Cumhuriyet Sci. J., 43(1) (2022) 82-87
DOI: https://doi.org/10.17776/csj.873398

Cumhuriyet Science Journal

| csj.cumhuriyet.edu.tr | Founded: 2002 ISSN: 2587-2680 e-ISSN: 2587-246X Publisher: Sivas Cumhuriyet University

On Involutes of Admissible Non-Lightlike Curves in Pseudo-Galilean 3-Space

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Research Article

History
Received: 02/02/2021
Accepted: 30/01/2022

ABSTRACT

This paper aims to investigate the theory of involutes of admissible non-lightlike curves in pseudo-Galilean 3-space. In the second section of this paper, we give fundamental concepts of pseudo-Galilean 3-space and curves over this space together with their casual properties. In section three, the involute of admissible non-lightlike curves in pseudo-Galilean 3-space is defined. Furthermore, the properties of involutes of admissible non-lightlike curves are also investigated by applying the fundamental properties provided in section 2. In the last part but not least, we give some numerical examples as applications of the theorems and corollaries which are derived in the previous section.

Keywords: Involutes, Admissible curve, Non-lightlike, Pseudo-Galilean.

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Introduction

In gravitational theory, one has inevitably to introduce some kind of hypothesis concerning the properties of space as a whole. The simplest and at the same time the most important of these hypotheses is the assumption that, at infinity, space is Galilean. Both properties of uniformity and the whole theory of Galilean space may be formulated in arbitrary coordinates. However, the privileged character of Galilean coordinates manifests itself and in particular simplicity (linearity) of the transformations that relate to the inertial coordinate system, that class of system within which the physical principle of relativity holds. Pseudo-Galilean space possesses a pseudo-Euclidean metric [6]. In pseudo-Galilean space, there are three types of curves, namely spacelike, timelike and null (lightlike) curves. The theory of curves and surfaces in pseudo-Galilean space can be seen in [2, 3, 4, 5, 7, 10, 11, 12, 13, 14].

The theory of involute and evolute was firstly introduced by C. Huygens in 1673 when he tried to build an accurate clock called the isochronous pendulum clock. He found that the isochronous curve is an arc of cycloid and that involute of cycloid [9]. In classical differential geometry, a curve $\alpha^*(s)$is called an involute of $\alpha(s)$ if it is lying in the tangent surface of $\alpha(s)$ and their tangent lines are perpendicular in all points on the curves. The theory of involutes of curves in Euclidean space has been provided in many books and articles. In this article the author refers to [1, 7, 8].

We organized our present work as follows: after the introduction in Section 1, we provide the basic properties of curves in pseudo-Galilean space in the second section. Then, we define and investigate the properties of the involutes of admissible non-lightlike curves in pseudo-Galilean space G_1^3 in the next section. To close this paper, we give numerical examples of admissible non-lightlike curves in pseudo-Galilean space with their properties.

Preliminaries

The Pseudo-Galilean geometry is one of the real Cayley-Klein spaces whose projective signature $(0,0,+,+)$ (see [10]). The absolute of the pseudo-Galilean geometry is an ordered triple $\{\omega, f, I\}$, where ω is the ideal (absolute) plane, f the line in ω and I the fixed elliptic involution of the points of f.

In appropriate affine coordinates for points and vectors (point pairs), the group B_6

$$\bar{x} = a + x,$$
$$\bar{y} = b + dx + y\cosh\varphi + z\sinh\varphi,$$
$$\bar{z} = c + ex + y\sinh\varphi + z\cosh\varphi$$

of pseudo-Galilean proper notions will preserve the absolute. Let the group

$$\overline{B_6} \coloneqq \left\langle B_6, \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right\rangle$$

be called the motion group of the pseudo-Galilean space G_3^1. The motion group $\overline{B_6}$ leaves invariant the absolute figure and defines the other invariants of this geometry.

In the affine coordinates, the group $\overline{B_6}$ acts as follows

$$\bar{x} = a + x,$$
$$\bar{y} = b + dx + y\eta\cos\varphi + z\eta\sin\varphi,$$
$$\bar{z} = d + ex + y\eta\sin\varphi + z\eta\cos\varphi,$$

where η is $+1$ or -1.

