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4-dimensional pseudo-Galilean geometry

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Abstract

According to F. Klein, Geometry is the study of invariant properties of figures, i.e., properties unchanged under all motions. In this article, we introduce 4-dimensional pseudo-Galilean transformations. Moreover, we study invariant properties under translation, shear and Minkowskian rotation motions. We have computed Frenet-Serret formulas of a curve and also we have found the fundamental theorem of curve theory in 4-dimensional pseudo-Galilean geometry.

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1. Introduction

Non-Euclidean geometry, literally is any geometry that is not the same as Euclidean geometry. The applications of Non-Euclidean geometry, [1] have been found in a lot of places of our life such as the theory of general relativity, celestial mechanics, cosmology. Another interesting application area is architecture. For example, in 2009, the Tote restaurant in Mumbai was designed with aid of the fractal geometry, [1]. Galilean geometry is a geometry of the Galilean Relativity or shortly a non-Euclidean geometry. It is a bridge from Euclidean geometry to special relativity. It is a theory that is invariant under Galilean transformations stated by Yaglom. Galilean geometry is worked in detail in [2-4].

In 1998, pseudo-Galilean geometry $\mathbb{G}_1^3$ as analog to [2] and [4] is defined by Divjak, [5]. This work [5] also includes the theory of curves in $\mathbb{G}_1^3$. Then, a lot of papers such as [5- 12] in pseudo-Galilean geometry $\mathbb{G}_1^3$, have been worked. In this paper, 4-dimensional pseudo-Galilean geometry $\mathbb{G}_1^4$ will be defined and the curves in $\mathbb{G}_1^4$ will be considered.

2. Minkowski Space $\mathbb{R}_1^3$

In this section, we give some fundamental information to construct a new geometry about 3- dimensional Minkowski space. Thus, we will be able to consider Galilean transformations in 3- dimensional Minkowski space. Let us consider $3-$dimensional Minkowski space $\mathbb{R}_1^3=\left[\mathbb{R},(+,+,-)\right]$ and let the Lorentzian inner product of $\mathbf{x}=(x_1,x_2,x_3)$ and $\mathbf{y}=(y_1,y_2,y_3)\in\mathbb{R}_1^3$, be

$$\langle\mathbf{x},\mathbf{y}\rangle=x_1y_1+x_2y_2-x_3y_3.$$

The norm of $\mathbf{x}\in\mathbb{R}_1^3$ is denoted by $\|\mathbf{x}\|$ and defined as

$$\|\mathbf{x}\|=\sqrt{|\langle\mathbf{x},\mathbf{x}\rangle|}.$$

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A vector $\mathbf{x} \in \mathbb{R}_1^3$ is called a spacelike, timelike and null (light-like) vector if $\langle \mathbf{x}, \mathbf{x} \rangle > 0$ or $\mathbf{x} = \mathbf{0}$, $\langle \mathbf{x}, \mathbf{x} \rangle < 0$ and $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ for $\mathbf{x} \neq \mathbf{0}$, respectively, [13,14]. A timelike vector is said to be positive (resp. negative) if and only if $x_3 > 0$ (resp. $x_3 < 0$).

Let $\mathbf{x}$ and $\mathbf{y}$ be positive (negative) timelike vectors in $\mathbb{R}_1^3$. There is a unique non-negative real number α such that

$$\langle \mathbf{x}, \mathbf{y} \rangle = \|\mathbf{x}\| \|\mathbf{y}\| \cosh \alpha.$$

This number is called the Lorentzian timelike angle between the vectors $\mathbf{x}$ and $\mathbf{y}$. Let $\mathbf{x}$ and $\mathbf{y}$ be spacelike vectors in $\mathbb{R}_1^3$ that span a spacelike vector subspace. There is a unique nonnegative real number α such that

$$\langle \mathbf{x}, \mathbf{y} \rangle = \|\mathbf{x}\| \|\mathbf{y}\| \cos \alpha.$$

Let $\mathbf{x}$ and $\mathbf{y}$ be spacelike vectors in $\mathbb{R}_1^3$ that span a timelike vector subspace. There is a unique nonnegative real number α such that

$$\langle \mathbf{x}, \mathbf{y} \rangle = \|\mathbf{x}\| \|\mathbf{y}\| \cosh \alpha.$$

Let $\mathbf{x}$ be a spacelike vector and $\mathbf{y}$ be a timelike vector in $\mathbb{R}_1^3$. Then, there is a unique real number $\alpha \geq 0$ such that

$$\langle \mathbf{x}, \mathbf{y} \rangle = \|\mathbf{x}\| \|\mathbf{y}\| \sinh \alpha.$$

Basic rotations (also called the elemental rotation) are rotations about one of the axes of a coordinate system. The following three basic rotation rotate vectors by an angle α about the x, y, or z axis, in $\mathbb{R}_1^3$. The rotation by angle α about the axes x is denoted by $R_x(\alpha)$ and is calculated as

$$R_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh \alpha & \sinh \alpha \\ 0 & \sinh \alpha & \cosh \alpha \end{bmatrix},$$

the rotation by angle α about the axes y is denoted by $R_y(\alpha)$ and is obtained by

$$R_y(\alpha) = \begin{bmatrix} \cosh \alpha & 0 & \sinh \alpha \\ 0 & 1 & 0 \\ \sinh \alpha & 0 & \cosh \alpha \end{bmatrix},$$

and the rotation by angle α about the axes z is denoted by $R_z(\alpha)$ and is calculated as:

$$R_z(\alpha) = \begin{bmatrix} \cos \alpha & \sin \alpha & 0 \\ -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

with the help of the article [15].

However, according to Euler's rotation theorem, any general rotations in space $\mathbb{R}^3$ may be described using three basic rotations. As you see, the elemental rotations can occur about the axes of the fixed coordinate system (extrinsic rotations) or about the axes of a rotating coordinate system, which is initially aligned with the fixed one, and modifies its orientation after each elemental rotation. Without considering the possibility of using two different conventions for the definition of the rotation axes (intrinsic or extrinsic), there exist twelve possible sequences of rotation axes, divided into two groups: by Proper Euler angles $(R_zR_xR_z,\ R_xR_yR_x,\ R_yR_zR_y,$ $R_zR_yR_z,\ R_xR_zR_x,\ R_yR_xR_y)$ and by Tait–Bryan angles $(R_xR_yR_z,\ R_yR_zR_x,\ R_zR_xR_y,\ R_xR_zR_y,\ R_zR_yR_x,$ $R_yR_xR_z)$. Similarly, the rotation matrices in $\mathbb{R}_1^3$ can be obtained from above three using matrix multiplication. For example, the product

$$R=R_z(\alpha)R_y(\beta)R_x(\gamma)=\begin{bmatrix}\cosh\beta\cos\alpha & \cosh\gamma\sin\alpha+\sinh\beta\sinh\gamma\cos\alpha & \sinh\gamma\sin\alpha+\cosh\gamma\sinh\beta\cos\alpha\\ -\cosh\beta\sin\alpha & \cosh\gamma\cos\alpha-\sinh\beta\sinh\gamma\sin\alpha & \sinh\gamma\cos\alpha-\cosh\gamma\sinh\beta\sin\alpha\\ \sinh\beta & \cosh\beta\sinh\gamma & \cosh\beta\cosh\gamma\end{bmatrix}$$

