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Research Article

Characterizations of a helicoid and a catenoid

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Abstract

In the present article, we consider a parametric surface generated by the Frenet frame of a curve, and study the minimality condition for the surface. As a result, we give characterizations of a helicoid and a catenoid. Finally we show some examples of minimal surfaces generated by a circle and a helix.

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1. Introduction

Minimal surfaces are one of main objects which have drawn geometers' interest for a very long time. A minimal surface is a surface with vanishing mean curvature. It is well known that the only minimal ruled surfaces in Euclidean 3-space $\mathbb{E}^3$ are planes and helicoids. Also, a plane and a catenoid are the only minimal surfaces of revolution in $\mathbb{E}^3$. Minimal surfaces have been studied in many research areas. In mathematics, the surfaces have wide applications in a surface design [1,4–7]. In physics, minimal surfaces are familiar as soap films. Besides the obvious application of a minimal surface theory to the study of soap films, there are a number of other physical systems in which the theory of minimal surfaces has a sometimes surprising applicability. The study of minimal surfaces generated by the its Frenet frame and a space curve appear attractive and is used many areas. In [4] Li, Wang and Zhu gave examples for approximation of minimal surface with a geodesic by using Dirichlet function. Also, in [6] author examined construction method of a minimal surface from a prescribed geodesic and drew minimal surfaces with a circle or a helix. Moreover, Riverros and Corro [5] analyzed the class of minimal surfaces parameterized by an isothermal coordinate and a geodesic coordinate. Several mathematician are studying minimal surfaces generated by a curve [2–6, 10], etc.

In this paper, we give minimal conditions of a parametric surface defined by the Frenet frame of a curve in terms of the marching-scale functions. Also, we present a new approach for obtaining minimal surfaces from a curve and give new examples of minimal surfaces. Finally, we characterize a helicoid and a catenoid generated by a circle and a helix in Euclidean 3-space.

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2. Conditions of minimal surfaces

Let γ be a curve parameterized by arc-length s in Euclidean 3-space $\mathbb{E}^3$. Denote by $\{T, N, B\}$ the Frenet frame of a curve γ with the curvature κ and the torsion τ.

Consider a parametric surface generated by the curve γ and its Frenet frame as following

$$X(s,t) = \gamma(s) + (f(s,t)\ g(s,t)\ h(s,t)) \begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix}, \qquad s_1 \le s \le s_2, \quad t_1 \le t \le t_2, \tag{2.1}$$

where $f(s,t), g(s,t)$ and $h(s,t)$ are smooth functions.
If we take the parameter t as time variable, then $f(s,t), g(s,t)$ and $h(s,t)$ can be viewed as directed marching distances of a point unit at the time t in the directions $T(s)$, $N(s)$ and $B(s)$, respectively. In the sense , $f(s,t), g(s,t)$ and $h(s,t)$ are said to be the marching-scale functions in the directions, respectively [8].

Some known simple examples are to be mentioned, namely

(1) If the marching-scale functions $f(s,t), g(s,t)$ and $h(s,t)$ are linear functions with the only parameter t, then the parametric surface $X(s,t)$ is a ruled surface.
(2) If γ is a circle and $f(s,t) = 0, g(s,t) = \tilde{g}(t), h(s,t) = \tilde{h}(t)$, then the surface $X(s,t)$ is a usual surface of revolution.
(3) If the marching-scale functions are given by $f(s,t) = 0, g(s,t) = r_0 \cos t, h(s,t) = r_0 \sin t$ with a constant r_0, then the surface is a tubular surface.

Definition 2.1. If $X(s,t)$ satisfies $E = G$ and $F = 0$, then $X(s,t)$ is called an isothermal surface, where E, F and G denote the coefficients of the first fundamental form of a surface $X(s,t)$.

Definition 2.2. If $X(s,t)$ satisfies $\frac{\partial^2 X}{\partial s^2} + \frac{\partial^2 X}{\partial t^2} = 0$, then $X(s,t)$ is called a harmonic surface.