Distance between two distinct proper points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is given by

$$d(P,Q) = \begin{cases} |x_2 - x_1| & if\ x_1 \neq x_2, \\ \sqrt{|(y_2 - y_1)^2 - (z_2 - z_1)^2} & if\ x_1 = x_2. \end{cases}$$

According to the group $\overline{B_6}$ there are non-isotropic and isotropic vectors. A vector $\boldsymbol{v}(x, y, z)$ is called non-isotropic if $x \neq 0$ and its unit vector can be expressed in $(1, y, z)$ form. On the other hand, vector $\boldsymbol{v}(x, y, z)$ is called isotropic if $x = 0$. In the case of isotropic vector, there are for types of vectors: spacelike $(y^2 - z^2 > 0)$, timelike $(y^2 - z^2 < 0)$ and two types of lightlike $(y = \pm z)$. A non-lightlike vector is a unit vector if $y^2 - z^2 = \pm 1$.

Scalar product of two vectors $\boldsymbol{v}(x_1, y_1, z_1)$ and $\boldsymbol{w}(x_2, y_2, z_2)$ in the pseudo-Galilean 3-space is defined by

$$g(\boldsymbol{v}, \boldsymbol{w}) = \begin{cases} x_1 x_2 & if\ x_1 \neq 0\ or\ x_2 \neq 0, \\ y_1 y_2 - z_1 z_2 & if\ x_1 = 0\ and\ x_2 = 0. \end{cases}$$

The norm of P is given by

$$\|x\| = \begin{cases} |x_1| & if\ x_1 \neq 0, \\ \sqrt{|y_1^2 - z_1^2|} & if\ x_1 = 0. \end{cases}$$

The vector product of $\boldsymbol{v}$ and $\boldsymbol{w}$ in G_3^1 is defined by

$$\boldsymbol{v} \times \boldsymbol{w} = \begin{vmatrix} 0 & -e_2 & e_3 \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$$

where $e_2 = (0, 1, 0)$ and $e_3 = (0,0,1)$.

A curve $\alpha(t) = \big(x(t), y(t), z(t)\big)$ is said admissible if it has no inflection points ($\alpha' \times \alpha'' \neq 0$), no isotropic tangent ($x' = 0$) or normal whose projections on the absolute plane would be lightlike vectors ($y'' = \pm z''$). For an admissible curve $\alpha(t)$, the curvature $\kappa(t)$ and the torsion $\tau(t)$ are defined by

$$\kappa(t) = \frac{\sqrt{\left|\big(y''(t)\big)^2 - \big(z''(t)\big)^2\right|}}{\big(x'(t)\big)^2},$$

$$\tau(t) = \frac{y''(t)z'''(t) - y'''(t)z''(t)}{\kappa^2(t)}.$$

An admissible curve $\alpha: I \subset \mathbb{R} \to G_3^1$ which is parametrized by arc length s can be written the form of

$$\alpha(s) = \big(s, y(s), z(s)\big).$$

Then, the curvature and torsion of $\alpha(s)$ are given by

$$\kappa(s) = \sqrt{\left|\big(y''(s)\big)^2 - \big(z''(s)\big)^2\right|},$$

$$\tau(s) = \frac{y''(s)z'''(s) - y'''(s)z''(s)}{\kappa^2(s)}.$$

The orthonormal trihedron of non-null cure $\alpha(s)$ in pseudo-Galilean 3-space is given by

$$T(s) = \alpha'(s), \qquad N(s) = \frac{\alpha''(s)}{\kappa(s)},$$
$$B(s) = \frac{\big(0, \epsilon z''(s), \epsilon y''(s)\big)}{\kappa(s)}$$

where $\epsilon = +1$ if $\alpha(s)$ is a spacelike curve and $\epsilon = -1$ if $\alpha(s)$ is a timelike curve. Here T, N, B are called the tangent, principal normal, and binormal vector fields of α, respectively. Indeed, curve $\alpha(s)$ is spacelike (resp. timelike) if $N(s)$ is spacelike (resp. timelike) vectors. Furthermore, the Frenet formulas of the curve are given by

$$T'(s) = \kappa(s)N(s), \qquad N'(s) = \tau(s)B(s),$$
$$B'(s) = \tau(s)N(s).$$

(see [2, 3, 11, 13].)

