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On the fine spectra of the Jacobi matrices on c_0, c, ℓ_p $(1 \le p \le \infty)$ and bv_p $(1 \le p < \infty)$

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Abstract

The spectrum and spectral divisions of band matrices are very new and popular topics of studies. In this paper, our aims are to investigate boundedness of Jacobi matrix which is a band matrix has important role in physics and give subdivisions of the spectra, which are approximate point spectrum, defect spectrum and compression spectrum, for a special type Jacobi matrix. Moreover, we will find the fine division of spectrum which is given by Goldberg with the help of it.

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1. Introduction

The band matrices are an interesting topic for researchers since they have important applications in applied mathematics. In the summability theory and functional analysis, there are applications of band matrices. Also, they are used in linear algebra, computation in classical and fractional situations and approximation theory. The spectrum and spectral divisions of band matrices are very new and popular topics of studies.

In recent years, some authors have investigated the spectral decomposition of generalized difference matrices on various sequence spaces. In 2011, Amirov, Durna and Yıldırım [1] calculated the approximate point spectrum, the defect spectrum, and the compression spectrum of the operators using the relationship between the spectral decompositions of the operators. Many researchers have benefited from this study and found the fine division of the operator. In the studies conducted so far, the approximate point spectrum, the defect spectrum and the compression spectrum were calculated using the fine spectrum of the operator. Generally, in order to examine the fine spectrum of operator, we investigate injectivity and surjectivity of its adjoint. Because it is well-known that "T has a dense range if and only if T^* is 1-1" and "T has a bounded inverse if and only if T^* is onto". But we can not always find adjoint operator. Even if we find it, we can not investigate the character of the series obtained while examining the injectivity and surjectivity of the adjoint operator. For example, it is not possible to talk about the adjoint of operator in general on ℓ_∞, because ℓ_∞ does not have the Schauder basis in the usual sense. And so, we will first calculate the approximate point spectrum, the defect spectrum and the compression spectrum of operator using the relationship between spectral division of operator and spectral division of its adjoint. Moreover, we will find the fine division of spectrum which is given by Goldberg with the help of it.

Firstly, we will recall basic definitions and properties of operator which are used by us.

Definition 1.1 Let $T: D(T) \to X$ be a linear operator, defined on $D(T) \subset X$, where $D(T)$ denote the domain of T and X is an infinite-dimensional complex normed space. Let $T_\lambda := \lambda I - T$ for $T \in B(X)$ and $\lambda \in \mathbb{C}$ where I is the identity operator, then different definitions and notations of spectra are defined as follows [2-3]:

(1) The spectrum: $\sigma(T, X) := \{\lambda \in \mathbb{C}: T_\lambda$ is not invertible$\}$,
(2) The resolvent set $\rho(T, X)$ is the complement of $\sigma(T, X)$ in $\mathbb{C}$,
(3) The point spectrum: $\sigma_p(T, X) := \{\lambda \in \mathbb{C}: T_\lambda$ is not injective$\}$,
(4) The continuous spectrum: $\sigma_c(T, X) := \{\lambda \in \mathbb{C}: T_\lambda$ is injective and $\overline{R(T_\lambda)} =$X but $R(T_\lambda) \neq$ X$\}$, where $R(T_\lambda)$ denote the domain of T_λ,

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(5) The residual spectrum: $\sigma_r(T,X) := \left\{\lambda \in \mathbb{C}: T_\lambda \text{ is injective but } \overline{R(T_\lambda)} \neq X\right\}$,
(6) The defect spectrum: $\sigma_\delta(T,X) := \{\lambda \in \sigma(T,X): R(T_\lambda) \neq X\}$,
(7) The compression spectrum: $\sigma_{co}(T,X) := \left\{\lambda \in \mathbb{C}: \overline{R(T_\lambda)} \neq X\right\}$,
(8) The approximate point spectrum: $\sigma_{ap}(T,X) :=\{ \lambda \in \mathbb{C}$: there exists a sequence (x_n) in X such that $\|x_n\| = 1$ for all $n \in \mathbb{N}$ and $lim_{n\to\infty}\|T_\lambda(x_n)\| = 0$.

In Banach spaces, Proposition 1.2 is frequently used for calculating the partition of the spectrum of the linear operator.

Proposition 1.2 [2] The spectra and subspectra of an operator $T \in B(X)$ and its adjoint $T^* \in B(X^*)$ are related by the following relations:

(a) $\sigma(T^*,X^*) = \sigma(T,X)$,

(b) $\sigma_c(T^*,X^*) \subseteq \sigma_{ap}(T,X)$,

(c) $\sigma_{ap}(T^*,X^*) = \sigma_\delta(T,X)$,

(d) $\sigma_\delta(T^*,X^*) = \sigma_{ap}(T,X)$,

(e) $\sigma_p(T^*,X^*) = \sigma_{co}(T,X)$,

(f) $\sigma_{co}(T^*,X^*) \supseteq \sigma_p(T,X)$,

(g) $\sigma(T,X) = \sigma_{ap}(T,X) \cup \sigma_p(T^*,X^*) = \sigma_p(T,X) \cup \sigma_{ap}(T^*,X^*)$.