represents a rotation whose yaw, pitch, and roll angles are α,β and γ about axes $z,y,x,$ respectively. Moreover, the product

$$R=R_z(\alpha)R_x(\gamma)R_z(\beta)=\begin{bmatrix}\cos\alpha\cos\beta-\cosh\gamma\sin\alpha\sin\beta & \cos\alpha\sin\beta+\cosh\gamma\cos\beta\sin\alpha & \sinh\gamma\sin\alpha\\ -\cos\beta\sin\alpha-\cosh\gamma\cos\alpha\sin\beta & \cosh\gamma\cos\alpha\cos\beta-\sin\alpha\sin\beta & \sinh\gamma\cos\alpha\\ -\sinh\gamma\sin\beta & \sinh\gamma\cos\beta & \cosh\gamma\end{bmatrix} \quad (1)$$

represents a rotation whose angles α,β,γ about axes $z,x,z.$

Also, rotations in Minkowski space, preserve the types of vectors. One can be found more information about Minkowski space in [13-21].

3. Pseudo-Galiean Geometry $\mathbb{G}_1^4$

Let $\{x,y,z\}$ and $\{x',y',z'\}$ be two referance frames in $\mathbb{R}_1^3$. We know that there is the relation

$$\begin{bmatrix}x'\\y'\\z'\end{bmatrix}=\begin{bmatrix}\cos\alpha\cos\beta-\cosh\gamma\sin\alpha\sin\beta & \cos\alpha\sin\beta+\cosh\gamma\cos\beta\sin\alpha & \sinh\gamma\sin\alpha\\ -\cos\beta\sin\alpha-\cosh\gamma\cos\alpha\sin\beta & \cosh\gamma\cos\alpha\cos\beta-\sin\alpha\sin\beta & \sinh\gamma\cos\alpha\\ -\sinh\gamma\sin\beta & \sinh\gamma\cos\beta & \cosh\gamma\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}+\begin{bmatrix}a\\b\\c\end{bmatrix}$$

between these two frames from (1). If the origin point O of referans frame $\{x,y,z\}$ with constant velocity $\mathbf{v}$ on a non-null line l moves according to relative frame $\{x',y',z'\}$, then there are two cases with respect to l for coordinates $a(t),b(t)$ and $c(t)$ of point O at the moment t where $x'\hat{O}l=\delta_1, y'\hat{O}l=\delta_2,$ and $z'\hat{O}l=\delta_3$ by aid of [13]:

Case 1 : if l is timelike, then one can be written

$$\begin{bmatrix}a(t)\\b(t)\\c(t)\end{bmatrix}=\begin{bmatrix}a+(v\sinh\delta_1)t\\b+(v\sinh\delta_2)t\\c+(v\cosh\delta_3)t\end{bmatrix},$$

where $\sinh^2\delta_1+\sinh^2\delta_2-\cosh^3\delta_3=1.$

So, the relation between the coordinates (x',y',z') and (x,y,z) of the point A is given by

$$\begin{bmatrix}x'\\y'\\z'\end{bmatrix}=\begin{bmatrix}\cos\alpha\cos\beta-\cosh\gamma\sin\alpha\sin\beta & \cos\alpha\sin\beta+\cosh\gamma\cos\beta\sin\alpha & \sinh\gamma\sin\alpha\\ -\cos\beta\sin\alpha-\cosh\gamma\cos\alpha\sin\beta & \cosh\gamma\cos\alpha\cos\beta-\sin\alpha\sin\beta & \sinh\gamma\cos\alpha\\ -\sinh\gamma\sin\beta & \sinh\gamma\cos\beta & \cosh\gamma\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}+\begin{bmatrix}a+(v\sinh\delta_1)t\\b+(v\sinh\delta_2)t\\c+(v\cosh\delta_3)t\end{bmatrix}.$$

By adding the relation $t' = t + d$ which expresses the possibility of shifting the time origin, we arrive at the formulas

$$\begin{cases} x' = a + x(\cos\alpha\cos\beta - \cosh\gamma\sin\alpha\sin\beta) + y(\cos\alpha\sin\beta + \cosh\gamma\cos\beta\sin\alpha) + t(v\sinh\delta_1) + z(\sinh\gamma\sin\alpha) \\ y' = b - x(\cos\beta\sin\alpha + \cosh\gamma\cos\alpha\sin\beta) - y(\sin\alpha\sin\beta - \cosh\gamma\cos\alpha\cos\beta) + t(v\sinh\delta_2) + z(\sinh\gamma\cos\alpha) \\ z' = c - x(\sinh\gamma\sin\beta) + y(\sinh\gamma\cos\beta) + z(\cosh\gamma) + t(v\cosh\delta_3) \\ t' = d + t, \end{cases}$$

which give the relation between two coordinate systems the pseudo-Galilean motions. The motions can be split into three motions: a rotation about the $t-$axis; a shear in the direction of vector $\mathbf{v} = (v\sinh\delta_1, v\sinh\delta_2, v\cosh\delta_3, 0)$, and a translation determined by the vector (a,b,c,d). If the motion is arranged as x instead of time parameter t and y,z,w instead of space parameter $x,y,z,$ respectively, we get

$$\begin{cases} x' = d + x \\ y' = a + (v\sinh\delta_1)x + (\cos\alpha\cos\beta - \cosh\gamma\sin\alpha\sin\beta)y + (\cos\alpha\sin\beta + \cosh\gamma\cos\beta\sin\alpha)z + (\sinh\gamma\sin\alpha)w \\ z' = b + (v\sinh\delta_2)x - (\cos\beta\sin\alpha + \cosh\gamma\cos\alpha\sin\beta y) - (\sin\alpha\sin\beta - \cosh\gamma\cos\alpha\cos\beta)z + (\sinh\gamma\cos\alpha)w \\ w' = c + (v\cosh\delta_3)x - (\sinh\gamma\sin\beta)y + (\sinh\gamma\cos\beta)z + (\cosh\gamma)w \end{cases}$$

where $\sinh^2\delta_1 + \sinh^2\delta_2 - \cosh^3\delta_3 = 1.$

Case 2: Similary, if l is spacelike, then there are four situations and it can be easily calculated such as above.

Finally, if we calculate the two cases then we obtain the following equations

$$\begin{cases} x' = d + x \\ y' = a + vex + (\cos\alpha\cos\beta - \cosh\gamma\sin\alpha\sin\beta)y + (\cos\alpha\sin\beta + \cosh\gamma\cos\beta\sin\alpha)z + (\sinh\gamma\sin\alpha)w \\ z' = b + vfx - (\cos\beta\sin\alpha + \cosh\gamma\cos\alpha\sin\beta)y - (\sin\alpha\sin\beta - \cosh\gamma\cos\alpha\cos\beta)z + (\sinh\gamma\cos\alpha)w \\ w' = c + vgx - (\sinh\gamma\sin\beta)y + (\sinh\gamma\cos\beta)z + (\cosh\gamma)w \end{cases} \tag{2}$$

where the coefficents e, f, g are angles such that $e^2 + f^2 - g^2 = 1.$

So, the above equations are called 4-dimensional pseudo-Galilean transformations. The invariant theory under 4-dimensional pseudo-Galilean transformations is called 4-dimensional pseudo-Galilean geometry and is denoted by $\mathbb{G}_1^4$.