Involutes of Admissible Curves in Pseudo-Galilean Space

In this section, we will define the involute of admissible curves in pseudo-Galilean 3-space and investigate the casual properties of the involute of admissible non-lightlike curves in pseudo-Galilean 3-space.

Definition 3.1. Let $\alpha: I \subset \mathbb{R} \to G_3^1$ and $\beta: I \subset \mathbb{R} \to G_3^1$ be curves in pseudo-Galilean space. For all $s \in I, \beta(s)$ is called the involute of $\alpha(s)$ if and only if the tangent of α at the point $\alpha(s)$ passes through the tangent at the point $\beta(s)$ of the curve β and

$$g(\bar{T}, T) = 0,$$

where T and $\bar{T}$ are the tangent of curves α and β, respectively.

Theorem 3.1. Let $\beta(s)$ be the involute of an admissible non-lightlike curve $\alpha(s)$ parametrized by arc length s and c be a constant real number. Then,

$$\beta(s) = \alpha(s) + (c - s)T(s) \tag{1}$$

Proof.

Let $\alpha(s)$ be an admissible non-lightlike curve in pseudo-Galilean G_3^1 space. The tangent line of curve $\alpha(s)$ will construct a tangent surface. If $\beta(s)$ is the involute of $\alpha(s)$, then $\beta(s)$ lies on the tangent surface and is orthogonal to all tangent line of $\alpha(s)$. Suppose $\bar{p}$ be the point of $\beta(s)$ which crosses the tangent line $T(s)$ of $\alpha(s)$ at point p. Then, $\bar{p} - p$ is proportional to $T(s)$. Consequently, $\beta(s)$ can be expressed in the form of

$$\beta(s) = \alpha(s) + \lambda(s)T(s). \tag{2}$$

Taking the derivative of (2) yields

$$\beta'(s) = \alpha'(s) + \lambda'(s)T(s) + \lambda(s)\kappa(s)N(s)$$

$= T(s) + \lambda'(s)T(s) + \lambda(s)\kappa(s)N(s)$

$= \big(1 + \lambda'(s)\big)T(s) + \lambda(s)N(s).$

Consequently, by definition 3.1, we have

$$g(\beta'(s), T(s)) = \big(1 + \lambda'(s)\big)g\big(T(s), T(s)\big) + \lambda(s)\kappa(s)\big(N(s), T(s)\big) = 1 + \lambda'(s) = 0. \quad (3)$$

Integrating (3) gives $\lambda(s) = -s + c$, where c is the real constant. Thus, by (1) there exist an infinite family of involutes of $\alpha(s)$ given by

$$\beta(s) = \alpha(s) + (c - s)T(s). \quad (4)$$

Theorem 3.2. Let $\beta(s)$ be the involute of an admissible non-lightlike curve $\alpha(s)$ parametrized by arc-length s in pseudo-Galilean G_3^1. Suppose $\{T(s), N(s), B(s)\}$ and $\{\bar{T}(s), \bar{N}(s), \bar{B}(s)\}$ are the orthonormal trihedron of the curve $\alpha(s)$ and $\beta(s)$ respectively. Then,

$$\begin{bmatrix} \bar{T}(s) \\ \bar{N}(s) \\ \bar{B}(s) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix} \quad (5)$$

Proof.
First, taking the derivative of (1) gives

$$\beta'(s) = T(s) - T(s) + (c - s)\kappa(s)N(s) = (c - s)\kappa(s)N(s).$$

So that,

$$\bar{T} = \frac{(c - s)\kappa(s)N(s)}{\|(c - s)\kappa(s)N(s)\|} = \frac{(c - s)\kappa(s)N(s)}{|(c - s)\kappa(s)|} = \pm N(s).$$