1.1. Goldberg's classification of spectrum

If X is a Banach space and $T \in B(X)$, then there are three possibilities for $R(T)$:

(I) $R(T) = X$, (II) $\overline{R(T)}$ =X, but $R(T)$≠X,

(III) $\overline{R(T)} \neq$ X

and three possibilities for T^{-1}:

(1) T^{-1} exists and continuous,

(2) T^{-1} exists but discontinuous,

(3) T^{-1} does not exist.

If these possibilities are combined in all possible ways, nine different states are created. These are labelled by: $I_1, I_2, I_3, II_1, II_2, II_3, III_1, III_2, III_3$. If an operator is in state III_2 for example, then $\overline{R(T)} \neq$ X and T^{-1} exist but is discontinuous (see [4]).

If λ is a complex number such that $T_\lambda \in I_1$ or $T_\lambda \in II_1$, then $\lambda \in \rho(T,X)$. All scalar values of λ not in $\rho(T,X)$ comprise the spectrum of T. The further classification of $\sigma(T,X)$ gives rise to the fine spectrum of T. That is, $\sigma(T,X)$ can be divided into the subsets $I_2\sigma(T,X) = \emptyset, I_3\sigma(T,X), II_2\sigma(T,X), II_3\sigma(T,X), III_1\sigma(T,X), III_2\sigma(T,X), III_3\sigma(T,X)$. For example, if T_λ is in a given state, III_2 (say), then we write $\lambda \in III_2\sigma(T,X)$.

Let us give a short survey concerning the spectrum and the fine spectrum and subdivision of the spectrum of the linear operators over certain sequence spaces.

First, the spectrum of the Cesàro operator of order one over the sequence space ℓ_2 has been examined by Brown, Halmos, and Shields [5] in 1965. In 1977, Cass and Rhoades [6], in 1978, Cardlidge [7] computed the spectrum of Weighted mean matrices.

Subdivisions of the spectrum for an operator on a sequence space were given by [8], [9] and [10] firstly.

Besides the above listed workers, the spectrum, fine spectrum and subdivision of the spectrum for various matrix operators have been investigated by many authors in the recent years, [11-25].

By the definitions given above, the following statements are obtained from the Table given by Durna and Yıldırım in [9]:

Table 1. Subdivisions of the spectrum of a linear operator

		1	2	3
		T_λ^{-1} exits and is bounded	T_λ^{-1} exits and is unbounded	T_λ^{-1} does not exits
I	$R(T_\lambda)=X$	$\lambda\in\rho(T,X)$ $\lambda\in\rho(T,X)$	–	$\lambda\in\sigma_p(T,X)$ $\lambda\in\sigma_{ap}(T,X)$
II	$\overline{R(T_\lambda)}$ =X	$\lambda\in\rho(T,X)$	$\lambda\in\sigma_c(T,X)$ $\lambda\in\sigma_{ap}(T,X)$ $\lambda\in\sigma_\delta(T,X)$	$\lambda\in\sigma_p(T,X)$ $\lambda\in\sigma_{ap}(T,X)$ $\lambda\in\sigma_\delta(T,X)$
III	$\overline{R(T_\lambda)}$ ≠X	$\lambda\in\sigma_r(T,X)$ $\lambda\in\sigma_\delta(T,X)$ $\lambda\in\sigma_{co}(T,X)$	$\lambda\in\sigma_r(T,X)$ $\lambda\in\sigma_{ap}(T,X)$ $\lambda\in\sigma_\delta(T,X)$ $\lambda\in\sigma_{co}(T,X)$	$\lambda\in\sigma_p(T,X)$ $\lambda\in\sigma_{ap}(T,X)$ $\lambda\in\sigma_\delta(T,X)$ $\lambda\in\sigma_{co}(T,X)$

In this paper, we computed subdivisions of the spectrum for constant Jacobi matrix.

2. Boundedness of Jacobi Matrix $J(s_n, r_n)$

A matrix of the form $J=(a_{ij})$ is called a Jacobi matrix, where $a_{ij}=0$ unless $|j-i|<1$. More specifically,

$$J(s_n,r_n)=\begin{pmatrix} s_0 & r_0 & 0 & 0 & \cdots \\ r_0 & s_1 & r_1 & 0 & \cdots \\ 0 & r_1 & s_2 & r_2 & \cdots \\ 0 & 0 & r_2 & s_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{1}$$

where all s_n, r_n are real. If we get some constant sequences such as $(s_n)=(s)$ and $(r_n)=(r)$, this $J(s_n,r_n)=J(s,r)$ matrix is called constant Jacobi matrix. The spectral results are clear when $r=0$, so for the sequel we will have $r\neq 0$.

Lemma 2.1 [25] Let T be an operator with the associated matrix $A=(a_{nk})$. Then the followings hold:

i. $T\in B(c)$ if and only if

$$\|A\|:=\sup_n \sum_{k=1}^{\infty}|a_{nk}|<\infty, \tag{2}$$

$$a_k:=\lim_{n\to\infty} a_{nk} \text{ exists for each } k, \tag{3}$$

$$a:=\lim_{n\to\infty}\sum_{k=1}^{\infty} a_{nk} \text{ exists} \tag{4}$$

are valid.

ii. $T\in B(c_0)$ if and only if (2) and (3) with $a_k=0$ for each k are valid.

iii. $T\in B(\ell_\infty)$ if and only if (2) is valid.