4. Basic Information About $\mathbb{G}_1^4$

Let $\mathbf{a} = (x,y,z,w)$ and $\mathbf{b} = (x_1,y_1,z_1,w_1)$ be vectors in the pseudo-Galilean space $\mathbb{G}_1^4$. The scalar product in the Pseudo-Galilean space $\mathbb{G}_1^4$ is defined by

$$<\mathbf{a},\mathbf{b}>_{\mathbb{G}} = xx_1.$$

A vector $\mathbf{a} = (x,y,z,w)$ is said to be isotropic or special vector if $x = 0$. Otherwise, $\mathbf{a} = (x,y,z,w)$ is called a non-isotropic. All unit non-isotropic vectors and isotropic vectors are of the form $\mathbf{a} = (x,y,z,w),\ x \neq 0$ and $\mathbf{p} = (0,y,z,w),$ respectively. Let $\mathbf{p} = (0,y,z,w)$ and $\mathbf{q} = (0,y_1,z_1,w_1)$ be two isotropic vectors. Then, the special scalar product of isotropic vectors $\mathbf{p}$ and $\mathbf{q}$ is defined by

$$<\mathbf{a},\mathbf{b}>_{\delta} = yy_1 + zz_1 - ww_1.$$

Along with the study, the special scalar product will be denoted by $\delta -$ product. The orthogonality of vectors in pseudo-Galilean space $\mathbb{G}_1^4$, $\mathbf{a} \perp_{\mathbb{G}} \mathbf{b}$, means that $<\mathbf{a},\mathbf{b}>_{\mathbb{G}} = 0$ for $<\mathbf{a},\mathbf{a}>_{\mathbb{G}} \neq 0$. So, all isotropic vectors are orthogonal to the non-isotropic vectors. Also, the δ-orthogonality of isotropic vectors $\mathbf{p}$ and $\mathbf{q}$ means that

$<\mathbf{p},\mathbf{q}>_{\delta} = 0.$

The norm of a vector $\mathbf{a}$ is defined by

$$\|\mathbf{a}\|_{\mathbb{G}} = |x|,$$

and $\mathbf{a}$ is called a unit vector if $\|\mathbf{a}\|_{\mathbb{G}} = 1$. The norm of an isotropic vector $\mathbf{p}$ is defined by

$$\|\mathbf{p}\|_{\delta} = \sqrt{\left|y^2 + z^2 - w^2\right|}$$

and p is called a unit isotropic vector if $\|\mathbf{p}\|_{\delta} = 1$. Briefly, the vectors in $\mathbb{G}_1^4$ are divided into two classes: the non-isotropic vector or the isotropic vectors which are spacelike, timelike or null.

Let $\mathbf{a} = (x, y, z, w)$, $\mathbf{b} = (x_1, y_1, z_1, w_1)$ and $\mathbf{c} = (x_2, y_2, z_2, w_2)$ be at least one non-isotropic vector in the pseudo-Galilean space $\mathbb{G}_1^4$, we introduce the vector product of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ as the following:

$$\mathbf{a} \times \mathbf{b} \times \mathbf{c} = -\begin{vmatrix} \mathbf{0} & \mathbf{e}_2 & \mathbf{e}_3 & -\mathbf{e}_4 \\ x & y & z & w \\ x_1 & y_1 & z_1 & w_1 \\ x_2 & y_2 & z_2 & z_2 \end{vmatrix}.$$

Especially, the vector product of isotropic vectors $\mathbf{p} = (0, y, z, w)$, $\mathbf{q} = (0, y_1, z_1, w_1)$ and $\mathbf{r} = (0, y_2, z_2, w_2)$ is introduced

$$\mathbf{p} \times \mathbf{q} \times \mathbf{r} = -\begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & -\mathbf{e}_4 \\ 0 & y & z & w \\ 0 & y_1 & z_1 & w_1 \\ 0 & y_2 & z_2 & z_2 \end{vmatrix}.$$

Here, $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and $\mathbf{e}_4$ are coordinate direction vectors which satisfy at follows:

$$\mathbf{e}_1 \times \mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_4,$$
$$\mathbf{e}_2 \times \mathbf{e}_3 \times \mathbf{e}_4 = \mathbf{e}_1,$$
$$\mathbf{e}_3 \times \mathbf{e}_4 \times \mathbf{e}_1 = \mathbf{e}_2,$$
$$\mathbf{e}_4 \times \mathbf{e}_1 \times \mathbf{e}_2 = -\mathbf{e}_3.$$

Let $\{\mathbf{D}, \mathbf{E}, \mathbf{F}, \mathbf{G}\}$ be vectors in $\mathbb{G}_1^4$.

i) If $\mathbf{D}$ is a unit non-isotropic vector and $\{\mathbf{E}, \mathbf{F}\}$ are unit isotropic spacelike vectors and $\mathbf{G}$ is a unit isotropic timelike vector, then $\{\mathbf{D}, \mathbf{E}, \mathbf{F}, \mathbf{G}\}$ is called an orthonormal basis of $\mathbb{G}_1^4$.

ii) If $\mathbf{D}$ is a unit non-isotropic vector and $\mathbf{E}$ is a unit isotropic spacelike vector, $\{\mathbf{F}, \mathbf{G}\}$ are unit isotropic lightlike vectors such that $\langle \mathbf{F}, \mathbf{G} \rangle_{\delta} = -1$, $\{\mathbf{D}, \mathbf{E}, \mathbf{F}, \mathbf{G}\}$ is called a null basis (or null frame) of $\mathbb{G}_1^4$.

5. Construction of Frenet-Serret Frame in $\mathbb{G}_1^4$

Let α be a curve in $\mathbb{G}_1^4$ given first by

$\alpha(t)=(x(t),y(t),z(t),w(t)),$

where $x(t),y(t),z(t),w(t)\in C^4$ (the set of four-times continuously differentiable functions) and t run through a real interval. If $\frac{dx(t)}{dt}\neq 0,$ then the curve α is called an admissible curve. Otherwise, the curve α is called a non-admissible curve. From now on, we denote differentiation with respect to t by a dash.

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An admissible curve given first by

$\alpha(t)=(x(t),y(t),z(t),w(t)),$

where $x^{'}(t)\neq 0$, the parameter of arc length is defined by

$ds=\left|x^{'}(t)dt\right|=\left|dx\right|.$

For briefly, we assume $ds=dx$ and $s=x$ as the arc length of the curve α . Let an admissible curve α of the class C^r $(r\geq 3)$ parameterized by arclength x, given in coordinate form $\alpha(x)=(x,y(x),z(x),w(x)).$ The first vector of the Frenet-Serret frame, namely the tangent vector of α is defined by

$\mathbf{T}(x)=\alpha^{'}(x)=\left(1,y^{'}(x),z^{'}(x),w^{'}(x)\right).$

Since $\mathbf{T}$ is a unit vector, so, we may express $\left\langle \mathbf{T},\mathbf{T}\right\rangle_{\mathbb{G}}=1.$ Differentiating the last equation with respect to x, we have $\left\langle \mathbf{T}^{'},\mathbf{T}\right\rangle_{\mathbb{G}}=0.$ Note that $\mathbf{T}^{'}(x)$ can be a timelike, spacelike or null vector:

So, we have computed Frenet-Serret formulas with respect to three conditions of $\mathbf{T}^{'}(x)$.