Since $\bar{T}$ and N are both unit vectors, then we can assume that

$$\bar{T}(s) = N(s). \quad (6)$$

Taking the derivative of (6) gives

$$\bar{T}'(s) = \tau(s)B(s),$$

and

$$\|\bar{T}(s)\| = |\tau(s)|. \quad (7)$$

So that

$$\bar{N}(s) = \frac{\tau(s)B(s)}{|\tau(s)|} = \pm B(s).$$

Again since $\bar{N}$ and B are both unit vectors, then we can assume that

$$\bar{N}(s) = B(s). \quad (8)$$

Since $\alpha(s)$ is an admissible curve, then it can be expressed as $\alpha(s) = \big(s, y(s), z(s)\big)$. As a consequence, we have

$$\bar{B}(s) = \bar{T}(s) \times \bar{N}(s) = N(s) \times B(s) = \begin{vmatrix} 0 & -e_2 & e_3 \\ 0 & \frac{y''(s)}{\kappa(s)} & \frac{z''(s)}{\kappa(s)} \\ 0 & \frac{\epsilon z''(s)}{\kappa(s)} & \frac{\epsilon y''(s)}{\kappa(s)} \end{vmatrix} \quad (9)$$

Remark 3.1. The theorem above shows that binormal vector of the curve $\beta(s)$ vanishes which imply that $\beta(s)$ is a plane curve.

Theorem 3.3. Let $\beta(s)$ be the involute of an admissible non-lightlike curve $\alpha(s)$ parametrized by arc-length s in pseudo-Galilean G_3^1. Suppose κ and τ are the curvature and torsion of α, respectively while $\bar{\kappa}$ and $\bar{\tau}$ are the curvature ad torsion of β, respectively, then

$$\bar{\kappa}(s) = |\tau(s)|, \quad \tau(s) = 0. \quad (10)$$

Proof.
It is easy to see that

$$\bar{\kappa}(s) = \|\bar{T}'(s)\| = |\tau(s)|.$$

Since $\beta(s)$ might be a curve not parametrized by arc length so as in the preliminary part,

$$\bar{\tau} = \frac{\det\big(\beta'(s), \beta''(s), \beta'''(s)\big)}{|\beta'|^5 \ \bar{\kappa}^2(s)}.$$

From the proof of theorem 3.2 we have

$$\beta' = (c - s)\kappa N.$$

Taking the derivative of $\beta'(s)$ twice again yields

$$\beta''(s) = \big((c - s)\kappa\big)' N + (c - s)\kappa\tau B = \Big(\big((c - s)\kappa\big)'' + (c - s)\kappa\tau^2\Big)N + \Big(\big((c - s)\kappa\big)'\tau + \big((c - s)\kappa\tau\big)'\Big)B.$$

So that

$$\beta'' \times \beta''' = -(c - s)\kappa\tau\Big(\big((c - s)\kappa\big)'' + (c - s)\kappa\tau^2\Big)T + \big((c - s)\kappa\big)'\Big(\big((c - s)\kappa\big)'' + (c - s)\kappa\tau^2\Big)T + \big((c - s)\kappa\big)'T$$

This implies

$$\det(\beta', \beta'', \beta''') = \beta' \cdot (\beta'' \times \beta''') = 0$$

Since $g(T, N) = 0$. Consequently, $\bar{\tau} = 0$.

Corollary 3.1. Let $\beta(s)$ be the involute of an admissible non-lightlike curve $\alpha(s)$ parametrized by arc-length s in pseudo-Galilean G_3^1, then $\beta(s)$ is a plane curve.

Corollary 3.2. Let $\alpha(s)$ be an admissible non-lightlike curve parametrized by arc length s in pseudo-Galilean G_3^1. $\alpha(s)$ has no involute if it is a plane curve.

Corollary 3.3. Let $\beta(s)$ be the involute of an admissible non-lightlike curve $\alpha(s)$ parametrized by arc length s in pseudo-Galilean G_3^1. If $\alpha(s)$ is a plane curve with constant torsion, then $\beta(s)$ is a circle with radius $r=\frac{1}{|\tau(s)|}$.

Theorem 3.4. Involute of the admissible curve parametrized by arc length s in pseudo-Galilean space is not admissible.

Proof.