In these cases, the operator norm of T is

$$\|T\|_{(\ell_\infty:\ell_\infty)}=\|T\|_{(c:c)}=\|T\|_{(c_0:c_0)}=\|A\|. \tag{5}$$

iv. $T\in B(\ell_1)$ if and only if

$$\|A^t\|:=\sup_k \sum_{n=1}^{\infty}|a_{nk}|<\infty \tag{6}$$

is valid.

In these cases, the operator norm of T is $\|T\|_{(\ell_1:\ell_1)} = \|A^t\|$.

Theorem 2.2 $\mu \in \{c_0, c, \ell_1, \ell_\infty\}$. $J(s,r) \in B(\mu)$ and $\|J(s,r)\|_{(\mu:\mu)} \leq 2|r| + |s|$.

Proof It is clear from Lemma 2.1.

Theorem 2.3 $J(s,r) \in B(\ell_p)$ $(1 < p < \infty)$ and $\|J(s,r)\|_{(\ell_p:\ell_p)} \leq 2|r| + |s|$.

Proof Since

$$\begin{aligned}
\|J(s,r)x\|_{\ell_p} &= \left(\sum_{n=1}^{\infty} |rx_{n-1} + sx_n + rx_{n+1}|^p\right)^{\frac{1}{p}} \\
&= \left(\sum_{n=1}^{\infty} |r(x_{n-1} + x_{n+1}) + sx_n|^p\right)^{\frac{1}{p}} \\
&\leq |r| \left(\sum_{n=1}^{\infty} |x_{n-1} + x_{n+1}|^p\right)^{\frac{1}{p}} + |s| \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}} \\
&\leq 2|r| \|x\|_{\ell_p} + |s| \|x\|_{\ell_p} \leq (2|r| + |s|) \|x\|_{\ell_p},
\end{aligned}$$

where $x_0 = 0$, we have $J(s,r) \in B(\ell_p)$ and $\|J(s,r)\|_{(\ell_p:\ell_p)} \leq 2|r| + |s|$.

Theorem 2.4 $J(s,r) \in B(bv_p)$ $(1 < p < \infty)$ and $\|J(s,r)\|_{(bv_p:bv_p)} \leq |s - r| + 3|r|$.

Proof We have

$$\begin{aligned}
\|J(s,r)x\|_{bv_p}^p &= |rx_1 + sx_2 + rx_3 - sx_1 - rx_2|^p + |rx_2 + sx_3 + rx_4 - rx_1 - sx_2 - rx_3|^p + \cdots \\
&= |(r-s)x_1 + (s-r)x_2 + rx_3|^p + |(r-s)x_2 + (s-r)x_3 + rx_4 - rx_1|^p + \cdots \\
&\leq \sum_{n=0}^{\infty} |(r-s)x_{n+1} + (s-r)x_{n+2} + r(x_{n+3} - x_{n+2} + x_{n+2} - x_{n+1} + x_{n+1} - x_n)|^p \\
&= \left[\left(\sum_{n=0}^{\infty} |(s-r)(x_{n+2} - x_{n+1}) + r(x_{n+3} - x_{n+2} + x_{n+2} - x_{n+1} + x_{n+1} - x_n)|^p\right)^{1/p}\right]^p \\
&\leq \left[\left(\sum_{n=0}^{\infty} |s-r|^p |x_{n+2} - x_{n+1}|^p\right)^{1/p}\right. \\
&\quad \left. + \left(\sum_{n=0}^{\infty} |r|^p |x_{n+3} - x_{n+2} + x_{n+2} - x_{n+1} + x_{n+1} - x_n|^p\right)^{1/p}\right]^p \\
&\leq \left[\left(|s-r| \sum_{n=0}^{\infty} |x_{n+2} - x_{n+1}|^p\right)^{1/p}\right. \\
&\quad \left. + \left(|r| \sum_{n=0}^{\infty} (|x_{n+3} - x_{n+2}| + |x_{n+2} - x_{n+1}| + |x_{n+1} - x_n|)^p\right)^{1/p}\right]^p
\end{aligned}$$

$$\leq \left[|s-r| \left(\sum_{n=0}^{\infty} |x_{n+2} - x_{n+1}|^p \right)^{1/p} + |r| \left[\left(\sum_{n=0}^{\infty} |x_{n+3} - x_{n+2}|^p \right)^{1/p} + \left(\sum_{n=0}^{\infty} |x_{n+2} - x_{n+1}|^p \right)^{1/p} + \left(\sum_{n=0}^{\infty} |x_{n+1} - x_n|^p \right)^{1/p} \right] \right]^p$$

$$\leq \left[|s-r| \|x\|_{bv_p} + |r| 3 \|x\|_{bv_p} \right]^p = [|s-r| + 3|r|]^p \|x\|_{bv_p}^p$$

where $x_0 = 0$. Then

$\|J(s,r)x\|_{bv_p} \leq (|s-r| + 3|r|)\|x\|_{bv_p}$.