A. Let $\mathbf{T}^{'}(x)$ be a timelike vector: The vector function $\mathbf{T}^{'}$ gives us the rotation measurement of the curve α . The real valued function

$$k_1=\left\|\mathbf{T}^{'}\right\|_{\delta}=\sqrt{-\left(y^{''}\right)^2-\left(z^{''}\right)^2+\left(w^{''}\right)^2} \tag{3}$$

is called the first curvature of the curve $\alpha.$ Now, we define the principal normal vector

$\mathbf{N}=\frac{\mathbf{T}^{'}}{k_1}$ or $\mathbf{N}(x)=\frac{1}{k_1(x)}\left(0,y^{''}(x),z^{''}(x),w^{''}(x)\right).$

Since $\mathbf{N}(x)$ is a timelike vector, $\left\langle \mathbf{N}(x),\mathbf{N}(x)\right\rangle_{\delta}=-1$ and $2\left\langle \mathbf{N}^{'}(x),\mathbf{N}(x)\right\rangle_{\delta}=0.$ So, $\mathbf{N}^{'}(x)$ is a spacelike vector. Then, $\mathbf{N}^{'}(x)\neq\mathbf{0}$ is a spacelike vector linearly independent with $\mathbf{N}(x).$ We define second curvature of the curve α as

$k_2(x)=\left\|\mathbf{N}^{'}(x)\right\|_{\delta}.$

The third vector field, namely binormal vector field of the curve α which is spacelike vector is defined by

$$\mathbf{B}_1(x)=\frac{1}{k_2(x)}\left(0,\left(\frac{y^{''}(x)}{k_1(x)}\right)^{'},\left(\frac{z^{''}(x)}{k_1(x)}\right)^{'},\left(\frac{w^{''}(x)}{k_1(x)}\right)^{'}\right).$$

Therefore, the vector $\mathbf{B}_1(x),$ is both orthogonal to $\mathbf{T}$ and $\mathbf{N}$. Hence, the fourth unit vector is defined by

$$\mathbf{B}_2(x) = \mathbf{T}(x) \times \mathbf{N}(x) \times \mathbf{B}_1(x).$$

The basis $\{\mathbf{T}, \mathbf{N}, \mathbf{B}_1, \mathbf{B}_2\}$ is positively oriented because $\det(\mathbf{T}, \mathbf{N}, \mathbf{B}_1, \mathbf{B}_2) = 1.$ We define the third curvature of the curve α by the inner product

$$k_3 = \langle \mathbf{B}_1', \mathbf{B}_2 \rangle_\delta.$$

Here, as well known, the set $\{\mathbf{T}, \mathbf{N}, \mathbf{B}_1, \mathbf{B}_2, k_1, k_2, k_3\}$ is called the Frenet-Serret apparatus of the curve α. And here, we know that the vectors are mutally orthogonal vectors satisfying

$$\langle \mathbf{T}, \mathbf{T} \rangle_{\mathbb{G}} = -\langle \mathbf{N}, \mathbf{N} \rangle_\delta = \langle \mathbf{B}_1, \mathbf{B}_1 \rangle_\delta = \langle \mathbf{B}_2, \mathbf{B}_2 \rangle_\delta = 1,$$

$$\langle \mathbf{T}, \mathbf{N} \rangle_{\mathbb{G}} = \langle \mathbf{T}, \mathbf{B}_1 \rangle_{\mathbb{G}} = \langle \mathbf{T}, \mathbf{B}_2 \rangle_{\mathbb{G}} = \langle \mathbf{N}, \mathbf{B}_1 \rangle_\delta = \langle \mathbf{N}, \mathbf{B}_2 \rangle_\delta = \langle \mathbf{B}_1, \mathbf{B}_2 \rangle_\delta = 0.$$

Now, let calculate Frenet Serret equations. Considering the definitions above, firstly, we know that

$$\mathbf{T}'(x) = k_1(x)\mathbf{N}(x).$$

It is possible to define the vector $\mathbf{N}'$ according to frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}_1, \mathbf{B}_2\}$ by

$$\mathbf{N}'(x) = \lambda_1(x)\mathbf{T}(x) + \lambda_2(x)\mathbf{N}(x) + \lambda_3(x)\mathbf{B}_1(x) + \lambda_4(x)\mathbf{B}_2(x),$$

$\lambda_i \in \mathbb{R},$ for $1 \leq i \leq 4.$ Multiply both sides by the vectors $\{\mathbf{T}, \mathbf{N}, \mathbf{B}_1, \mathbf{B}_2\}$ and considering above the equations, we have, respectively

$$\lambda_1(x) = \langle \mathbf{N}'(x), \mathbf{T}(x) \rangle_{\mathbb{G}} = 0$$

$$\lambda_2(x) = \langle \mathbf{N}'(x), \mathbf{N}(x) \rangle_\delta = 0$$

$$\lambda_3(x) = \langle \mathbf{N}'(x), \mathbf{B}_1(x) \rangle_\delta = k_2(x)$$

By the definiton the the third vector field $\mathbf{B}_1,$ we easily obtain

$$\lambda_4(x) = \langle \mathbf{N}', \mathbf{B}_2 \rangle_\delta = 0.$$

We immediately arrive at

$$\mathbf{N}' = k_2(x)\mathbf{B}_1(x).$$

In order to compute the vector function $\mathbf{B}_1'$, let us decompose

$$\mathbf{B}_1' = \mu_1(x)\mathbf{T}(x) + \mu_2(x)\mathbf{N}(x) + \mu_3(x)\mathbf{B}_1(x) + \mu_4(x)\mathbf{B}_2(x),$$

where $\mu_i \in \mathbb{R},$ for $1 \leq i \leq 4.$ Similiar to $\mathbf{N}'$, we express

$$\mu_1(x) = \langle \mathbf{B}_1'(x), \mathbf{T}(x) \rangle_{\mathbb{G}} = 0$$

$$\mu_2(x) = \langle \mathbf{B}_1'(x), \mathbf{N}(x) \rangle_\delta = k_2(x)$$

$$\mu_3(x) = \langle \mathbf{B}_1'(x), \mathbf{B}_1(x) \rangle_\delta = 0$$

$$\mu_4(x) = \langle \mathbf{B}_1'(x), \mathbf{B}_2(x) \rangle_\delta = k_3(x)$$

so we get,

$$\mathbf{B}_1'(x) = k_2(x)\mathbf{N}(x) + k_3(x)\mathbf{B}_2(x). \tag{4}$$

In an analogous way, we can write

$$\mathbf{B}_2'(x) = \xi_1(x)\mathbf{T}(x) + \xi_2(x)\mathbf{N}(x) + \xi_3(x)\mathbf{B}_1(x) + \xi_4(x)\mathbf{B}_2(x),$$

where $\xi_i \in \mathbb{R},$ for $1 \le i \le 4.$ Then, with the aid of the equaiton (4), we can find

$$\xi_1(x) = \left\langle \mathbf{B}_2'(x), \mathbf{T}(x) \right\rangle_{\mathbb{G}} = 0$$

$$\xi_2(x) = \left\langle \mathbf{B}_2'(x), \mathbf{N}(x) \right\rangle_{\delta} = 0$$

$$\xi_3(x) = \left\langle \mathbf{B}_2'(x), \mathbf{B}_1(x) \right\rangle_{\delta} = -k_3(x)$$

$$\xi_3(x) = \left\langle \mathbf{B}_2'(x), \mathbf{B}_2(x) \right\rangle_{\delta} = 0.$$