Let $\alpha(s)$ be an admissible non-lightlike curve parametrized by arc length s in pseudo-Galilean space. Therefore, $\alpha(s)$ can be written as

$$\alpha(s)=\big(s,y(s),z(s)\big).$$

Then, by (1) we have

$$\begin{aligned}\beta(s)&=\alpha(s)+(c-s)T(s)\\&=\big(s,y(s),z(s)\big)+(c-s)\big(1,y'(s),z'(s)\big)\\&=\big(c,y(s)-(c-s)y(s),z(s)-(c-s)z'(s)\big).\end{aligned}$$

Taking the first derivative of the last equation above yields zero in the first component. Hence, its tangent vector is isotropic, and it implies the curve $\beta(s)$ is a non-admissible curve.

Theorem 3. 5.

The involute of an admissible spacelike curve in pseudo-Galilean space is a timelike curve and the involute of the timelike curve in pseudo-Galilean space is a spacelike curve.

Proof.

Let $\alpha(s)$ be an admissible spacelike curve parametrized by arc length s in pseudo-Galilean space and expressed by $\alpha(s)=\big(s,y(s),z(s)\big)$. Then, the principal normal vector field $N(s)$ of $\alpha(s)$ is spacelike and $\big(y''(s)\big)^2-\big(z''(z)\big)^2>0$. It implies the binormal vector field $B(s)$ of $\alpha(s)$ become timelike since $\big(\epsilon z''(s)\big)^2-\big(\epsilon y''(s)\big)^2=\big(z''(s)\big)^2-\big(y''(s)\big)^2<0$. By equation (5) we have $\bar{N}(s)=B(s)$ which means that the principal normal vector field $\bar{N}(s)$ of $\beta(s)$ is also timelike. Hence, by definition $\beta(s)$ is a timelike curve. In the same way, if we set $\alpha(s)$ be timelike curve parametrized in pseudo-Galilean space then its involute will be spacelike.

Numerical Examples

Example 4.1. Let $\alpha: I\subset\mathbb{R}\to G_3^1$ be an admissible non-lightlike curve parametrized by arc length s in pseudo-Galilean space and defined by

$$\alpha(s)=(s,\cosh s,\sinh s).$$

Taking the derivative of $\alpha(s)$ three times yields

$$\begin{aligned}\alpha'(s)&=(1,\sinh s,\cosh s),\\\alpha''(s)&=(0,\cosh s,\sinh s),\\\alpha'''(s)&=(0,\sinh s,\cosh s).\end{aligned}$$

Since $\big(\alpha''(s),\alpha''(s)\big)=\cosh^2 s-\sinh^2 s=1>0$ then $\alpha(s)$ is a spacelike curve. The curvature and torsion of $\alpha(s)$ are given by

$$\kappa(s)=\sqrt{\left|\big(y''(s)\big)^2-\big(z''(z)\big)^2\right|}=\sqrt{|\cosh^2 s-\sinh^2 s|}=1$$

$$\tau(s)=\frac{y''(s)z'''(z)-y'''(s)z''(s)}{\kappa^2(s)}=\frac{\cosh^2 s-\sinh^2 s}{1}=1$$

and the orthonormal trihedron of $\alpha(s)$ are

$$T(s)=\alpha'(s)=(1,\sinh s,\cosh s)$$

$$N(s)=\frac{1}{\kappa(s)}\alpha''(s)=(0,\cosh s,\sinh s)$$

$$B(s)=\frac{1}{\kappa(s)}\big(0,\epsilon z''(s),\epsilon\, y''(s)\big)=(0,\sinh s,\cosh s).$$

Note that $\epsilon=1$ since $\alpha(s)$ is a spacelike curve.

Consequently, the involute of the curve $\alpha(s)$ is given by

$$\begin{aligned}\beta(s)&=\alpha(s)+(c-s)T(s)\\&=(s,\cosh s,\sinh s)+(c-s)(1,\sinh s,\cosh s)\\&=\begin{pmatrix}c,\cosh s+(c-s)\sinh s,\sinh s+\\(c-s)\cosh s\end{pmatrix}.\end{aligned}$$

The orthogonal dihedron of $\beta(s)$ are

$$\bar{T}(s)=N(s)=(0,\cosh s,\sinh s)$$

$$\bar{N}(s)=B(s)=(0,\sinh s,\cosh s)$$

And the curvature of $\beta(s)$ is $\bar{\kappa}(s)=|\tau(s)|=1$.