Lemma 2.3. (cf. [9]) *The surface with an isothermal parameter is minimal if and only if it is a harmonic surface.*

For the future analysis of a parametric surface, we now consider the marching-scale functions $f(s,t)$, $g(s,t)$ and $h(s,t)$ expressed by

$$f(s,t) = l(s) + u(t), \quad g(s,t) = m(s) + v(t), \quad h(s,t) = n(s) + w(t), \tag{2.2}$$

where $l(s), m(s), n(s), u(t), v(t), w(t)$ are smooth functions. In this case, the surface (2.1) does not pass through the curve $\gamma(s)$.

The following theorem is useful to construct minimal surfaces of a parametric surface $X(s,t)$ with the marching-scale functions given as (2.2).

Theorem 2.4. *Let γ be a unit speed curve with the Frenet frame $\{T, N, B\}$ in Euclidean 3-space. A surface parameterized by*

$$X(s,t) = \gamma(s) + f(s,t)T(s) + g(s,t)N(s) + h(s,t)B(s) \tag{2.3}$$

with the marching-scale functions f, g and h given by (2.2) *is minimal if and only if the functions f, g and h satisfy the following conditions:*

$$\begin{aligned} &[1 + l'(s) - \kappa(s)(m(s) + v(t))]^2 + [m'(s) + \kappa(s)(l(s) + u(t)) - \tau(s)(n(s) + w(t))]^2 \\ &\quad + [n'(s) + \tau(s)(m(s) + v(t))]^2 - u'^2(t) - v'^2(t) - w'^2(t) = 0, \end{aligned} \tag{2.4}$$

$$\begin{aligned} &u'(t)[1 + l'(s) - \kappa(s)(m(s) + v(t))] \\ &\quad + v'(t)[m'(s) + \kappa(s)(l(s) + u(t)) - \tau(s)(n(s) + w(t))] \\ &\quad + w'(t)[n'(s) + \tau(s)(m(s) + v(t))] = 0, \end{aligned} \tag{2.5}$$

$$l''(s) + u''(t) - \kappa'(s)(m(s) + v(t)) - 2\kappa(s)m'(s) - \kappa^2(s)(l(s) + u(t)) + \kappa(s)\tau(s)(n(s) + w(t)) = 0, \tag{2.6}$$

$$m''(s) + v''(t) + \kappa'(s)(l(s) + u(t)) - \tau'(s)(n(s) + w(t)) + 2\kappa(s)l'(s) - 2\tau(s)n'(s) - (\kappa^2(s) + \tau^2(s))(m(s) + v(t)) + \kappa(s) = 0, \tag{2.7}$$

$$n''(s) + w''(t) + \tau'(s)(m(s) + v(t)) + 2\tau(s)m'(s) + \kappa(s)\tau(s)[l(s) + u(t)] - \tau^2(s)[n(s) + w(t)] = 0, \tag{2.8}$$

where $\kappa(s)$ and $\tau(s)$ are the curvature and the torsion of the curve $\gamma(s)$, respectively.

Proof. After computations of the first fundamental form and the second derivative of the surface (2.3), if we apply the conditions of the isothermal surface and the harmonic surface, equations (2.4)–(2.8) are obtained. □

If we are able to solve the system of ordinary differential equations, we can find the minimal surface generated by a curve. But it is not easy for us to find exact solutions satisfying (2.4)–(2.8) for minimal surfaces. So we will consider partial solutions in terms of the curvature $\kappa(s)$ and the torsion $\tau(s)$ of the curve $\gamma(s)$.

3. Minimal surfaces generated by a circle

Let γ be a unit speed curve in Euclidean 3-space and X be a minimal surface parameterized by

$$X(s,t) = \gamma(s) + f(s,t)T(s) + g(s,t)N(s) + h(s,t)B(s), \tag{3.1}$$

where f, g and h satisfy (2.2).