So, we have $\mathbf{B}_2' = -k_3\mathbf{B}_1.$ And we obtain the Frenet equations in matrix form

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}_1' \\ \mathbf{B}_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ 0 & 0 & k_2 & 0 \\ 0 & k_2 & 0 & k_3 \\ 0 & 0 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix}.$$

B.Let $\mathbf{T}'(x)$ be a spacelike vector: The first curvature of α is defined by

$$k_1 = \sqrt{(y'')^2 + (z'')^2 - (w'')^2}.$$

We define the principal normal vector $\mathbf{N}(x) = \dfrac{\mathbf{T}'(x)}{k_1(x)}$ and $\left\langle \mathbf{N}(x), \mathbf{N}(x) \right\rangle_{\delta} = 1.$ So, we get $2\left\langle \mathbf{N}'(x), \mathbf{N}(x) \right\rangle_{\delta} = 0.$ Since $\mathbf{N}'(x)$ is orthogonal to the spacelike vector $\mathbf{N}(x),$ $\mathbf{N}'(x)$ may be spacelike, timelike or lightlike.

i.Assume that $\mathbf{N}'(x)$ is a spacelike vector. Again we write the second curvature

$$k_2 = \left\| \mathbf{N}' \right\|_{\delta},$$

and

$$\mathbf{B}_1(x) = \frac{1}{k_2(x)}\left(0, \left(\frac{y''(x)}{k_1(x)}\right)', \left(\frac{z''(x)}{k_1(x)}\right)', \left(\frac{w''(x)}{k_1(x)}\right)'\right).$$

Also,

$$\mathbf{B}_2(x) = \mathbf{T}(x) \times \mathbf{N}(x) \times \mathbf{B}_1(x)$$

and

$$k_3 = \left\langle \mathbf{B}_1', \mathbf{B}_2 \right\rangle_{\delta}.$$

$\mathbf{B}_2(x)$ is a timelike vector. Similarly, the Frenet equations are

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}_1' \\ \mathbf{B}_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ 0 & 0 & k_2 & 0 \\ 0 & -k_2 & 0 & k_3 \\ 0 & 0 & k_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix}. \tag{5}$$

ii.Assume that $\mathbf{N}'(x)$ ***is a* timelike vector.** The second curvature is

$$k_2(x) = \left\| \mathbf{N}'(x) \right\|_\delta$$

and

$$\mathbf{B}_1(x) = \frac{1}{k_2(x)} \left(0, \left(\frac{y''(x)}{k_1(x)} \right)', \left(\frac{z''(x)}{k_1(x)} \right)', \left(\frac{w''(x)}{k_1(x)} \right)' \right).$$

Moreover,

$$\mathbf{B}_2(x) = \mathbf{T}(x) \times \mathbf{N}(x) \times \mathbf{B}_1(x)$$

and

$$k_3 = \left\langle \mathbf{B}_1', \mathbf{B}_2 \right\rangle_\delta . \tag{6}$$

$\mathbf{B}_2(x)$ is a spacelike vector. The Frenet equations can be easily seen

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}_1' \\ \mathbf{B}_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ 0 & 0 & k_2 & 0 \\ 0 & k_2 & 0 & k_3 \\ 0 & 0 & k_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix}. \tag{7}$$

iii.Assume that $\mathbf{N}'(x)$ ***is a lightlike vector.*** We define the third vector field as $\mathbf{B}_1(x) = \mathbf{N}'(x)$, which is linearly independent with $\mathbf{N}(x)$. Let $\mathbf{B}_2(x)$ be the unique lightlike vector such that $\langle \mathbf{B}_1, \mathbf{B}_2 \rangle_\delta = -1$ and it is orthogonal to $\mathbf{N}(x)$. The vector $\mathbf{B}_2(x)$ is the second binormal vector of α. The third curvature of the curve α $k_3 = -\langle \mathbf{B}_1', \mathbf{B}_2 \rangle_\delta$. The Frenet formulas are similar to above

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}_1' \\ \mathbf{B}_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & 1 & 0 & -k_3 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix}.$$

C. Let $\mathbf{T}'(x)$ ***be a lightlike vector:*** The normal vector as $\mathbf{N}(x) = \mathbf{T}'(x)$ and define the first binormal vector as $\mathbf{B}_1(x) = \mathbf{N}'(x)$, which is a unit spacelike vector. The second binormal vector $\mathbf{B}_2(x)$ is unique lightlike vector which is orthogonal to $\mathbf{B}_1(x)$ such that $\langle \mathbf{N}(x), \mathbf{B}_2(x) \rangle_\delta = -1$. Thus, $\{\mathbf{T}, \mathbf{N}, \mathbf{B}_1, \mathbf{B}_2\}$ is null frame. The Frenet formulas are

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}_1' \\ \mathbf{B}_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & k_3 & 0 & 1 \\ 0 & 0 & k_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix}. \tag{8}$$

The third curvature of α is $k_3 = -\langle \mathbf{B}_1', \mathbf{B}_2 \rangle_\delta$.

Corollary 5.1 The admissible curve $\alpha(x)$ in $\mathbb{G}_1^4$ classifies in the following cases:

1) a curve with timelike normal vector $\mathbf{N}$.

2) three curves with spacelike normal vector $\mathbf{N}$ and binormal vector $\mathbf{B}_1$ which is spacelike, timelike or null.

3)a curve with null normal vector $\mathbf{N}$.

II.

A non-admissible curve α is given by the parametrization $\alpha(t) = (c, y(t), z(t), w(t))$, where $c =$ constant. So, a non-admissible curve α classify in the three kinds, spacelike, timelike, null curve, on $3-$dimensional Minkowski Space $x = c$ in $\mathbb{G}_1^4$. Finally, with the help of [19], we can easily find the construction of Frenet-Serret frames for a non-admissible curve α.

6. The Fundamental Theorem

Until now, we can construct the Frenet-Serret apparatus for a given curve. But, we have not yet addressed to what extent we can do inverse. Given some k_1, k_2 and k_3, we would like to know if it is possible to construct a curve to fit these functions. The fundamental theorem of curves says that it is possible to reconstruct the curve from only the curvature functions.