Hence we get $J(s,r) \in B(bv_p)$ and $\|J(s,r)\|_{(bv_p:bv_p)} \leq |s-r| + 3|r|$.

Theorem 2.5 $J(s_n, r_n) \in B(\mu)$ and $\| J(s_n, r_n)\|_{(\mu:\mu)} \leq 2\|r\|_\infty + \|s\|_\infty$ where $\mu \in \{c_0, c, \ell_1, \ell_\infty\}$, (s_n), $(r_n) \in \mu$.

Proof It is clear from Lemma 2.1.

Theorem 2.6 $J(s_n, r_n) \in B(\ell_p)$ $(1 < p < \infty)$ and $\| J(s_n, r_n)\|_{(\ell_p:\ell_p)} \leq 2\|r\|_p + \|s\|_p$ where (s_n), $(r_n) \in \ell_p$.

Proof Since

$$\begin{aligned} \|J(s_n, r_n)x\|_p &= \left(\sum_{n=1}^{\infty} |r_{n-1}x_{n-1} + s_n x_n + r_n x_{n+1}|^p \right)^{\frac{1}{p}} \\ &\leq \left(\sum_{n=1}^{\infty} |r_{n-1}x_{n-1} + s_n x_n|^p \right)^{\frac{1}{p}} + \left(\sum_{n=1}^{\infty} |r_n x_{n+1}|^p \right)^{\frac{1}{p}} \\ &\leq \left(\sum_{n=1}^{\infty} |r_{n-1}x_{n-1}|^p \right)^{\frac{1}{p}} + \left(\sum_{n=1}^{\infty} |s_n x_n|^p \right)^{\frac{1}{p}} + \left(\sum_{n=1}^{\infty} |r_n x_{n+1}|^p \right)^{\frac{1}{p}} \\ &\leq \left(2\|r\|_p + \|s\|_p\right)\|x\|_p, \end{aligned}$$

we have $J(s_n, r_n) \in B(\ell_p)$ and $\| J(s_n, r_n)\|_{(\ell_p:\ell_p)} \leq 2\|r\|_p + \|s\|_p$.

3. Spectrum of Jacobi Matrix $J(s,r)$ with constant entries

In this section, we will give the spectral decomposition of Jacobi Matrix $J(s,r)$ with constant entries with the help of the spectrum and the fine spectrum, which were previously studied in [27] and [18].

3.1. Subdivision of the spectrum of $J(s,r)$ on c_0

Theorem 3.1 $\sigma_{ap}(J(s,r), c_0) = \sigma(J(s,r), c_0) = [s-2r, s+2r]$.

Proof From Table 1, we know

$\sigma_{ap}(J(s,r), c_0) = \sigma(J(s,r), c_0) \backslash III_1\sigma(J(s,r), c_0)$.

Since $\sigma_r(J(s,r),c_0)=\emptyset$ from [27, Theorem 3.3], we have $III_1\sigma(J(s,r),c_0)=\emptyset$ and we know $\sigma(J(s,r),c_0)=[s-2r,s+2r]$ from [27, Theorem 2.5]. Hence $\sigma_{ap}(J(s,r),c_0)=[s-2r,s+2r]$.

Theorem 3.2 $\sigma_\delta(J(s,r),c_0)=\sigma(J(s,r),c_0)=[s-2r,s+2r]$.

Proof We have

$$\sigma_\delta(J(s,r),c_0)=\sigma(J(s,r),c_0)\backslash I_3\sigma(J(s,r),c_0)$$

from Table 1. Since $\sigma_p(J(s,r),c_0)=\emptyset$ from [27, Theorem 2.1], we get $I_3\sigma(J(s,r),c_0)=\emptyset$ and we know $\sigma(J(s,r),c_0)=[s-2r,s+2r]$ from [27, Theorem 2.5]. Hence $\sigma_\delta(J(s,r),c_0)=[s-2r,s+2r]$.

Theorem 3.3 $\sigma_{co}(J(s,r),c_0)=\emptyset$.

Proof From Table 1, we get

$$\sigma_{co}(J(s,r),c_0)=III_1\sigma(J(s,r),c_0)\cup III_2\sigma(J(s,r),c_0)\cup III_3\sigma(J(s,r),c_0).$$

Since $\sigma_p(J(s,r),c_0)=\emptyset$ from [27, Theorem 2.1] and $\sigma_r(J(s,r),c_0)=\emptyset$ from [27, Theorem 3.3], we have

$$III_1\sigma(J(s,r),c_0)=III_2\sigma(J(s,r),c_0)=III_3\sigma(J(s,r),c_0)=\emptyset.$$

Therefore $\sigma_{co}(J(s,r),c_0)=\emptyset$.

3.2. The fine spectrum and subdivision of the spectrum of $J(s,r)$ on c

Theorem 3.4 $III_2\sigma(J(s,r),c)=\{s+2r\}$.