Suppose that the curve γ is a circle with $\kappa = 1$ and $\tau = 0$. Then, (2.8) implies

$$n''(s) + w''(t) = 0.$$

Since $n = n(s)$ and $w = w(s)$, it follows that there exists a constant c_1 such that

$$n''(s) = c_1, \quad w''(t) = -c_1,$$

that is,

$$n(s) = \frac{1}{2}c_1 s^2 + c_2 s + c_3, \qquad w(t) = -\frac{1}{2}c_1 t^2 + c_4 t + c_5, \tag{3.2}$$

where c_i $(i = 1, ..., 5)$ are constants. Also, (2.7) gives

$$m''(s) + 2l'(s) - m(s) + v''(t) - v(t) + 1 = 0,$$

which implies that there is a constant b_1 such that

$$m''(s) + 2l'(s) - m(s) = b_1, \qquad v''(t) - v(t) + 1 = -b_1. \tag{3.3}$$

Also, equation (2.6) can be rewritten as

$$l''(s) - 2m'(s) - l(s) + u''(t) - u(t) = 0,$$

it follows that there is a constant a_1 such that

$$l''(s) - 2m'(s) - l(s) = a_1, \qquad u''(t) - u(t) = -a_1. \tag{3.4}$$

The solutions of the second equations of (3.3) and (3.4) are

$$v(t) = b_2 e^t + b_3 e^{-t} + 1 + b_1, \qquad u(t) = a_2 e^t + a_3 e^{-t} + a_1 \tag{3.5}$$

for constants a_i and b_i, $i = 1, 2, 3$, respectively.

After taking the second derivative of the first equation of (3.3) and the first derivative of the first equation of (3.4) if we combine the two equations, then one finds

$$m^{(4)}(s) + 3m''(s) + 2l'(s) = 0.$$

Thus, the last equation with the help of the first equation of (3.3) becomes

$$m^{(4)}(s) + 2m''(s) + m(s) + b_1 = 0,$$

and its solution is given by

$$m(s) = (d_1 + d_2 s)\cos s + (d_3 + d_4 s)\sin s - b_1, \tag{3.6}$$

where d_i $(i = 1, ..., 4)$ are constants. Applying the same method in (3.3) and (3.4) for a function $l(s)$, one finds

$$l^{(4)}(s) + 2l''(s) + l(s) + a_1 = 0,$$

it follows that its general solution is

$$l(s) = (d_5 + d_6 s)\cos s + (d_7 + d_8 s)\sin s - a_1, \tag{3.7}$$

where d_i $(i = 5, ..., 8)$ are constants.

If we substitute (3.6) and (3.7) into the first equations of (3.3) and (3.4), we get

$$d_5 = -d_3,\ d_6 = -d_4,\ d_7 = d_1,\ d_8 = d_2,\ a_1 = 0,\ b_1 = 0, \tag{3.8}$$

it follows that the functions $m(s)$ and $l(s)$ can be written as

$$\begin{aligned} m(s) &= (d_1 + d_2 s)\cos s + (d_3 + d_4 s)\sin s, \\ l(s) &= -(d_3 + d_4 s)\cos s + (d_1 + d_2 s)\sin s. \end{aligned} \tag{3.9}$$

Now, we must check that the marching-scale functions determined by (2.2) satisfy (2.4) and (2.5). If we first substitute (3.2), (3.5) and (3.9) into (2.5), we get the following equations to be satisfied:

$$\begin{aligned} c_1 &= 0, \\ c_2 c_4 + 2a_3 b_2 - 2a_2 b_3 &= 0 \end{aligned} \tag{3.10}$$

as the coefficients of st term and constant term, respectively, and we also obtain

$$\begin{aligned} b_2 d_2 + a_2 d_4 &= 0, \\ a_2 d_2 - b_2 d_4 &= 0, \end{aligned} \tag{3.11}$$

$$\begin{aligned} b_3 d_2 + a_3 d_4 &= 0, \\ a_3 d_2 - b_3 d_4 &= 0, \end{aligned} \tag{3.12}$$

$$\begin{aligned} a_2(d_2 + 2d_3) + b_2(2d_1 - d_4) &= 0, \\ a_2(-2d_1 + d_4) + b_2(d_2 + 2d_3) &= 0, \end{aligned} \tag{3.13}$$

$$\begin{aligned} a_3(d_2 + 2d_3) + b_3(2d_1 - d_4) &= 0, \\ a_3(-2d_1 + d_4) + b_3(d_2 + 2d_3) &= 0 \end{aligned} \tag{3.14}$$

because the coefficients of the exponential function and the trigonometric function are all zero.

In order to solve the system (3.11)−(3.14), we split it into two cases.

Case 1: $a_2 \neq 0$ or $b_2 \neq 0$.