Theorem 6.1 Let $k_1(x) > 0,$ $k_2(x) > 0$ and $k_3(x),$ $x \in I,$ be three differentiable maps. Then, there exist three differential regular parametrized curves $\alpha : I \to \mathbb{G}_1^4,$ $\alpha = \alpha(x),$ with curvatures $k_1(x)$, $k_2(x)$ and $k_3(x)$.

Proof. Let $x_0 \in I$ and let $\{\mathbf{D}, \mathbf{E}, \mathbf{F}, \mathbf{G}\}$ be an orthonormal basis, which it will be the initial conditions of an ordinary differential equation (ODE) system. Depending on the causal character of the vectors $\mathbf{E}$ and $\mathbf{F}$, we obtain three different cases:

Firstly, if we want to obtain a curve with timelike normal $\mathbf{N}$ and curvatures $k_1(x)$, $k_2(x)$ and $k_3(x)$, respectively, then we consider that $\{\mathbf{D}, \mathbf{E}, \mathbf{F}, \mathbf{G}\}$ is orthonormal basis positively oriented and $\mathbf{E}$ is timelike. In such case, we solve the ODE system of equations

$$\mathbf{T}'(x_0) = k_1(x_0)\mathbf{N}(x_0)$$
$$\mathbf{N}'(x_0) = k_2(x_0)\mathbf{B}_1(x_0)$$
$$\mathbf{B}_1'(x_0) = k_2(x_0)\mathbf{N}(x_0) + k_3(x_0)\mathbf{B}_2(x_0)$$
$$\mathbf{B}_2'(x_0) = -k_3(x_0)\mathbf{B}_1(x_0)$$

with initial condition

$\mathbf{T}(x_0)=\mathbf{D}$
$\mathbf{N}(x_0)=\mathbf{E}$
$\mathbf{B}_1(x_0)=\mathbf{F}$
$\mathbf{B}_2(x_0)=\mathbf{G}.$

Let $\{\mathbf{T},\mathbf{N},\mathbf{B}_1,\mathbf{B}_2\}$ be the unique solutions and define $\alpha(x)=\int_{x_0}^{x}\mathbf{T}(u)du.$ We prove that this curve is with timelike normal $\mathbf{N}$ and curvatures $k_1(x)$, $k_2(x)$ and $k_3(x)$, respectively. We show that

$\{\mathbf{T},\mathbf{N},\mathbf{B}_1,\mathbf{B}_2\}$ is an orthonormal basis with the same causal properties that initial basis $\{\mathbf{D},\mathbf{E},\mathbf{F},\mathbf{G}\}$. Consider the ODE system:

$$\langle\mathbf{T},\mathbf{T}\rangle'_{\mathbb{G}}=2k_1\langle\mathbf{N},\mathbf{T}\rangle_{\mathbb{G}}$$
$$\langle\mathbf{N},\mathbf{N}\rangle'_{\delta}=2k_2\langle\mathbf{B}_1,\mathbf{N}\rangle_{\delta}$$
$$\langle\mathbf{B}_1,\mathbf{B}_1\rangle'_{\delta}=2k_2\langle\mathbf{N},\mathbf{B}_1\rangle_{\delta}+2\langle\mathbf{B}_2,\mathbf{B}_1\rangle_{\delta}$$
$$\langle\mathbf{B}_2,\mathbf{B}_2\rangle'_{\delta}=-2k_3\langle\mathbf{B}_1,\mathbf{B}_2\rangle_{\delta}$$
$$\langle\mathbf{T},\mathbf{N}\rangle'_{\mathbb{G}}=k_1\langle\mathbf{N},\mathbf{N}\rangle_{\mathbb{G}}+k_2\langle\mathbf{T},\mathbf{B}_1\rangle_{\mathbb{G}}$$
$$\langle\mathbf{T},\mathbf{B}_1\rangle'_{\mathbb{G}}=k_1\langle\mathbf{N},\mathbf{B}_1\rangle_{\mathbb{G}}$$
$$\langle\mathbf{T},\mathbf{B}_2\rangle'_{\mathbb{G}}n=k_1\langle\mathbf{N},\mathbf{B}_1\rangle_{\mathbb{G}}$$
$$\langle\mathbf{N},\mathbf{B}_1\rangle'_{\delta}=k_2\langle\mathbf{B}_1,\mathbf{B}_1\rangle_{\delta}+k_2\langle\mathbf{N},\mathbf{N}\rangle_{\delta}+k_3\langle\mathbf{N},\mathbf{B}_2\rangle_{\delta}$$
$$\langle\mathbf{N},\mathbf{B}_2\rangle'_{\delta}n=k_2\langle\mathbf{B}_1,\mathbf{B}_2\rangle_{\delta}-k_3\langle\mathbf{N},\mathbf{B}_1\rangle_{\delta}$$
$$\langle\mathbf{B}_1,\mathbf{B}_2\rangle'_{\delta}=k_2\langle\mathbf{N},\mathbf{B}_2\rangle_{\delta}+k_3\langle\mathbf{B}_2,\mathbf{B}_2\rangle_{\delta}-k_3\langle\mathbf{B}_1,\mathbf{B}_1\rangle_{\delta}$$

with initial conditions at $x=x_0$ given by $(1,-1,1,1,0,0,0,0,0,0)$. On the other hand, the functions

$f_1=1,\ f_2=-1,\ f_3=1,\ f_4=1,\ f_5=0,\ f_6=0,\ f_7=0,\ f_8=0,\ f_9=0,\ f_{10}=0$ satisfy the same ODE system and initial conditions. By uniqueness,

$$\langle\mathbf{T},\mathbf{T}\rangle_{\mathbb{G}}=-\langle\mathbf{N},\mathbf{N}\rangle_{\delta}=\langle\mathbf{B}_1,\mathbf{B}_1\rangle_{\delta}=\langle\mathbf{B}_2,\mathbf{B}_2\rangle_{\delta}=1$$
$$\langle\mathbf{T},\mathbf{N}\rangle_{\mathbb{G}}=\langle\mathbf{T},\mathbf{B}_1\rangle_{\mathbb{G}}=\langle\mathbf{T},\mathbf{B}_2\rangle_{\mathbb{G}}=\langle\mathbf{N},\mathbf{B}_1\rangle_{\delta}=\langle\mathbf{N},\mathbf{B}_2\rangle_{\delta}=\langle\mathbf{B}_1,\mathbf{B}_2\rangle_{\delta}=0.$$

So, $\{\mathbf{T},\mathbf{N},\mathbf{B}_1,\mathbf{B}_2\}$ is an orthonormal basis of $\mathbb{G}_1^4$, where $\mathbf{N}$ is timelike. From the definition of α,

$\alpha'(x)=\mathbf{T}(x)$ and so α is a curve with timelike normal parametrized by arc lengh and curvatures of α are k_1, k_2 and k_3.

Secondly, if we want to obtain a curve with spacelike normal vector $\mathbf{N}$ and spacelike binormal vector $\mathbf{B}_1$ and curvatures k_1, k_2 and k_3, consider the initial conditions

$\mathbf{T}(x_0)=\mathbf{D}$
$\mathbf{N}(x_0)=\mathbf{E}$
$\mathbf{B}_1(x_0)=\mathbf{F}$
$\mathbf{B}_2(x_0)=\mathbf{G},$

where $\{\mathbf{D},\mathbf{E},\mathbf{F},\mathbf{G}\}$ is orthonormal basis and $\mathbf{G}$ is timelike. Considering that the ODE system that we solve is (5), the proof is clear.

