the popularity of the special defined curves these are numerous works related to this subject in different aspects [11-12].

Inspired by the above studies , we have focused on the associated curves of a Frenet curve in R_1^4 which is another famous research area for mathematicians.

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Preliminaries

Let R_1^4 be 4-dimensional vector space endowed with the scalar product $<,>$ defined as

$$< x, y >= x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4 \qquad (1)$$

where (x_1, x_2, x_3, x_4) is a rectangular coordinate system in R_1^4
R_1^4 is 4-dimensional vector space equipped with the scalar product $< , >$ then R_1^4 is called Lorentzian 4-space or 4-dimensional Minkowski space. A vector $v \in R_1^4$ can have one of the three casual characters called space-like ($< v, v >> 0$ or $v = 0$), time-like ($< v, v >< 0$) and light-like(or null) ($< v, v >= 0$ and $v \neq 0$) Similarly, an arbitrary curve α $\alpha = \alpha(s)$ in R_1^4 is called space-like, time-like or light-like respectively.

If all of velocity vector $\alpha'(s)$ are space-like, time-like or light-like respectively. The norm of a vector $v \in {R_1}^4$ is given by $\|v\| = \sqrt{(| < v, v > |)}$. Therefore, v is a unit vector $< v, v > \pm 1$. A curve (space-like, or time-like) is parametrized by the arc length if $\alpha'(s)$ is unit vector for any s. Also, we say that the vectors $v, w \in R_1^4$ are orthogonal if $< v, w >= 0$. [10]

For any three vectors $a = (a_1, a_2, a_3, a_4), b = (b_1, b_2, b_3, b_4), c = (c_1, c_2, c_3, c_4) \in R_1^4$ the Lorentzian vector product is defined by

$$a \times b \times c = \begin{vmatrix} -e_1 & e_2 & e_3 & e_4 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix}$$

Here e_1, e_2, e_3 and e_4 are orthogonal vectors satisfying equations

$e_1 \wedge e_2 \wedge e_3 = e_4$, $e_2 \wedge e_3 \wedge e_4 = e_1$, $e_3 \wedge e_4 \wedge e_1 = e_2$, $e_4 \wedge e_1 \wedge e_2 = -e_3$. [12].
Let γ be a space-like curve in R_1^4 with the curvatures k_1, k_2, k_3.Then Frenet formulaes are given as follows

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ -k_1 & 0 & k_2 & 0 \\ 0 & -k_2 & 0 & k_3 \\ 0 & 0 & k_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix}$$

Let γ be a time-like curve in R_1^4 with the curvatures k_1, k_2, k_3. Then Frenet formulaes are given as follows:

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ k_1 & 0 & k_2 & 0 \\ 0 & -k_2 & 0 & k_3 \\ 0 & 0 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix}$$

For a detailed information we refer to [10].

Associated Curves of a Frenet Curve in ${R_1}^4$

In this section we have focused on the associated curves of a Frenet curve in ${R_1}^4$.

Definition 3.1. Let us consider an admissible γ Frenet curve $\{T, N, B_1, B_2\}$ with Frenet frame in R_1^4.

The integral curve of the principal normal vector field of γ is defined as principal direction curve of γ .

The integral curve of the first binormal vector field of γ is defined as B_1 −direction curve of γ .

The integral curve of the second binormal vector field of γ is defined B_2 −direction curve of γ .

Theorem 3.1. Let γ be a space-like curve whose curvatures are k_1, k_2, k_3 and $\bar{\gamma}$ be the principal direction curve of γ in R_1^4 .Then the curvatures of γ are as follows

$$\overline{k_1}(s) = |k_1 - k_2|$$
$$\overline{k_2}(s) = 0$$

Proof. Let $\{\bar{T}, \bar{N}, \overline{B_1}, \overline{B_2}, \overline{k_1}, \overline{k_2}, \overline{k_3}\}$ be the Frenet elements of $\bar{\gamma}$. We find from the definition (3.1.a) as, we can easily obtain ,

