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New extension of beta, Gauss and confluent hypergeometric functions

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Abstract

There are many extensions and generalizations of Gamma and Beta functions in the literature. However, a new extension of the extended Beta function $B^{\alpha_2;\, m_1,\, m_2}_{\zeta,\ \alpha_1}(a_1, a_2)$ was introduced and presented here because of its important properties. The new extended Beta function has symmetric property, integral representations, Mellin transform, inverse Mellin transform and statistical properties like Beta distribution, mean, variance, moment and cumulative distribution which ware also presented. Finally, the new extended Gauss and Confluent Hypergeometric functions with their propertied were introduced and presented.

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1. Introduction

Functions like factorial and others attracted the attention of Mathematicians for a long period of time. For example, in 1729 a Swiss Mathematician, Leonard Euler generalized factorial function from the domain of natural numbers to the domain over the positive complex plane. Also, in 1811, French Mathematician Adrien-Marie Legendre decomposed Euler's Gamma function into incomplete gamma functions and later in 1814 he introduced the notation of Γ for gamma function. In 1730, Euler also introduced beta function, $B(a_1, a_2)$ for a pair of complex numbers a_1 and a_2 with real positive parts through the integrand. Later on, various extensions of classical gamma and beta functions were studied by renowned Mathematicians and proved to be significantly important in different areas of Applied Mathematics, Statistics, Physics and Engineering such as heat conduction, probability theory, Fourier, Laplace, K-transforms and so on [1-20].

Definition 1. [21] Oraby et al., proposed the following extended beta function:

$$B^{\alpha_2;\ m_1}_{\zeta,\ \alpha_1}(a_1,\ a_2) = \int_0^1 t^{a_1-1}(1-t)^{a_2-1}\, E_{\alpha_1,\alpha_2}\left(-\frac{\zeta}{t^{m_1}(1-t)^{m_1}}\right)dt, \tag{1}$$

$$(Re(a_1) > 0, Re(a_2) > 0, Re(\zeta) \geq 0, Re(\alpha_1) > 0, Re(\alpha_2) > 0,\ Re(m_1) > 0),$$

$E_{\alpha_1,\alpha_2}(;)$ is two parameters Mittag-leffler function.

Definition 2. [22, 23] Wiman function or two parameters Mittag-Leffler function is defined by

$$E_{\alpha_1,\ \alpha_2}(z) = \sum_{\varkappa=0}^{\infty}\frac{z^{\varkappa}}{\Gamma(\varkappa\alpha_1+\alpha_2)},\ (\alpha_1,\ \alpha_2 \in \mathbb{C},\ Re(\alpha_1) > 0,\ Re(\alpha_2) > 0). \tag{2}$$

Definition 3. [24 -26] Classical Mittag-Leffler or one parameter Mittag-Leffler function is defined by

$$E_{\alpha_1}(z) = \sum_{\varkappa=0}^{\infty}\frac{z^{\varkappa}}{\Gamma(\varkappa\alpha_1+1)},\ (\alpha_1 \in \mathbb{C},\ Re(\alpha_1) > 0). \tag{3}$$

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Definition 4. [27] Classical gamma function is defined using integral representation as

$$\Gamma(\alpha_1) = \int_0^\infty t^{\alpha_1 - 1} e^{-t}\, dt, \quad (Re(\alpha_1) > 0). \tag{4}$$

With the Euler reflection formula

$$\Gamma(\alpha_1)\,\Gamma(1-\alpha_1) = \frac{\pi}{sin\pi\alpha_1}, \qquad (\alpha_1 > 0). \tag{5}$$

Definition 5. [27] Classical beta function is defined as

$$B(\alpha_1,\ \alpha_2) = \begin{cases} \int_0^1 t^{\alpha_1 - 1}(1-t)^{\alpha_2 - 1}\, dt, & (Re(\alpha_1) > 0,\ Re(\alpha_2) > 0), \\ \frac{\Gamma(\alpha_1)\ \Gamma(\alpha_2)}{\Gamma(\alpha_1+\alpha_2)}, & (\alpha_1,\ \alpha_2 \in \mathbb{C}\backslash\mathbb{Z}_0^-). \end{cases} \tag{6}$$