Finally, if we are looking for a curve with spacelike normal and timelike binormal vector, the initial condition is an orthonormal basis $\{\mathbf{D},\mathbf{E},\mathbf{F},\mathbf{G}\}$, where $\mathbf{F}$ is timelike and the ODE system (7). And the proof is similar.

Theorem 6.2 Let $k_1(x)>0$, and $k_3(x)$, $x\in I$, be two smooth maps. Then, there exist a curve with spacelike normal $\mathbf{N}$ and lightlike binormal $\mathbf{B}_1$ with curvatures $k_1(x)$ and $k_3(x)$.

Proof. If we want to obtain a curve with spacelike normal $\mathbf{N}$ and lightlike binormal $\mathbf{B}_1$ with curvatures

$k_1(x)$ and $k_3(x)$, respectively, then we consider that $\{\mathbf{D},\mathbf{E},\mathbf{F},\mathbf{G}\}$ be the null frame of $\mathbb{G}_1^4$ such that $\mathbf{E}$ is spacelike and $\mathbf{F},\mathbf{G}$ are unit isotropic lightlike vectors such that $\langle\mathbf{F},\mathbf{G}\rangle_\delta=-1$. We pose the ODE system (8) with initial conditions

$$\mathbf{T}(x_0)=\mathbf{D},$$
$$\mathbf{N}(x_0)=\mathbf{E},$$
$$\mathbf{B}_1(x_0)=\mathbf{F},$$
$$\mathbf{B}_2(x_0)=\mathbf{G}.$$

Let $\{\mathbf{T},\mathbf{N},\mathbf{B}_1,\mathbf{B}_2\}$ be the unique solution and define $\alpha(x)=\int_{x_0}^{x}\mathbf{T}(u)du.$ We prove that α is a curve with spacelike normal $\mathbf{N}$ and null binormal vector $\mathbf{B}_1$. First, we consider the next ODE system of 10 equations:

$$\langle\mathbf{T},\mathbf{T}\rangle_{\mathbb{G}}'=2k_1\langle\mathbf{N},\mathbf{T}\rangle_{\mathbb{G}}$$
$$\langle\mathbf{N},\mathbf{N}\rangle_\delta'=2\langle\mathbf{N},\mathbf{B}_1\rangle_\delta$$
$$\langle\mathbf{B}_1,\mathbf{B}_1\rangle_\delta'=2k_3\langle\mathbf{B}_1,\mathbf{B}_1\rangle_\delta$$
$$\langle\mathbf{B}_2,\mathbf{B}_2\rangle_\delta'=2\langle\mathbf{N},\mathbf{B}_1\rangle_\delta-2k_3\langle\mathbf{B}_2,\mathbf{B}_2\rangle_\delta$$
$$\langle\mathbf{T},\mathbf{N}\rangle_{\mathbb{G}}'=k_1\langle\mathbf{N},\mathbf{N}\rangle_{\mathbb{G}}+k_2\langle\mathbf{T},\mathbf{B}_1\rangle_{\mathbb{G}}$$
$$\langle\mathbf{T},\mathbf{B}_1\rangle_{\mathbb{G}}'=k_1\langle\mathbf{N},\mathbf{B}_1\rangle_{\mathbb{G}}$$
$$\langle\mathbf{T},\mathbf{B}_2\rangle_{\mathbb{G}}'=k_1\langle\mathbf{N},\mathbf{B}_1\rangle_{\mathbb{G}}$$
$$\langle\mathbf{N},\mathbf{B}_1\rangle_\delta'=\langle\mathbf{B}_1,\mathbf{B}_1\rangle_\delta+k_3\langle\mathbf{N},\mathbf{B}_1\rangle_\delta$$
$$\langle\mathbf{N},\mathbf{B}_2\rangle_\delta'=\langle\mathbf{B}_1,\mathbf{B}_2\rangle_\delta+\langle\mathbf{N},\mathbf{N}\rangle_\delta-k_3\langle\mathbf{N},\mathbf{B}_2\rangle_\delta$$
$$\langle\mathbf{B}_1,\mathbf{B}_2\rangle_\delta'=k_3\langle\mathbf{B}_1,\mathbf{B}_2\rangle_\delta+\langle\mathbf{B}_1,\mathbf{N}\rangle_\delta-k_3\langle\mathbf{B}_1,\mathbf{B}_2\rangle_\delta$$

with initial conditions at $x=x_0$ given by $(1,1,0,0,0,0,0,0,0,-1)$. On the other hand, the functions

$$f_1=1,\ f_2=1,\ f_3=0,\ f_4=0,\ f_5=0,\ f_6=0,\ f_7=0,\ f_8=0,\ f_9=0,\ f_{10}=-1$$

satisfy the same ODE system and initial conditions. By uniqueness,

$$\langle\mathbf{T},\mathbf{T}\rangle_{\mathbb{G}}=\langle\mathbf{N},\mathbf{N}\rangle_\delta=1,\ \langle\mathbf{B}_1,\mathbf{B}_1\rangle_\delta=\langle\mathbf{B}_2,\mathbf{B}_2\rangle_\delta=0$$
$$\langle\mathbf{T},\mathbf{N}\rangle_{\mathbb{G}}=\langle\mathbf{T},\mathbf{B}_1\rangle_{\mathbb{G}}=\langle\mathbf{T},\mathbf{B}_2\rangle_{\mathbb{G}}=\langle\mathbf{N},\mathbf{B}_1\rangle_\delta=\langle\mathbf{N},\mathbf{B}_2\rangle_\delta=0,\ \langle\mathbf{B}_1,\mathbf{B}_2\rangle_\delta=-1.$$

This implies that $\{\mathbf{T},\mathbf{N},\mathbf{B}_1,\mathbf{B}_2\}$ is a null basis of $\mathbb{G}_1^4$, where $\mathbf{N}$ is spacelike. From the definition of α, $\alpha'(x)=\mathbf{T}(x)$ and so α is a curve with spacelike normal $\mathbf{N}$ and lightlike binormal $\mathbf{B}_1$ with curvatures $k_1(x)$ and $k_3(x)$.

Theorem 6.3 Let $k_3(x)$, $x\in I$, be a smooth function. Then, there is a curve with null normal vector $\mathbf{N}(x)$ and curvature $k_3(x)$.

Proof. It can be easily proved from the ODE system (8) as above the theorem.

As we see from Theorem 6.1, we have two different curves having the same curvatures. So, there is not a unique curve with the same curvatures. And also, these curves are not equivalent under pseudo-Galilean motions. Because we don't have any of these motions. But for any two same types orthonormal frame in $\mathbb{G}_1^4$ there is a pseudo-Galilean motion which transforms one frame into the other one and a space curve in $\mathbb{G}_1^4$ under proper pseudo-Galilean motions is transformed in the same type curve. So, we can give the following theorem:

Theorem 6.4 Two admissible same type curves in $\mathbb{G}_1^4$ are equivalent under pseudo-Galilean motions if only if they have the same natural equations for $k_i(x), i=1,2,3$.