$$N(s)|_{\bar{\gamma}(s)} = \bar{\gamma}'(s) = \bar{T}(s)$$

Then,

$$\bar{N}(s) = \frac{\bar{\gamma}''(s)}{\|\bar{\gamma}(s)\|} = \frac{N'(s)}{\|N'(s)\|} = \frac{-k_1 T + k_2 B_1}{\|-k_1 T + k_2 B_1\|} = \frac{-k_1 T + k_2 B_1}{|k_1 - k_2|}$$

$$\overline{B_2} = \frac{\bar{\gamma}' \times \bar{\gamma}'' \times \bar{\gamma}'''}{\|\bar{\gamma}' \times \bar{\gamma}'' \times \bar{\gamma}'''\|} = \frac{k_1 B_1 + k_2 T}{k_1 + k_2}$$

and ,

$$\overline{B_1} = \overline{B_2} \times \bar{T} \times \bar{N} = \frac{k_1^2 + k_2^2}{|k_1 + k_2||k_1 - k_2|} B_2$$

Then the curvatures of $\bar{\gamma}$ are given by

$$\overline{k_1}(s) = \langle \bar{T}', \bar{N} \rangle = \frac{k_1^2 + k_2^2}{|k_1 - k_2|}$$
$$\overline{k_2}(s) = \langle \bar{N}', \overline{B_1} \rangle = 0$$

Theorem 3.2. Let γ be a space-like curve whose curvatures are k_1, k_2, k_3 and γ be the B_1 − direction curve of γ in R_1^4, the curvatures of γ are as follows

$$\overline{k_1}(s) = \sqrt{k_2^2 - k_3^2}$$
$$\overline{k_2}(s) = -\frac{k_1 k_2 (k_2^2 + k_3^2)}{(k_3^2 - k_2^2)^{\frac{3}{2}}}$$

Proof. The proof is similar to the proof of Theorem 3.1 , so it is omitted.

Theorem 3.3. Let γ be a space-like curve whose curvatures are k_1, k_2, k_3 and $\bar{\gamma}$ be the B_2 − direction curve of γ, in R_1^4, the curvatures of $\bar{\gamma}$ are as follows

$$\overline{k_1}(s) = k_3\, sgn(k_3)$$
$$\overline{k_2}(s) = k_2 sgn(k_3)$$

Proof. Let $\{\bar{T}, \bar{N}, \overline{B_1}, \overline{B_2}, \overline{k_1}, \overline{k_2}, \overline{k_3}\}$ be the Frenet elements of $\bar{\gamma}$. We find from the definition (3.1.c) , we get,

$$B_2(s) = \bar{\gamma}'(s) = \bar{T}(s)$$

Using the Frenet vector fields, we find,

$$\bar{N}(s) = sgn(k_3) B_1$$
$$\overline{B_2}(s) = T$$
$$\overline{B_1} = -sgn(k_3) N$$

Then, the curvatures of $\bar{\gamma}$ are given by

$$\overline{k_1}(s) = k_3 sgn(k_3)$$
$$\overline{k_2}(s) = k_2 sgn(k_3)$$

Theorem 3.4. Let γ be a space-like curve in R_1^4 and $\bar{\gamma}$ be the principal direction curve of γ is a slant helix ⇔ $\bar{\gamma}$ is a general helix.

Proof. Let $\{T, N, B_1, B_2\}$ be the Frenet frame of γ. From the definition (3.1.a) , we can write,

$$N(s)|_{\bar{\gamma}(s)} = \bar{\gamma}'(s) = \bar{T}(s)$$

thus,

γ is a slant helix ⇔ $\langle N, u \rangle = c$ here u is a constant vector and $c = const.$

⇔ $\langle \bar{T}, u \rangle = c$
⇔ γ is a general helix.

Theorem 3.5. Let γ be a space-like curve in R_1^4 and $\bar{\gamma}$ be the B_2 − direction of γ. Then γ is a B_2 − slant helix ⇔ $\bar{\gamma}$ is a general helix.