The relation also holds

$$B(\alpha_1,\ \alpha_2 - \alpha_1) = \frac{\alpha_2}{\alpha_1} B(\alpha_1 + 1,\ \alpha_2 - \alpha_1), \quad (Re(\alpha_2) > Re(\alpha_1) > 0). \tag{7}$$

Definition 6. [20, 28] Classical pochhammer symbol is defined as

$$(\alpha_1)_\varkappa = \frac{\Gamma(\alpha_1+\varkappa)}{\Gamma(\alpha_1)} = \begin{cases} \alpha_1(\alpha_1+1)(\alpha_1+2)\cdots(\alpha_1+\varkappa-1), & (\varkappa \geq 1), \\ 1, & (\varkappa = 0,\ \alpha_1 \neq 0), \end{cases} \tag{8}$$

with the well-known binomial theorem

$$\sum_{\varkappa=0}^{\infty} (\alpha_1)_\varkappa \frac{(zt)^{\alpha_1}}{\varkappa!} = (1-zt)^{-\alpha_1}. \tag{9}$$

Definition 7. [29, 30] The Mellin transform of integrable function $f(z)$ with index l is defined by

$$f^*(l) = M\{f(\zeta); l\} = \int_0^\infty \zeta^{l-1} f(\zeta) d\zeta. \tag{10}$$

The inverse Mellin transform is defined by

$$f(\zeta) = M^{-1}\{f(\zeta); l\} = \frac{1}{2\pi i}\int_{BS} \zeta^{-l} f^*(l) dl. \tag{11}$$

Definition 8. [31] Classical Gauss hypergeometric function is defined

$$F(\alpha_1,\ \alpha_2;\ \alpha_3;\ z) = \sum_{\varkappa=0}^{\infty} \frac{(\alpha_1)_\varkappa(\alpha_2)_\varkappa}{(\alpha_3)_\varkappa}\frac{z^\varkappa}{\varkappa!}, \tag{12}$$

$$(Re(\alpha_1) > 0, Re(\alpha_2) > 0, Re(\alpha_3) > 0, |z| < 1).$$

And

$$F(\alpha_1, \alpha_2;\ \alpha_3;\ z) = \frac{1}{B(\alpha_2, \alpha_3 - \alpha_2)} \int_0^1 t^{\alpha_2 - 1}(1-t)^{\alpha_3 - \alpha_2 - 1}\,(1-zt)^{-\alpha_1} dt, \tag{13}$$

$$(Re(\alpha_3) > Re(\alpha_2) > 0, |\arg(1-z)| < 1).$$

Definition 9. [31] Classical confluent hypergeometric function is defined as

$$\Phi(\alpha_2;\ \alpha_3;\ z) = \sum_{\varkappa=0}^{\infty} \frac{(\alpha_2)_\varkappa}{(\alpha_3)_\varkappa} \frac{z^\varkappa}{\varkappa!}, \quad (Re(\alpha_2) > 0, Re(\alpha_3) > 0, |z| < 1). \tag{14}$$

And

$$\Phi(\ \alpha_2;\ \alpha_3;\ z) = \frac{1}{B(\alpha_2,\ \alpha_3-\alpha_2)} \int_0^1 t^{\alpha_2-1}(1-t)^{\alpha_3-\alpha_2-1}\, e^{zt} dt, \tag{15}$$

$$(Re(\alpha_3) > Re(\alpha_2) > 0, |\arg(1-z)| < 1).$$

Definition 10. [32] The relations between Mittag-Leffler and gamma function is

$$\int_0^\infty u^{l-1} E_{\alpha_1,\alpha_2}^{\alpha_3}(-\mu u) du = \frac{\Gamma(l)\,\Gamma(\alpha_3-l)}{\mu^l\,\Gamma(\alpha_3)\,\Gamma(\alpha_2-l\alpha_3)}. \tag{16}$$

Setting $\alpha_3 = \mu = 1$ in equation (16), becomes

$$\int_0^\infty u^{l-1} E_{\alpha_1,\alpha_2}(-u) du = \frac{\Gamma(l)\,\Gamma(1-l).}{\Gamma(,\alpha_2-l)} \tag{17}$$