7. Applications

Now, we illustrate examples of presented method.

Example 7.1 Let us consider the following curve with spacelike normal vector $\mathbf{N}(x)$ and timelike binormal vector $\mathbf{B}_1(x)$ in the space $\mathbb{G}_1^4$

$$\alpha(x)=(x,1,\cosh x,\sinh x). \tag{9}$$

By differentiating both sides of (9) with respect to arc length x, we have

$$\alpha'(x)=(1,0,\sinh x,\cosh x).$$

Thus, we decompose tangent vector of α as follows:

$$\mathbf{T}(x)=(1,0,\sinh x,\cosh x).$$

And considering the equation (3),

$$k_1(x) = \left\|\mathbf{T}'(x)\right\|_\delta = 1.$$

Thereafter, we arrive at $\mathbf{N}(x)=(0,0,\cosh x,\sinh x)$. So, the curve is a curve with spacelike normal vector.

Moreover, one more differentiating of the normal vector equation, we have

$$\mathbf{N}'(x)=(0,0,\sinh x,\cosh x).$$

By the aid of the this equation, we have the second curvature function

$$k_2(x)=1$$

and timelike binormal vector $\mathbf{B}_1(x)$ is obtained

$$\mathbf{B}_1(x)=(0,0,\sinh x,\cosh x).$$

Furthermore, the cross product of tangent, principal normal, and binormal vectors is formed

$$\mathbf{B}_2(x)=\mathbf{T}(x)\times\mathbf{N}(x)\times\mathbf{B}_1(x)=-\begin{vmatrix}\mathbf{0} & \mathbf{e}_2 & \mathbf{e}_3 & -\mathbf{e}_4\\ 1 & 0 & \cosh x & \sinh x\\ 0 & 0 & \sinh x & \cosh x\\ 0 & 0 & \cosh x & \sinh x\end{vmatrix}.$$

Thus, we have

$$\mathbf{B}_2(x)=(0,1,0,0).$$

In order to determine the third curvature of the curve, considering the equation (6), we have

$$k_3(x)=0.$$

So, the curve is a curve with spacelike normal vector $\mathbf{N}(x)$ and timelike binormal vector $\mathbf{B}_1(x)$. Also, the following equations provide

$$\begin{bmatrix}\mathbf{T}'\\ \mathbf{N}'\\ \mathbf{B}_1'\\ \mathbf{B}_2'\end{bmatrix}=\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\mathbf{T}\\ \mathbf{N}\\ \mathbf{B}_1\\ \mathbf{B}_2\end{bmatrix}.$$

So, we construct the Frenet-Serret apparatus for the given curve $\alpha(x)$. Now, let reconstruct the curve $\alpha(x)$ from only the curvature functions.

Example 7.2 Let $k_1(x)=1,\ k_2(x)=1$ and $k_3(x)=0,\ x\in I,$ and consider the following ordinary differential equation system

$$\mathbf{T}'(x_0)=k_1(x_0)\mathbf{N}(x_0)$$
$$\mathbf{N}'(x_0)=k_2(x_0)\mathbf{B}_1(x_0)$$
$$\mathbf{B}_1'(x_0)=k_2(x_0)\mathbf{N}(x_0)+k_3(x_0)\mathbf{B}_2(x_0)$$
$$\mathbf{B}_2'(x_0)=-k_3(x_0)\mathbf{B}_1(x_0)$$

with initial condition

$$\mathbf{T}(x_0)=(1,0,0,1)$$
$$\mathbf{N}(x_0)=(0,0,1,0)$$
$$\mathbf{B}_1(x_0)=(0,0,0,1)$$
$$\mathbf{B}_2(x_0)=(0,1,0,0).$$

Then we have

$$t_i'=n_i$$
$$n_i'=b_i$$
$$b_{1i}'=n_i$$
$$b_{2i}'=0,$$

for $i=1,2,3,4$. If we solve this ODE with method of Laplace transformation, then we obtain

$\mathbf{T}(x) = (1, 0, \sinh x, \cosh x),$
$\mathbf{N}(x) = (0, 0, \cosh x, \sinh x),$
$\mathbf{B}_1(x) = (0, 0, \sinh x, \cosh x),$
$\mathbf{B}_2(x) = (0, 1, 0, 0).$

Now, let define $\alpha(x) = \int_0^x \mathbf{T}(u)du.$ So, we get $\alpha(x) = (x, 0, \cosh x - 1, \sinh x).$

Actually, we find the same curve in the equation (9)

$\alpha(x) = (x, 1, \cosh x, \sinh x)$

under the translation determined with $\mathbf{u} = (0, -1, 1, 0)$. Finally, the curve is a curve with spacelike normal vector $\mathbf{N}(x)$ and timelike binormal vector $\mathbf{B}_1(x)$ and $k_1(x) = 1,\ k_2(x) = 1$ and $k_3(x) = 0,\ \ x \in I$ (See **Figure 1**).

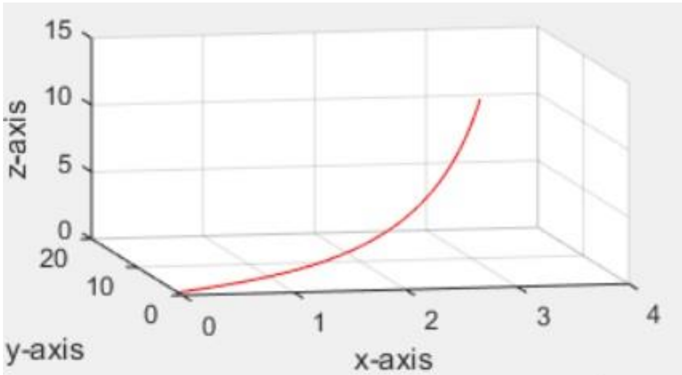

Figure 1. The image of the curve α in 3-dimensional $y=1$-pseudo-Galilean space or 3-dimensional $xzw-$ pseudo-Galilean space in 4-dimensional pseudo-Galiean geometry

8. Conclusion and Further Remarks

Throughout the presented paper, we define pseudo-Galilean motions and pseudo-Galilean geometry $\mathbb{G}_1^4$. That is, we introduce a new geometry. Also, we present the curve theory in $\mathbb{G}_1^4$. Here, using vector product, we give formulas of frame vectors for curves.

Unlike Euclidean, Minkowskian, and Galilean geometries, the curves in pseudo-Galilean geometry $\mathbb{G}_1^4$ are not enough to classify the curves according to their tangent vectors $\mathbf{T}$. It is also necessary to classify with respect to their normal vector $\mathbf{N}$ and binormal vector $\mathbf{B}_1$. There are actually 8-kinds of curves in $\mathbb{G}_1^4$. So, differences according to other geometries are observed in the calculation of the fundamental theorem of curve theory in $\mathbb{G}_1^4$. However, we can construct the Frenet-Serret apparatus for a given curve and also reconstruct the curve from only the curvature functions.

Via this method, some of classical differential geometry topics can be treated. We hope these results will helpful to mathematicians who are specialized in mathematical modeling.

Conflicts of interest

The authors declared no conflict of interests.

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