Proof By [27, Theorem 3.5], $\sigma_r(J(s,r),c)=\{s+2r\}$. So that $s+2r\in III_1\sigma(J(s,r),c)\cup III_2\sigma(J(s,r),c)$. Now we investigate either $s+2r\in III_1\sigma(J(s,r),c)$ or $s+2r\in III_2\sigma(J(s,r),c)$. For this we must show that $\left((s+2r)I-J(s,r)\right)^{-1}$ is whether bounded or not.

$$\begin{aligned}(s+2r)I-J(s,r)&=\begin{pmatrix}s+2r&0&0&\cdots\\0&s+2r&0&\cdots\\0&0&s+2r&\cdots\\\vdots&\vdots&\vdots&\ddots\end{pmatrix}-\begin{pmatrix}s&r&0&0&\cdots\\r&s&r&0&\cdots\\0&r&s&r&\cdots\\\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}\\&=r\begin{pmatrix}2&-1&0&0&\cdots\\-1&2&-1&0&\cdots\\0&-1&2&-1&\cdots\\\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}.\end{aligned}$$

The inverse of above matrix is

$$\frac{1}{r}\begin{pmatrix}1&1&1&1&\cdots\\1&2&2&2&\cdots\\1&2&3&3&\cdots\\1&2&3&4&\cdots\\\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}$$

which is unbounded matrix. Therefore we get $s+2r\in III_2\sigma(J(s,r),c)$. Hence $III_2\sigma(J(s,r),c)=\{s+2r\}$, from $III_2\sigma(J(s,r),c)\subset\sigma_r(J(s,r),c)$.

Corollary 3.5 $III_1\sigma(J(s,r),c)=\emptyset$.

Proof By [27, Theorem 3.5], $\sigma_r(J(s,r),c)=\{s+2r\}$ and $III_1\sigma(J(s,r),c)=\emptyset$ from Theorem 3.4

Corollary 3.6 $\sigma_{ap}(J(s,r),c) = [s-2r, s+2r]$.

Proof From Table 1, we get

$\sigma_{ap}(J(s,r),c) = \sigma(J(s,r),c) \backslash III_1\sigma(J(s,r),c)$.

Since $III_1\sigma(J(s,r),c) = \emptyset$ from Corollary 3.5 and $\sigma(J(s,r),c) = [s-2r, s+2r]$ from [27, Theorem 2.5], we have $\sigma_{ap}(J(s,r),c) = [s-2r, s+2r]$.

Corollary 3.7 $\sigma_{\delta}(J(s,r),c) = [s-2r, s+2r]$.

Proof We know

$\sigma_{\delta}(J(s,r),c) = \sigma(J(s,r),c) \backslash I_3\sigma(J(s,r),c)$

from Table 1. Since $\sigma_p(J(s,r),c) = \emptyset$ from [27, Theorem 2.1] and $\sigma(J(s,r),c) = [s-2r, s+2r]$ from [27, Theorem 2.5], we get $\sigma_{\delta}(J(s,r),c) = [s-2r, s+2r]$.

Corollary 3.8 $\sigma_{co}(J(s,r),c) = \{s+2r\}$.

Proof By Table 1, we get

$\sigma_{co}(J(s,r),c) = III_1\sigma(J(s,r),c) \cup III_2\sigma(J(s,r),c) \cup III_3\sigma(J(s,r),c)$.

Using $\sigma_p(J(s,r),c) = \emptyset$ from [27, Theorem 2.1], we get $\sigma_{co}(J(s,r),c) = \{s+2r\}$ from Theorem 3.4 and Corollary 3.5.

3.3. The fine spectrum and subdivision of the spectrum of $J(s,r)$ on ℓ_1

Theorem 3.9 $III_1\sigma(J(s,r),\ell_1) = \emptyset$.

Proof We know

$\sigma_{ap}(J^*(s,r), c^* \equiv \ell_1) = \sigma(J^*(s,r),\ell_1) \backslash III_1\sigma(J^*(s,r),\ell_1)$

by Table 1 and we have

$\sigma_{ap}(J^*(s,r) = J(s,r),\ell_1) = \sigma_{\delta}(J(s,r),c_0)$

by the Proposition 1.2. We get

$\sigma_{\delta}(J(s,r),c_0) = \sigma(J(s,r),c_0) \backslash I_3\sigma(J(s,r),c_0)$

by Table 1. Since $\sigma(J(s,r),c_0) = [s-2r, s+2r]$ from [27, Theorem 2.5], and $I_3\sigma(J(s,r),c_0) = \emptyset$, we have $\sigma_{\delta}(J(s,r),c_0) = [s-2r, s+2r]$. Therefore $\sigma_{ap}(J^*(s,r) = J(s,r),\ell_1) = [s-2r, s+2r]$ and since $\sigma(J^*(s,r),\ell_1) = \sigma(J(s,r),\ell_1) = [s-2r, s+2r]$ from Proposition 1.2, we get $III_1\sigma(J(s,r),\ell_1) = \emptyset$.

Corollary 3.10 $III_2\sigma(J(s,r),\ell_1) = (s-2r, s+2r)$.

Proof We know $\sigma_r(J(s,r),\ell_1) = (s-2r, s+2r)$ by [27, Theorem 3.4] and we have $III_2\sigma(J(s,r),\ell_1) = (s-2r, s+2r)$ by Theorem 3.9.