Example 4.2. Let $\gamma: I\subset\mathbb{R}\to G_3^1$ be an admissible non-lightlike curve parametrized by arc length s in pseudo-Galilean space and defined by

$$\gamma(s)=\left(s,\frac{s^5}{80}+\frac{1}{2s},\frac{s^5}{80}-\frac{1}{2s}\right).$$

Taking the derivative of $\gamma(s)$ three times yields

$$\begin{aligned}\gamma'(s)&=\left(1,\frac{s^4}{16}-\frac{1}{2s^2},\frac{s^4}{16}+\frac{1}{2s^2}\right),\\\gamma''(s)&=\left(0,\frac{s^3}{4}+\frac{1}{s^3},\frac{s^3}{4}-\frac{1}{s^3}\right),\\\gamma'''(s)&=\left(0,\frac{3s^2}{4}-\frac{3}{2s^4},\frac{3s^2}{4}+\frac{3}{2s^4}\right).\end{aligned}$$

Since

$$\big(\gamma''(s),\gamma''(s)\big)=\left(\frac{s^3}{4}+\frac{1}{s^3}\right)^2-\left(\frac{s^3}{4}-\frac{1}{s^3}\right)^2=1>0,$$

then $\alpha(s)$ is a spacelike curve. The curvature and torsion of $\gamma(s)$ are given by

$$\begin{aligned}\kappa(s)&=\sqrt{\left|\big(y''(s)\big)^2-\big(z''(z)\big)^2\right|}\\&=\sqrt{\left|\left(\frac{s^3}{4}+\frac{1}{s^3}\right)^2-\left(\frac{s^3}{4}-\frac{1}{s^3}\right)^2\right|}=1\end{aligned}$$

$\tau(s) = \frac{y''(s)z'''(z)-y'''(s)z''(s)}{\kappa^2(s)}$

$= \left(\frac{s^3}{4}+\frac{1}{s^3}\frac{s^3}{4}+\frac{1}{s^3}\right)\left(\frac{3s^2}{4}+\frac{3}{2s^4}\right)-\left(\frac{s^3}{4}-\frac{1}{s^3}\right)\left(\frac{3s^2}{4}-\frac{3}{2s^4}\right)=\frac{3}{s}$

and the orthonormal trihedron of $\gamma(s)$ are

$T(s) = \gamma'(s) = \left(1,\frac{s^4}{16}-\frac{1}{2s^2},\frac{s^4}{16}+\frac{1}{2s^2}\right)$

$N(s) = \frac{1}{\kappa(s)}\gamma''(s) = \left(0,\frac{s^3}{4}+\frac{1}{s^3},\frac{s^3}{4}-\frac{1}{s^3}\right)$

$B(s) = \frac{1}{\kappa(s)}\left(0,\epsilon z''(s),\epsilon\, y''(s)\right) = \left(0,\frac{s^3}{4}-\frac{1}{s^3},\frac{s^3}{4}+\frac{1}{s^3}\right).$

Note that $\epsilon = 1$ since $\gamma(s)$ is a spacelike curve. Consequently, the involute of the curve $\gamma(s)$ is given by

$\bar{\gamma}(s) = \gamma(s) + (c-s)T(s)$

$= \left(s,\frac{s^5}{80}+\frac{1}{2s},\frac{s^5}{80}-\frac{1}{2s}\right) + (c-s)\left(1,\frac{s^4}{16}-\frac{1}{2s^2},\frac{s^4}{16}+\frac{1}{2s^2}\right)$

$= \left(c,\frac{s^5}{80}+\frac{1}{2s}+(c-s)\left(\frac{s^4}{16}-\frac{1}{2s^2}\right),\frac{s^5}{80}-\frac{1}{2s}+(c-s)\left(\frac{s^4}{16}+\frac{1}{2s^2}\right)\right).$