Definition 11. The extended beta function is defined as

$$B_{\zeta,\ \alpha_1}^{\alpha_2;\,m_1,m_2}(a_1,a_2) = \int_0^1 t^{a_1-1}(1-t)^{a_2-1}\, E_{\alpha_1,\alpha_2}\left(-\frac{\zeta}{t^{m_1}(1-t)^{m_2}}\right) dt, \tag{18}$$

$$(Re(a_1) > 0, Re(a_2) > 0, Re(\zeta) \geq 0, Re(\alpha_1) > 0, Re(\alpha_2) > 0, Re(m_1) > 0, Re(m_2) > 0),$$

$E_{\alpha_1,\ \alpha_2}(;)$ is two parameters Mittag-Leffler function.

Definition 12. The extended Gauss hypergeometric function is defined as

$$F_{\zeta,\ \alpha_1}^{\alpha_2;\,m_1,m_2}(a_1,\ a_2;\ a_3;\ z) = \sum_{\varkappa=0}^{\infty} (a)_\varkappa \frac{B_{\zeta,\ \alpha_1}^{\alpha_2;\,m_1,m_2}(a_2+\varkappa,\ a_3-a_2)}{B(a_2,\ a_3-a_2)} \frac{z^\varkappa}{\varkappa!}, \tag{19}$$

$$(Re(m_1) > 0, Re(m_2) > 0, Re(\alpha_1) > 0, Re(\alpha_2) > 0, Re(a_1) > 0, Re(a_3) > Re(a_2) > 0, Re(\zeta) \geq 0).$$

Definition 13. The extended Gauss hypergeometric function is defined as

$$\Phi_{\zeta,\ \alpha_1}^{\alpha_2;\,m_1,m_2}(\ a_2;\ a_3;\ z) = \sum_{\varkappa=0}^{\infty} \frac{B_{\zeta,\ \alpha_1}^{\alpha_2;\,m_1,m_2}(a_2+\varkappa,\ a_3-a_2)}{B(a_2,\ a_3-a_2)} \frac{z^\varkappa}{\varkappa!}. \tag{20}$$

$$(Re(m_1) > 0, Re(m_2) > 0, Re(\alpha_1) > 0, Re(\alpha_2) > 0, Re(a_3) > Re(a_2) > 0, Re(\zeta) \geq 0).$$

2. Special Cases

Some special cases of the new extended beta function are

Cases 1: When $m_1 = m_2$, then the new extended beta function reduces to the beta function [21]:

$$B_{\zeta,\ \alpha_1}^{\alpha_2;\, m_1,\, m_1}(a_1,a_2) = B_{\zeta,\ \alpha_1}^{\alpha_2;\, m_1}(a_1,\ a_2) = \int_0^1 t^{a_1-1}(1-t)^{a_2-1}\, E_{\alpha_1,\alpha_2}\left(-\frac{\zeta}{t^{m_1}(1-t)^{m_1}}\right) dt, \tag{21}$$

$$(Re(a_1) > 0, (Re(a_2) > 0, Re(\zeta) \geq 0, Re(\alpha_1) > 0, Re(\alpha_2) > 0, Re(m_1) > 0).$$

Cases 2: If $m_1 = m_2$ and $\alpha_2 = 1$, then the new extended beta function reduces to the beta function [33]:

$$B_{\zeta,\ \alpha_1}^{1;\, m_1,\ m_1}(a_1,a_2) = B_{\zeta,\ \alpha_1}^{\alpha_1;\, m_1}(a_1,a_2) = \int_0^1 t^{a_1-1}(1-t)^{a_2-1} E_{\alpha_1}\left(-\frac{\zeta}{t^{m_1}(1-t)^{m_1}}\right) dt, \tag{22}$$

$$(Re(a_1) > 0, (Re(a_2) > 0, Re(\zeta) \geq 0, Re(\alpha_1) > 0, Re(m_1) > 0).$$