Corollary 3.11 $\sigma_{ap}(J(s,r),\ell_1) = [s-2r, s+2r]$.

Proof From Table 1, we know

$\sigma_{ap}(J(s,r),\ell_1) = \sigma(J(s,r),\ell_1)\backslash III_1\sigma(J(s,r),\ell_1)$.

Since $\sigma(J(s,r),\ell_1) = [s-2r, s+2r]$ from [27, Theorem 2.5] and $III_1\sigma(J(s,r),\ell_1) = \emptyset$ from Theorem 3.9, we have $\sigma_{ap}(J(s,r),\ell_1) = [s-2r, s+2r]$.

Corollary 3.12 $\sigma_\delta(J(s,r),\ell_1) = [s-2r, s+2r]$.

Proof Since $\sigma_p(J(s,r),\ell_1) = \emptyset$ from [27, Theorem 2.1], we have $I_3\sigma(J(s,r),\ell_1) = \emptyset$. And since

$\sigma_\delta(J(s,r),\ell_1) = \sigma(J(s,r),\ell_1)\backslash I_3\sigma(J(s,r),\ell_1)$

by Table 1, we have $\sigma_\delta(J(s,r),\ell_1) = [s-2r, s+2r]$.

Corollary 3.13 $\sigma_{co}(J(s,r),\ell_1) = (s-2r, s+2r)$.

Proof From Table 1, we have

$\sigma_{co}(J(s,r),\ell_1) = III_1\sigma(J(s,r),\ell_1) \cup III_2\sigma(J(s,r),\ell_1) \cup III_3\sigma(J(s,r),\ell_1)$.

Using $\sigma_r(J(s,r),\ell_1) = (s-2r, s+2r)$ from [27, Theorem 3.4] and $\sigma_p(J(s,r),\ell_1) = \emptyset$ from [27, Theorem 2.1], we have $\sigma_{co}(J(s,r),\ell_1) = (s-2r, s+2r)$ from Theorem 3.9 and Corollary 3.10.

3.4. Subdivision of the spectrum of $J(s,r)$ on ℓ_p

Theorem 3.14 $\sigma_{ap}(J(s,r),\ell_p) = \sigma(J(s,r),\ell_p) = [s-2r, s+2r]$.

Proof We know

$\sigma_{ap}(J(s,r),\ell_p) = \sigma(J(s,r),\ell_p)\backslash III_1\sigma(J(s,r),\ell_p)$

by the Table 1. Since $\sigma_r(J(s,r),\ell_p) = \emptyset$ from [18, Corollary 3.5], we find $III_1\sigma(J(s,r),\ell_p) = \emptyset$ and we have $\sigma(J(s,r),\ell_p) = [s-2r, s+2r]$ from [18, Theorem 3.2]. Therefore $\sigma_{ap}(J(s,r),\ell_p) = [s-2r, s+2r]$.

Theorem 3.15 $\sigma_\delta(J(s,r),\ell_p) = \sigma(J(s,r),\ell_p) = [s-2r, s+2r]$.

Proof From Table 1, we have

$\sigma_\delta(J(s,r),\ell_p) = \sigma(J(s,r),\ell_p)\backslash I_3\sigma(J(s,r),\ell_p)$.

We have $I_3\sigma(J(s,r),\ell_p) = \emptyset$ and $\sigma(J(s,r),\ell_p) = [s-2r, s+2r]$ from [18, Theorem 3.2] since $\sigma_p(J(s,r),\ell_p) = \emptyset$ from [18, Theorem 3.3]. Hence $\sigma_\delta(J(s,r),\ell_p) = [s-2r, s+2r]$.

Theorem 3.16 $\sigma_{co}(J(s,r),\ell_p) = \emptyset$.

Proof We know

$\sigma_{co}(J(s,r),\ell_p) = III_1\sigma(J(s,r),\ell_p) \cup III_2\sigma(J(s,r),\ell_p) \cup III_3\sigma(J(s,r),\ell_p)$

by the Table 1. Since $\sigma_p(J(s,r),\ell_p) = \emptyset$ from [18, Theorem 3.3] and $\sigma_r(J(s,r),\ell_p) = \emptyset$ from [18, Corallary 3.5], we get $III_1\sigma(J(s,r),\ell_p) = III_2\sigma(J(s,r),\ell_p) = III_3\sigma(J(s,r),\ell_p) = \emptyset$. Hence $\sigma_{co}(J(s,r),\ell_p) = \emptyset$.

3.5. The fine spectrum and subdivision of the spectrum of $J(s,r)$ on ℓ_∞

Theorem 3.17 $I_3\sigma(J(s,r),\ell_\infty)=\emptyset$.

Proof From Table 1, we have

$\sigma_\delta(J(s,r),\ell_\infty)=\sigma(J(s,r),\ell_\infty)\backslash I_3\sigma(J(s,r),\ell_\infty)$.