In this case, (3.11) implies $d_2 = 0$ and $d_4 = 0$. It follows that from (3.13) we also obtain $d_1 = 0$ and $d_3 = 0$. Therefore, the functions $m(s)$ and $l(s)$ are identically zero. Thus, (2.4) leads to the condition that the constants satisfy

$$4a_2 a_3 + 4b_2 b_3 + c_2^2 - c_4^2 = 0.$$

Case 2: $a_2 = 0$ and $b_2 = 0$.

In the case, the second equation of (3.10) gives $c_2c_4 = 0$ and (3.11) implies that d_2 and d_4 are arbitrary constants. It follows that from (3.12) one finds $a_3 = 0$ and $b_3 = 0$. Equation (2.4) with the help of $a_2 = 0$, $a_3 = 0$, $b_2 = 0$ and $b_3 = 0$ gives

$$d_2 = 0,\ d_4 = 0,$$
$$4d_1^2 + 4d_3^2 + c_2^2 - c_4^2 = 0,$$

It follows that the marching-scale functions are reduced to

$$\begin{aligned} f(s,t) &= d_1 \sin s - d_3 \cos s, \\ g(s,t) &= d_1 \cos s + d_3 \sin s + 1, \\ h(s,t) &= c_2 s + c_4 t + c_3 + c_5, \end{aligned} \tag{3.15}$$

and thus c_4 must be a non-zero constant. Since $c_2c_4 = 0$, one find $c_2 = 0$. In such a case, the coefficients of the first fundamental form of the surface are given by $E = 0, F = 0$ and $G = c_4^2$. Therefore, there exist no surfaces for Case 2.

Consequently, since the functions $m(s)$ and $l(s)$ vanish, by renaming the constants we have following theorem.

Theorem 3.1. *Let γ be a circle parameterized by arc-length with radius 1 in Euclidean 3-space and let X be a regular surface parameterized by*

$$X(s,t) = \gamma(s) + f(s,t)T(s) + g(s,t)N(s) + h(s,t)B(s) \tag{3.16}$$

with $f(s,t) = l(s) + u(t), g(s,t) = m(s) + v(t)$ and $h(s,t) = n(s) + w(t)$. Then the surface X is minimal if and only if the marching-scale functions f, g and h are expressed in the form:

$$\begin{cases} f(s,t) &= a_1 e^t + a_2 e^{-t}, \\ g(s,t) &= b_1 e^t + b_2 e^{-t} + 1, \\ h(s,t) &= c_1 s + c_2 t + c_3, \end{cases} \tag{3.17}$$

where constants a_i, b_i, c_i $(i = 1, 2)$ satisfy the following equations:

$$\begin{gathered} c_1c_2 - 2a_1b_2 + 2a_2b_1 = 0, \\ 4a_1a_2 + 4b_1b_2 + c_1^2 - c_2^2 = 0, \\ (a_1, a_2) \neq (0,0),\ (b_1, b_2) \neq (0,0),\ (c_1, c_2) \neq (0,0). \end{gathered} \tag{3.18}$$

Example 3.2. Consider a circle with radius 1 on xy-plane and take

$$a_1 = 0, \quad a_2 = 1, \quad b_1 = -\frac{1}{2}, \quad b_2 = 0, \quad c_1 = 1, \quad c_2 = 1, \quad c_3 = 0$$

in Theorem 3.1. Then the minimal surface $X(s,t)$ with the help of (3.17) is parameterized as

$$X(s,t) = \left(\frac{1}{2}e^t \cos s - e^{-t} \sin s, \frac{1}{2}e^t \sin s + e^{-t} \cos s, s + t\right).$$

This surface is given in Figure 1.

By using Theorem 3.1 with the following theorem, we can characterize a catenoid and a helicoid as minimal surfaces.

Theorem 3.3. *Let $X(s,t)$ be a minimal surface determined by Theorem 3.1 with the marching-scale functions* (3.17)*. If $c_1 = 0$ and a_1, a_2, b_1, b_2 are nonzero constants with the relations*

$$a_1 = a_2 = \frac{c_2}{2} \cos\theta_0, \quad b_1 = b_2 = \frac{c_2}{2} \sin\theta_0 \tag{3.19}$$

for a constant θ_0, then the surface is part of a catenoid.