Cases 3: If $m_1 = m_2 = 1$ and $\alpha_2 = 1$, then the new extended beta functions reduce to the beta function [34]:

$$B_{\zeta,\ \alpha_1}^{1;\, 1,\ 1}(a_1,a_2) = B_{\zeta,\ \alpha_1}^{\alpha_1}(a_1,a_2) = \int_0^1 t^{a_1-1}(1-t)^{a_2-1} E_{\alpha_1}\left(-\frac{\zeta}{t(1-t)}\right) dt, \tag{23}$$

$$(Re(a_1) > 0, (Re(a_2) > 0, Re(\zeta) \geq 0, Re(\alpha_1) > 0).$$

Cases 4: When $\alpha_1 = \alpha_2 = 1$, then the new extended beta function reduces to the beta function as in [35]:

$$B_{\zeta,\ 1}^{1;\, m_1,\ m_2}(a_1,a_2) = B_{\zeta}^{m_1}(a_1,a_2) = \int_0^1 t^{a_1-1}(1-t)^{a_2-1}\, exp\left(-\frac{\zeta}{t^{m_1}(1-t)^{m_2}}\right) dt, \tag{24}$$

$$(Re(a_1) > 0, (Re(a_2) > 0, Re(\zeta) \geq 0, Re(m_1) > 0, Re(m_2) > 0).$$

Cases 5: When $m_1 = m_2$ and $\alpha_1 = \alpha_2 = 1$, then the new extended beta function reduces to the beta function as in [36]:

$$B_{\zeta,\ 1}^{1;\, m_1, m_1}(a_1,a_2) = B_{\zeta}^{m_1}(a_1,a_2) = \int_0^1 t^{a_1-1}(1-t)^{a_2-1}\, exp\left(-\frac{\zeta}{t^{m_1}(1-t)^{m_1}}\right) dt, \tag{25}$$

$$(Re(a_1) > 0, (Re(a_2) > 0, Re(\zeta) \geq 0, Re(m_1) > 0).$$

Cases 6: If $m_1 = m_2 = \alpha_1 = \alpha_2 = 1$, then the new extended beta function reduces to the beta function as in [37]:

$$B_{\zeta,\ 1}^{1;\, 1,\ 1}(a_1,a_2) = B_{\zeta}(a_1,a_2) = \int_0^1 t^{a_1-1}(1-t)^{a_2-1}\, exp(-\frac{\zeta}{t(1-t)}) dt, \tag{26}$$

$$(Re(a_1) > 0, (Re(a_2) > 0, Re(\zeta) \geq 0).$$

Cases 7: If $\xi = 0$ and $m_1 = m_2 = \alpha_1 = \alpha_2 = 1$, then the new extended beta function reduces to the classical beta function as in [27]:

$$B_{0,\ 1}^{1;\ 1,\ 1}(a_1, a_2) = B(a_1, a_2) = \int_0^1 t^{a_1-1}(1-t)^{a_2-1}\,dt, \tag{27}$$

$$(Re(a_1) > 0, (Re(a_2) > 0).$$

3. Generalized Beta Function

Theorem 1.

$$B_{\zeta,\ \alpha_1}^{\alpha_2;\ m_1,\ m_2}(a_1+1, a_2) + B_{\zeta,\ \alpha_1}^{\alpha_2;\ m_1, m_2}(a_1, a_2+1) = B_{\zeta,\ \alpha_1}^{\alpha_2;\ m_1, m_2}(a_1, a_2). \tag{28}$$

Proof. On setting left hand side of (28) to be L and direct calculation

$$L = \int_0^1 t^{a_1}(1-t)^{a_2}\,E_{\alpha_1,\,\alpha_2}\left(-\frac{\zeta}{t^{m_1}(1-t)^{m_2}}\right)\{(1-t)^{-1} + t^{-1}\}dt. \tag{29}$$

On simplification of the equation (29),

$$L = \int_0^1 t^{a_1}(1-t)^{a_2}\,E_{\alpha_1,\,\alpha_2}\left(-\frac{\zeta}{t^{m_1}(1-t)^{m_2}}\right)\{(1-t)^{-1}t^{-1}\}dt. \tag{30}$$

Applying equation (18) to (30), the desired result in (28) is obtained.