Proof: Let $\{T, N, B_1, B_2\}$ be the Frenet frame of γ.From the definition (3.1.c), we may write,

$$B_2(s)|_{\bar{\gamma}(s)} = \bar{\gamma}'(s) = \bar{T}(s)$$

Thus,
γ is B_2 slant helix $\Leftrightarrow \langle B_2, v\rangle = c$ here v is a constant vector and $c = const.$

$\Leftrightarrow \langle \bar{T}, v\rangle = c$ here v is a constant vector and $c = const.$

$\Leftrightarrow$ γ is a general helix.

From now on we have focused on the time-like curve in R_1^4.

Theorem 3.6. Let be a time-like curve in R_1^4 with the curvatures k_1, k_2, k_3 and $\bar{\gamma}$ be the principal direction curve of γ.Then the curvatures of $\bar{\gamma}$ are as follows

$$\overline{k_1}(s) = \sqrt{{k_2}^2 - {k_1}^2}$$
$$\overline{k_2}(s) = -({k_1}^2 - {k_2}^2)^{3/2}$$

Proof. Let $\{\bar{T}, \bar{N}, \overline{B_1}, \overline{B_2}, \overline{k_1}, \overline{k_2}, \overline{k_3}\}$ be the Frenet elements of $\bar{\gamma}$. We find from the definition (3.1.a) , we can easily obtain ,

$$N(s)|_{\bar{\gamma}(s)} = \bar{\gamma}'(s) = \bar{T}(s)$$

Then,

$$\bar{N}(s) = \frac{\bar{\gamma}''(s)}{\|\bar{\gamma}(s)\|} = \frac{N'(s)}{\|N'(s)\|} = \frac{k_1T + k_2B_1}{\|k_1T + k_2B_1\|} = \frac{k_1T + k_2B_1}{\sqrt{{k_2}^2 - {k_1}^2}}$$

$$\overline{B_2} = \frac{\bar{\gamma}' \times \bar{\gamma}'' \times \bar{\gamma}'''}{\|\bar{\gamma}' \times \bar{\gamma}'' \times \bar{\gamma}'''\|} = \frac{-k_1B_1 + k_2T}{\sqrt{{k_1}^2 - {k_2}^2}}$$

and ,

$$\overline{B_1} = \overline{B_2} \times \bar{T} \times \bar{N} = \frac{k_1^2 + k_2^2}{({k_2}^2 - {k_1}^2)i}B_2$$

Then the curvatures of $\bar{\gamma}$ are given by

$$\overline{k_1}(s) = \langle \bar{T}', \bar{N}\rangle = \frac{k_1^2 + k_2^2}{\sqrt{{k_2}^2 - {k_1}^2}}$$
$$\overline{k_2}(s) = \langle \bar{N}', \overline{B_1}\rangle = -({k_1}^2 - {k_2}^2)^{3/2}$$

Theorem 3.7. Let γ be a time-like curve whose curvatures are k_1, k_2, k_3 and γ be the $B_1 -$ direction curve of γ in R_1^4.Then the curvatures of γ are as follows

$$\overline{k_1}(s) = k_2 - k_3$$

$$\overline{k_2}(s) = |k_2 - k_3|$$

Proof. The proof is similar to the proof of Theorem 3.7, so it is omitted.

Theorem 3.8. Let γ be a time-like curve whose curvatures are k_1, k_2, k_3 and $\bar{\gamma}$ be the $B_2 -$ direction curve of γ, in R_1^4, the curvatures of $\bar{\gamma}$ are as follows

$$\overline{k_1}(s) = k_3 sgn(k_3)$$
$$\overline{k_2}(s) = (k_2 + k_3)sgn(k_3)$$

Proof. Let $\{\bar{T}, \bar{N}, \overline{B_1}, \overline{B_2}, \overline{k_1}, \overline{k_2}, \overline{k_3}\}$ be the Frenet elements of $\bar{\gamma}$. From the (3.1.c), we get,

$$B_2(s) = \bar{\gamma}'(s) = \bar{T}(s)$$

Using the Frenet vector fields, we find,

$$\bar{N}(s) = -sgn(k_3)B_1$$
$$\overline{B_2}(s) = T$$
$$\overline{B_1} = sgn(k_3)N$$

Finally , the curvatures of $\bar{\gamma}$ are given by

$$\overline{k_1}(s) = k_3 sgn(k_3)$$
$$\overline{k_2}(s) = (k_2 + k_3)sgn(k_3)$$

Theorem 3.9. Let γ be a time-like curve in R_1^4 and $\bar{\gamma}$ be the principal direction curve of γ is a slant helix $\Leftrightarrow$ $\bar{\gamma}$ is a general helix.