We get $\sigma_\delta(J(s,r),\ell_\infty)=\sigma_{ap}(J^*(s,r),\ell_1)$ from Proposition 1.2. Using Corollary 3.11, $\sigma_{ap}(J(s,r),\ell_1)=[s-2r,s+2r]$ and $\sigma(J(s,r),\ell_\infty)=[s-2r,s+2r]$ from [27, Theorem 2.5], we have $I_3\sigma(J(s,r),\ell_\infty)=\emptyset$.

Theorem 3.18 $III_1\sigma(J(s,r),\ell_\infty)=\emptyset$.

Proof We know

$\sigma_{ap}(J(s,r),\ell_\infty)=\sigma(J(s,r),\ell_\infty)\backslash III_1\sigma(J(s,r),\ell_\infty)$

by Table 1. $\sigma_{ap}(J^*(s,r)=J(s,r),\ell_\infty)=\sigma_\delta(J(s,r),\ell_1)$ from Proposition 1.2. Since $\sigma_\delta(J(s,r),\ell_1)=[s-2r,s+2r]$ from Corollary 3.12 and $\sigma(J(s,r),\ell_\infty)=[s-2r,s+2r]$ from [27, Theorem 2.5], we have $III_1\sigma(J(s,r),\ell_\infty)=\emptyset$.

Corollary 3.19 $\sigma_{ap}(J(s,r),\ell_\infty)=[s-2r,s+2r]$.

Proof From Table 1, we get

$\sigma_{ap}(J(s,r),\ell_\infty)=\sigma(J(s,r),\ell_\infty)\backslash III_1\sigma(J(s,r),\ell_\infty)$.

We have $\sigma_{ap}(J(s,r),\ell_\infty)=[s-2r,s+2r]$ since $\sigma(J(s,r),\ell_\infty)=[s-2r,s+2r]$ from [27 Theorem 2.5] and $III_1\sigma(J(s,r),\ell_\infty)=\emptyset$ from Theorem 3.18.

Corollary 3.20 $\sigma_\delta(J(s,r),\ell_\infty)=[s-2r,s+2r]$.

Proof We know

$\sigma_\delta(J(s,r),\ell_\infty)=\sigma(J(s,r),\ell_\infty)\backslash I_3\sigma(J(s,r),\ell_\infty)$

by the Table 1. Since $\sigma(J(s,r),\ell_\infty)=[s-2r,s+2r]$ from [27, Theorem 2.5] and $I_3\sigma(J(s,r),\ell_\infty)=\emptyset$ from Theorem 3.17, we have $\sigma_\delta(J(s,r),\ell_\infty)=[s-2r,s+2r]$.

3.6. Subdivision of the spectrum of $J(s,r)$ on bv_p

Theorem 3.21 $\sigma_{ap}(J(s,r),bv_p)=\sigma(J(s,r),bv_p)=[s-2r,s+2r]$.

Proof From Table 1, we get

$\sigma_{ap}(J(s,r),bv_p)=\sigma(J(s,r),bv_p)\backslash III_1\sigma(J(s,r),bv_p)$.

Since $\sigma_r(J(s,r),bv_p)=\emptyset$ from [18, Theorem 4.3 (iii)], we have $III_1\sigma(J(s,r),bv_p)=\emptyset$ and $\sigma(J(s,r),bv_p)=[s-2r,s+2r]$ from [18, Theorem 4.2]. Therefore $\sigma_{ap}(J(s,r),bv_p)=[s-2r,s+2r]$.

Theorem 3.22 $\sigma_\delta(J(s,r),bv_p)=\sigma(J(s,r),bv_p)=[s-2r,s+2r]$.

Proof We know

$$\sigma_\delta\big(J(s,r), bv_p\big) = \sigma\big(J(s,r), bv_p\big)\backslash I_3\sigma\big(J(s,r), bv_p\big)$$

by the Table 1. Since $\sigma_p\big(J(s,r), bv_p\big) = \emptyset$ from [18, Theorem 4.3 (i)], we have $I_3\sigma\big(J(s,r), bv_p\big) = \emptyset$ and we have $\sigma\big(J(s,r), bv_p\big) = [s-2r, s+2r]$ from [18, Theorem 4.2]. Hence $\sigma_\delta\big(J(s,r), bv_p\big) = [s-2r, s+2r]$.

Theorem 3.16 $\sigma_{co}\big(J(s,r), bv_p\big) = \emptyset$.

Proof From Table 1, we get

$$\sigma_{co}\big(J(s,r), bv_p\big) = III_1\sigma\big(J(s,r), bv_p\big) \cup III_2\sigma\big(J(s,r), bv_p\big) \cup III_3\sigma\big(J(s,r), bv_p\big)$$

Since $\sigma_p\big(J(s,r), bv_p\big) = \emptyset$ from [18, Theorem 4.3 (i)] and $\sigma_r\big(J(s,r), bv_p\big) = \emptyset$ from [18, Theorem 4.3 (iii)], we have $III_1\sigma\big(J(s,r), bv_p\big) = III_2\sigma\big(J(s,r), bv_p\big) = III_3\sigma\big(J(s,r), bv_p\big) = \emptyset$. Hence $\sigma_{co}\big(J(s,r), bv_p\big) = \emptyset$.

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Conflicts of interest

The authors state that did not have a conflict of interests.