Theorem 2.

$$B_{\zeta,\ \alpha_1}^{\alpha_2;\ m_1,\ m_2}(a_1, 1-a_2) = \sum_{\varkappa=0}^{\infty}\frac{(a_2)_{\varkappa}}{\varkappa!}B_{\zeta,\ \alpha_1}^{\alpha_2;\ m_1,\ m_2}(a_1+\varkappa, 1). \tag{31}$$

Proof. By direct calculation

$$B_{\zeta,\ \alpha_1}^{\alpha_2;\ m_1,\ m_2}(a_1, 1-a_2) = \int_0^1 t^{a_1-1}(1-t)^{\,a_2}\,E_{\alpha_1,\,\alpha_2}\left(-\frac{\zeta}{t^{m_1}(1-t)^{m_2}}\right)dt. \tag{32}$$

Applying equation (9) to (32), yield

$$B_{\zeta,\ \alpha_1}^{\alpha_2;\ m_1, m_2}(a_1, 1-a_2) = \int_0^1 t^{a_1-1}\sum_{\varkappa=0}^{\infty}\,(a_2)_{\varkappa}\frac{t^{\varkappa}}{\varkappa!}E_{\alpha_1,\alpha_2}\left(-\frac{\zeta}{t^{m_1}(1-t)^{m_2}}\right)dt. \tag{33}$$

On interchanging the order of summation and integration in equation (33),

$$B_{\zeta,\ \alpha_1}^{\alpha_2;\, m_1, m_2}(a_1, 1-a_2) = \sum_{\varkappa=0}^{\infty} (a_2)_{\varkappa} \frac{t^{\varkappa}}{\varkappa!} \int_0^1 t^{a_1-1} E_{\alpha_1,\ \alpha_2}\left(-\frac{\zeta}{t^{m_1}(1-t)^{m_2}}\right) dt. \qquad (34)$$

Applying equation (18) to (34), the desired result is obtained.

Theorem 3.

$$B_{\zeta,\ \alpha_1}^{\alpha_2;\, m_1,\, m_2}(a_1, a_2) = \sum_{\varkappa=0}^{\infty} B_{\zeta,\ \alpha_1}^{\alpha_2;\, m_1, m_2}(a_1+\varkappa, a_2+1). \qquad (35)$$

Proof. By direct calculation

$$B_{\zeta,\ \alpha_1}^{\alpha_2;\, m_1,\, m_2}(a_1, a_2) = \int_0^1 t^{a_1-1}(1-t)^{a_2}(1-t)^{-1} E_{\alpha_1, \alpha_2}\left(-\frac{\zeta}{t^{m_1}(1-t)^{m_2}}\right) dt. \qquad (36)$$

Applying equation (9) to (36), yield

$$B_{\zeta,\ \alpha_1}^{\alpha_2;\, m_1, m_2}(a_1, a_2) = \int_0^1 t^{a_1-1}(1-t)^{a_2} \sum_{n=0}^{\infty} \mathrm{t}^{\aleph} E_{\alpha_1,\ \alpha_2}\left(-\frac{\zeta}{t^{m_1}(1-t)^{m_2}}\right) dt. \qquad (37)$$

On interchanging the order of summation and integration in equation (37), we have

$$B_{\zeta,\ \alpha_1}^{\alpha_2;\, m_1, m_2}(a_1, a_2) = \sum_{\varkappa=0}^{\infty} \int_0^1 t^{a_1+\varkappa-1}(1-t)^{a_2} E_{\alpha_1, \alpha_2}\left(-\frac{\zeta}{t^{m_1}(1-t)^{m_2}}\right) dt. \qquad (38)$$

Applying equation (18) to (38), gives the desired result.

Theorem 4. For the new extended beta function,

$$B_{\zeta,\ \alpha_1}^{\alpha_2;\, m_1,\, m_2}(a_1, a_2) = B_{\zeta,\ \alpha_1}^{\alpha_2;\, m_2, m_1}(a_2,\ a_1). \qquad (39)$$

Proof. Setting $t \to 1-t$ in equation (18), gives the required result in (39).