Proof. Let $\{T, N, B_1, B_2\}$ be the Frenet frame of γ. From the definition (3.1.a) , we also know,

$$N(s)|_{\bar{\gamma}(s)} = \bar{\gamma}'(s) = \bar{T}(s)$$

Thus,

γ is a slant helix $\Leftrightarrow$ $\langle N, u\rangle = c$ here u is a constant vector and $c = const.$

$\Leftrightarrow \langle \bar{T}, u\rangle = c$
$\Leftrightarrow$ γ is a general helix.

Theorem 3.10. Let γ be a space-like curve in R_1^4 and $\bar{\gamma}$ be the $B_2 -$ direction of γ. Then γ is a $B_2 -$ slant helix $\Leftrightarrow$ $\bar{\gamma}$ is a general helix.

Proof: Let $\{T, N, B_1, B_2\}$ be the Frenet frame of γ.From the definition (3.1.c), we may write,

$$B_2(s)|_{\bar{\gamma}(s)} = \bar{\gamma}'(s) = \bar{T}(s)$$

Thus,

γ is B_2 slant helix $\Leftrightarrow \langle B_2, v\rangle = c$ here v is a constant vector and $c = const.$

$\Leftrightarrow \langle \bar{T}, v\rangle = c$ here v is a constant vector and $c = const.$
$\Leftrightarrow$ γ is a general helix.

Example

In this section , an example of directional associated curves of space-like curve are given as follows:
Consider a space-like curve

$$\alpha(s) = (\sin 2s, \cos 2s, \sqrt{3}s, \sqrt{3})$$

The Frenet frame vectors and curvatures are obtained by

$$T = (2\cos 2s, -2\sin 2s, \sqrt{3}, 0)$$

$N = (-\sin 2s, -\cos 2s, 0, 0)$

$B_1 = \frac{1}{\sqrt{113}}(8\cos 2s - \sin 2s, -\cos 2s - 8\sin 2s, 4\sqrt{3}, 0)$

$B_2 = (0,0,0,0)$

$k_1 = 4, \quad k_2 = \sqrt{113}$

We obtain principal-direction , B_1 −direction and B_2 −direction ,

$$N(s)|_{\bar{\gamma}(s)} = \bar{\gamma}'(s) = \bar{T}(s) = \left(\frac{1}{2}\cos 2s, -\frac{1}{2}\sin 2s, \frac{\sqrt{3}}{4}, 0\right)$$

$$\bar{N}(s) = \frac{\bar{\gamma}''(s)}{\|\bar{\gamma}(s)\|} = (-\sin 2s, -\cos 2s, 0, 0)$$

$$\overline{B_2} = \frac{\bar{\gamma}' \times \bar{\gamma}'' \times \bar{\gamma}'''}{\|\bar{\gamma}' \times \bar{\gamma}'' \times \bar{\gamma}'''\|} = \left(0,0,0,-\frac{\sqrt{3}}{2}\right)$$

$$\overline{B_1} = \overline{B_2} \times \bar{T} \times \bar{N} = \left(\frac{\sqrt{3}}{4}\cos 2s, \frac{\sqrt{3}}{2}\sin 2s, -\frac{\sqrt{3}}{3}, 0\right)$$

Then we get the curvatures of $\bar{\gamma}$ as

$$\overline{k_1}(s) = \langle \bar{T}', \bar{N} \rangle = 1$$

$$\overline{k_2}(s) = \langle \bar{N}', \overline{B_1} \rangle = -\frac{\sqrt{3}}{2}\cos^2 2s + \sqrt{3}\sin^2 2s$$

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Conflicts of interest

The authors state that did not have conflict of interests.

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