References

[1] Amirov R. Kh., Durna N., Yildirim M., Subdivisions of the spectra for cesaro, rhaly and weighted mean operators on c_0, c and ℓ_p, *IJST,* A3 (2011) 175-183.

[2] Appell J., Pascale E.D, Vignoli A., Nonlinear Spectral Theory, Berlin, New York: Walter de Gruyter, (2002).

[3] Stone M.H., Linear transformations in Hilbert space and their applications to analysis, *New York (NY): American Mathematical Society*; 1932.

[4] Goldberg S., Unbounded Linear Operators, *New York:McGraw Hill,* (1966).

[5] Brown A., Halmos P.R., Shields A.L., Cesàro operators, *Acta Sci. Math. (Szeged)* 26(1-2) (1965) 125-137.

[6] Cass, F.P.; Rhoades, B. E., Mercerian theorems via spectral theory, *Pacific J. Math*. 73(1) (1977) 63-71.

[7] Cartlidge J.P., Weighted Mean Matrices as Operator on ℓ_p, *Ph.D. Dissertation*, Indiana University, 1978.

[8] Başar F., Durna N., Yildirim M., Subdivisions of the Spectra for Generalized Difference Operator over Certain Sequence Spaces, *Thai J. Math*. 9(2) (2011), 285-295.

[9] Durna N, Yıldırım M., Subdivision of the spectra for factorable matrices on c_0, *GUJ Sci*, 24(1) (2011) 45-49.

[10] Durna N, Yıldırım M., Subdivision of the spectra for factorable matrices on c and ℓ_p, *Math. Commun*., 16 (2011) 519-530.

[11] Durna, N. , Yıldırım, M., Kılıç, R., Partition of the Spectra for the Generalized Difference Operator B(r,s) on the Sequence Space cs,. *Cumhuriyet Sci. J.*, 39 (1) (2018), 7-15.

[12] Akhmedov A.M., El-Shabrawy S.R., Spectra and Fine Spectra of Lower Triangular Double-Band Matrices as Operators on L_p $(1 \leq p < \infty)$, *Mathematica Slovaca* 65(5) (2015) 1137-1152.

[13] Bilgiç H., Furkan H., On the fine spectrum of the generalized difference operator $B(r,s)$ over the sequence spaces ℓ_p and bv_p $(1 < p < \infty)$, *Nonlinear Anal*. 68(3) (2008) 499-506.

[14] Coskun C., The spectra and fine spectra for p-Cesàro operators, *Turkish J. Math.,* 21 (1997) 207-212.

[15] Das R., On the spectrum and fine spectrum of the lower triangular matrix $B(\pm r, \pm s)$ over the sequence space c_0, *International Journal of Mathematical Archive* 6(1) (2015) 229-240.

[16] Das R., On The fine spectra of the lower triangular matrix $\Delta(r,s)$ over the sequence space ℓ_1,

Internat. J. Functional Analysis, Operator Theory and Applications, 7(1) (2015) 1-18.

[17] Fathi J., On the ne spectrum of generalized upper triangular double-band matrices uv over the sequence spaces c_0 and c, *Int. J. Nonlinear Anal. Appl*, 7(1) (2015) 31-43.

[18] Karakaya V., Manafov M., Şimşek N., On the fine spectrum of the second order difference operator over the sequence spaces ℓ_p and bv_p $(1 < p < \infty)$, *Mathematical and Computer Modelling* 55(3-4) (2012) 426-431.

[19] Rhoades B.E, Yıldırım M., Spectra and fine spectra for factorable matrices, *Integr.Equ. Oper. Theory*, 53 (2005) 127-144.

[20] El-Shabrawy S.R., Spectra and fine spectra of certain lower triangular double-band matrices as operators on c_0, *J. Inequal. Appl.*, 2014 (2014) 1-9.

[21] Srivastava P.D., Kumar S., Fine spectrum of the generalized difference operator Δ_{uv} on sequence space ℓ_1, *Appl. Math. Comput.*, 218(11) (2012) 6407-6414.

[22] Tripathy B.C., Das R., Spectrum and fine spectrum of the upper triangular matrix $U(r, s)$ over the sequence space cs, *Proyecciones Journal of Mathematics* 34(2) (2015) 107-125.

[23] Tripathy B.C., Das R., Spectrum and fine spectrum of the lower triangular matrix $B(r, 0, s)$ over the sequence space cs, *Appl. Math. Inf. Sci.* 9(4) (2015) 2139-2145.

[24] Yıldırım M., On the spectrum and fine spectrum of the compact Rhally operators, *Indian J. Pure Appl. Math.*,27(8) (1996) 779-784.

[25] Yıldırım M.E., The spectrum and fine spectrum of q-Cesaro matrices with $0 < q < 1$ on c_0, *Numerical functional analysis and optimization*, 41(3) (2020) 361-377.

[26] Wilansky A., Summability through Functional Analysis, *vol. 85 of North-Holland Mathematics Studies, North-Holland, The Netherlands, Amsterdam* 1984.

[27] Altun M., Fine Spectra of Tridiagonal Symmetric Matrices, *International Journal of Mathematics and Mathematical Sciences* (2011) 161209.