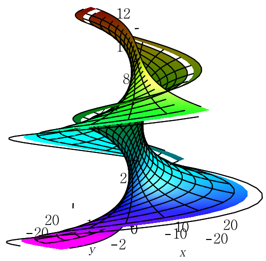

Fig. 1: A minimal surface generated by the circle with radius 1.

Proof. Since the relations in (3.18) are satisfied by (3.19), the marching-scale functions f, g and h are reduced to

$$\begin{aligned} f(s,t) &= c_2 \cos\theta_0 \cosh t, \\ g(s,t) &= c_2 \sin\theta_0 \cosh t + 1, \\ h(s,t) &= c_2 t + c_3. \end{aligned} \tag{3.20}$$

Suppose that $\gamma(s)$ is a unit circle in Euclidean 3-space. By a rigid motion, we consider γ parameterized by

$$\gamma(s) = (\cos s, \sin, 0)\,. \tag{3.21}$$

Thus, the minimal surface $X(s,t)$ with the help of (3.20) and (3.21) is expressed as

$$X(s,t) = (-c_2 \cosh t \sin(s+\theta_1), c_2 \cosh t \cos(s+\theta_1), c_2 t + c_3)$$

with a constant θ_1. By a rigid motion, the surface is obtained by rotating the curve $y = c_2 \cosh z$ in the yz-plane around the z-axis and it is a catenoid. Thus, the theorem is proved. □

Theorem 3.4. *Let $X(s,t)$ be a minimal surface determined by Theorem 3.1 with the marching-scale functions* (3.17)*. If $c_2 = 0$ and a_1, a_2, b_1, b_2 are nonzero constants with the relations*

$$a_1 = -a_2 = \frac{c_1}{2}\cos\theta_0, \quad b_1 = -b_2 = \frac{c_1}{2}\sin\theta_0 \tag{3.22}$$

for a constant θ_0, then the surface is part of a helicoid.

Proof. A similar computation as in Theorem 3.2 gives

$$X(s,t) = (-c_1 \sinh t \sin(s+\theta_1), c_1 \sinh t \cos(s+\theta_1), c_1 s + c_3)$$

with constant θ_1. It is a ruled surface and a helicoid. Thus, the theorem is proved. □

4. A minimal surface generated by a helix

We mentioned in Chapter 2 that it is difficult to find the exact solution of the system of the ordinary differential equations in Theorem 2.4. So, we want to find a partial solution of the system for a minimal surface in some special cases.
In this section, we consider a helix parameterized by

$$\gamma(s) = (\frac{1}{\sqrt{2}}\cos s, \frac{1}{\sqrt{2}}\sin s, \frac{1}{\sqrt{2}}s), \tag{4.1}$$

then the helix has the curvature $\kappa = \frac{1}{\sqrt{2}}$ and the torsion $\tau = \frac{1}{\sqrt{2}}$.

4.1. Case $f(s,t) = h(s,t)$

In this case, equation (2.8) implies

$$l''(s) + \sqrt{2}l'(s) + u''(t) = 0.$$

It follows that there exists a constant c_1 such that

$$\begin{aligned} l''(s) + \sqrt{2}l'(s) &= c_1, \\ u''(t) &= -c_1, \end{aligned}$$

and its general solutions of ODEs are

$$\begin{aligned} l(s) &= -\frac{\sqrt{2}}{2}d_1 e^{-\sqrt{2}s} + \frac{\sqrt{2}}{2}c_1 s + d_2, \\ u(t) &= -\frac{1}{2}c_1 t^2 + c_2 t + c_3, \end{aligned} \tag{4.2}$$

for some constants c_2, c_3, d_1, d_2, respectively. Also, equation (2.7) leads to

$$m''(s) - m(s) + v''(t) - v(t) + \frac{1}{\sqrt{2}} = 0,$$

from this, there exists a constant a_1 satisfying

$$\begin{aligned} m''(s) - m(s) &= a_1, \\ v''(t) - v(t) + \frac{1}{\sqrt{2}} &= -a_1. \end{aligned}$$