The orthogonal dihedron of $\bar{\gamma}(s)$ are

$\bar{T}(s) = N(s) = \left(0,\frac{s^3}{4}+\frac{1}{s^3},\frac{s^3}{4}-\frac{1}{s^3}\right)$

$\bar{N}(s) = B(s) = \left(0,\frac{s^3}{4}-\frac{1}{s^3},\frac{s^3}{4}+\frac{1}{s^3}\right)$

And the curvature of $\bar{\gamma}(s)$ is $\bar{\kappa}(s) = |\tau(s)| = \left|\frac{3}{s}\right|$

Example 4.3. Let $r: I \subset \mathbb{R} \to G_3^1$ be an admissible non-lightlike curve parametrized by arc length s in pseudo-Galilean space and defined by

$$r(s) = \left(s,\frac{e^{2s}}{4}-\frac{s^3}{6},\frac{e^{2s}}{4}+\frac{s^3}{6}\right).$$

Taking the derivative of $r(s)$ three times yields

$r'(s) = \left(1,\frac{e^{2s}}{2}-\frac{s^2}{2},\frac{e^{2s}}{2}+\frac{s^2}{2}\right),$

$r''(s) = (0, e^{2s}-s, e^{2s}+s),$

$r'''(s) = (0, 2e^{2s}, 2e^{2s}).$

Since $\left(r''(s), r''(s)\right) = (e^{2s}-s)^2 - (e^{2s}+s)^2 = -4e^{2s} < 0$ then $\alpha(s)$ is a timelike curve. The curvature and torsion of $r(s)$ are given by

$\kappa(s) = \sqrt{\left|\left(y''(s)\right)^2 - \left(z''(z)\right)^2\right|} = \sqrt{|(e^{2s}-s)^2 - (e^{2s}+s)^2|} = \sqrt{|-4e^{2s}|} = 2e^s$

$\tau(s) = \frac{y''(s)z'''(z)-y'''(s)z''(s)}{\kappa^2(s)}$

$= \frac{(e^{2s}-s)(2e^s)-(2e^s)(e^{2s}+s)}{4e^{2s}} = -s$

and the orthonormal trihedron of $r(s)$ are

$T(s) = r'(s) = \left(1,\frac{e^{2s}}{2}-\frac{s^2}{2},\frac{e^{2s}}{2}+\frac{s^2}{2}\right)$

$N(s) = \frac{1}{\kappa(s)}r''(s) = \left(0,\frac{e^{2s}-s}{2e^s},\frac{e^{2s}+s}{2e^s}\right)$

$B(s) = \frac{1}{\kappa(s)}\left(0,\epsilon z''(s),\epsilon\, y''(s)\right) = \left(0,-\frac{e^{2s}+s}{2e^s},-\frac{e^{2s}-s}{2e^s}\right).$

Note that $\epsilon = -1$ since $r(s)$ is a timelike curve.

Consequently, the involute of the curve $r(s)$ is given by

$\bar{r}(s) = r(s) + (c-s)T(s)$

$= \left(s,\frac{e^{2s}}{4}-\frac{s^3}{6},\frac{e^{2s}}{4}+\frac{s^3}{6}\right) + (c-s)\left(1,\frac{e^{2s}}{2}-\frac{s^2}{2},\frac{e^{2s}}{2}+\frac{s^2}{2}\right)$

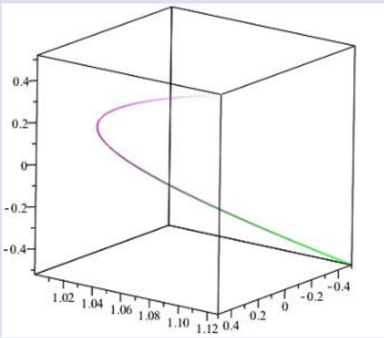
$= \left(c,\frac{e^{2s}}{4}-\frac{s^3}{6}+(c-s)\left(\frac{e^{2s}}{2}-\frac{s^2}{2}\right),\frac{e^{2s}}{4}+\frac{s^3}{6}+(c-s)\left(\frac{e^{2s}}{2}+\frac{s^2}{2}\right)\right).$