Then, its general solutions are given by

$$\begin{aligned} m(s) &= a_2 e^s + a_3 e^{-s} - a_1, \\ v(t) &= a_4 e^t + a_5 e^{-t} + a_1 + \frac{\sqrt{2}}{2}, \end{aligned} \tag{4.3}$$

for constants a_2, a_3, a_4, a_5, respectively. Substituting (4.2) and (4.3) into (2.6), we have

$$a_2 = 0,\ a_3 = 0,\ c_1 = 0,\ d_1 = 0.$$

From this, (2.5) implies $c_2 = 0$ and (2.4) also gives $8a_4a_5 + 1 = 0$. If $a_4 \neq 0$, we have the marching-scale functions in the form

$$\begin{aligned} f(s,t) &= d_2 + c_3 = c, \\ g(s,t) &= a_4 e^t - \frac{1}{8a_4}e^{-t} + \frac{\sqrt{2}}{2}. \end{aligned}$$

Thus, a surface (3.1) is parameterized as

$$X(s,t) = \left(-(a_4e^t - \frac{1}{8a_4}e^{-t})\cos s, -(a_4e^t - \frac{1}{8a_4}e^{-t})\sin s, \frac{1}{\sqrt{2}}s + \sqrt{2}c\right) \tag{4.4}$$

and it is a helicoid as a minimal ruled surface.

4.2. Case $f(s,t) = -h(s,t)$

By using the same method used in the previous part, equation (2.8) implies

$$\begin{aligned} l(s) &= -a_2 e^{\frac{\sqrt{6}-\sqrt{2}}{2}s} - a_3 e^{-\frac{\sqrt{6}+\sqrt{2}}{2}s} + a_1, \\ u(t) &= -a_4 e^t - a_5 e^{-t} - a_1, \end{aligned} \tag{4.5}$$

where a_i $(i = 1, ..., 5)$ are constants. Also, equation (2.7) gives

$$\begin{aligned} m(s) &= b_2 e^s + b_3 e^{-s} - 2a_2 e^{\frac{\sqrt{6}-\sqrt{2}}{2}s} - 2a_3 e^{-\frac{\sqrt{6}+\sqrt{2}}{2}s} - b_1, \\ v(t) &= b_4 e^t + b_5 e^{-t} + \frac{1}{\sqrt{2}} + b_1 \end{aligned} \tag{4.6}$$

for some constants b_i $(i = 1, ..., 5)$.
On the other hand, we can determine constants a_i and b_i $(i = 1, ..., 5)$ in (2.4), (2.5) and (2.6) with the help of (4.5) and (4.6), and they are given by

$$\begin{aligned} &a_2 = 0,\ a_3 = 0,\ b_2 = 0,\ b_3 = 0, \\ &a_4 b_5 - a_5 b_4 = 0,\ 2a_4 a_5 + b_4 b_5 = 0. \end{aligned} \tag{4.7}$$

• If $a_4 \neq 0$ and $b_4 \neq 0$, there exists a constant k such that $a_5 = ka_4$ and $b_5 = kb_4$. It follows that $k(2a_4^2 + b_4^2) = 0$, from this $k = 0$, that is, $a_5 = 0$ and $b_5 = 0$. In this case, the marching-scale functions f and g are reduced to

$$f(s,t) = -a_4 e^t,\ g(s,t) = b_4 e^t + \frac{1}{\sqrt{2}}.$$

Thus, a surface (3.1) is parameterized as

$$X(s,t) = \left(p_0 e^t \cos(s+\theta_0), p_0 e^t \sin(s+\theta_0), \frac{1}{\sqrt{2}} s\right)$$

for some constants p_0 and θ_0, and it is a helicoid.

• If $a_4 = 0$, equation (4.7) implies that either $b_4 = 0$ or $a_5 = 0$ and $b_5 = 0$. In both cases, we can show that a surface (3.1) is also a helicoid.

Thus, we have the following result.

Theorem 4.1. *Let γ be a helix given by* (4.1) *in Euclidean 3-space and let X be a surface parameterized by*

$$X(s,t) = \gamma(s) + f(s,t)T(s) + g(s,t)N(s) + \varepsilon f(s,t)B(s), \tag{4.8}$$

where $\varepsilon = \pm 1$ and $f(s,t) = l(s) + u(t)$, $g(s,t) = m(s) + v(t)$. If the surface X is minimal, it is part of a helicoid.

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