The orthogonal dihedron of $\bar{r}(s)$ are

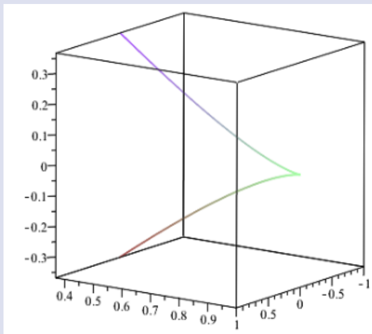
$\bar{T}(s) = N(s) = \left(0,\frac{e^{2s}-s}{2e^s},\frac{e^{2s}+s}{2e^s}\right)$

$\bar{N}(s) = B(s) = \left(0,-\frac{e^{2s}+s}{2e^s},-\frac{e^{2s}-s}{2e^s}\right)$

And the curvature of $\bar{\gamma}(s)$ is $\bar{\kappa}(s) = |\tau(s)| = |s|$

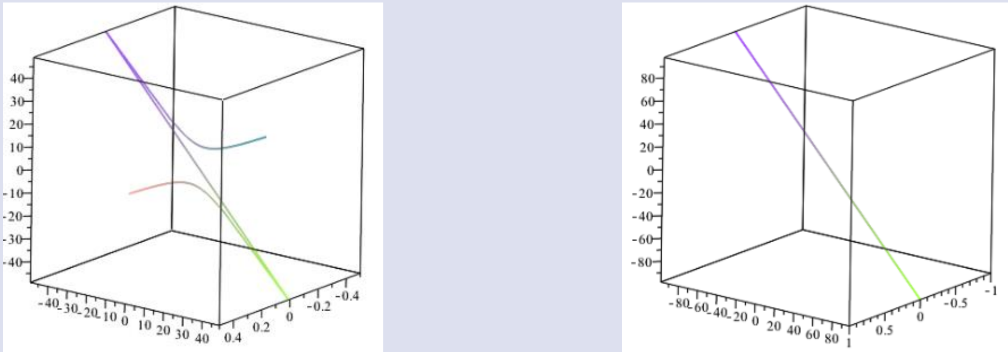


(a)



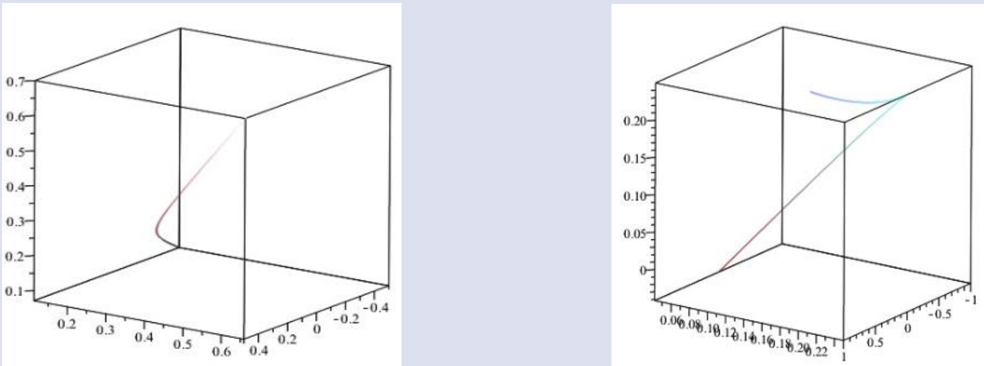
(b)

Figure 1. Plot of curves in example 4.1, (a) plot of $\alpha(s)$, (b) plot of $\beta(s)$



(a) (b)

Figure 2. Plot of curves in example 4.2, (a) plot of $\gamma(s)$, (b) plot of $\bar{\gamma}(s)$.



(a) (b)

Figure 3. Plot of curves in example 4.3, (a) plot of r(s), (b) plot of $\bar{r}$(s).

Acknowledgment

The author would like to thank the referees for all the useful suggestions

Conflicts of interest

The author states that he has no conflict of interest to declare that are relevant to the content of this article.

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