4. Integral Representations

Theorem 5.

$$B_{\zeta,\ \alpha_1}^{\alpha_2;\, m_1,\, m_2}(a_1, a_2) = 2\int_0^{\frac{\pi}{2}} cos^{2a_1-1}\varphi\, sin^{2a_2-1}\varphi E_{\alpha_1, \alpha_2}\left(-\frac{\zeta}{sin^{2m_1}\varphi\, cos^{2m_2}\varphi}\right) d\varphi, \qquad (40)$$

$$B_{\zeta,\ \alpha_1}^{\alpha_2;\, m_1,\, m_2}(a_1, a_2) = \int_0^{\infty} \frac{t^{a_1-1}}{(1+t)^{a_1+a_2}} E_{\alpha_1,\ \alpha_2}\left(-\frac{\zeta(1+t)^{m_1+m_2}}{t^{m_1}}\right) dt, \qquad (41)$$

$$B_{\zeta,\ \alpha_1}^{\alpha_2;\, m_1,\, m_2}(a_1, a_2) = 2^{1-(a_1+a_2)} \int_{-1}^{1} (1+t)^{a_1-1}(1-t)^{a_2-1} E_{\alpha_1, \alpha_2}\left(-\frac{\zeta^{m_1+m_2}}{(1+t)^{m_1}(1-t)^{m_2}}\right) dt, \qquad (42)$$

$$B_{\zeta,\ \alpha_1}^{\alpha_2;m_1,\ m_2}(a_1,a_2) = \int_{a'}^{c'} \frac{(t-a')^{a_1-1}(c'-t)^{a_2-1}}{(c'-a')^{a_1+a_2-1}} E_{\alpha_1,\alpha_2}\left(-\frac{\zeta(c'-a')^{m_1+m_2}}{(t-a_{,})^{m_1}(c'-t_{,})^{m_2}}\right) dt. \tag{43}$$

Proof. Equations (40), (41), (42) and (43) can be obtained by putting $t = cos^2\varphi$, $t = u(1+u)^{-1}$, $t = 2^{-1}(1+u)$ and $t = (u-a')(c'-a')^{-1}$, respectively in equation (18) and by changing of variable.

Theorem 6.

$$B_{\zeta,\ \alpha_1}^{\alpha_2;\ m_1,\ m_2}(a_1,a_2) = n\int_0^1 t^{na_1-1}(1-t^n)^{a_2-1} E_{\alpha_1,\alpha_2}\left(-\frac{\zeta}{t^{nm_1}(1-t^n)^{m_2}}\right)dt, \tag{44}$$

$$B_{\zeta,\ \alpha_1}^{\alpha_2;\ m_1,\ m_2}(a_1,a_2) = \int_0^{a'} t^{a_1-1}(a'-t)^{a_2-1} E_{\alpha_1,\ \alpha_2}\left(-\frac{\zeta\ a'^{m_1+m_2}}{t^{m_1}(a'-t)^{m_2}}\right)dt. \tag{45}$$

Proof. Equations (44) and (45) can be obtained by putting $t = u^n$ and $t = ua'^{-1}$, respectively in equation (18) and change of variable.

5. Mellin Transform

Theorem 7.

$$M\{B_{\zeta,\ \alpha_1}^{\alpha_2;\ m_1,\ m_2}(a_1,a_2);l\} = \frac{\pi}{\Gamma(\alpha_2-l)sin\pi l} B(a_1+m_1 l,\ a_2+m_2 l). \tag{46}$$

Proof. Using definition of Mellin transform in equation (10), we have

$$M\{B_{\zeta,\ \alpha_1}^{\alpha_2;\ m_1,\ m_2}(a_1,\ a_2);\ l\} = \int_0^\infty \zeta^{l-1} B_{\zeta,\ \alpha_1}^{\alpha_2;\ m_1,\ m_2}(a_1,\ a_2)d\zeta. \tag{47}$$

Substituting equation (18) into (47), we get

$$M\{B_{\zeta,\ \alpha_1}^{\alpha_2;\ m_1,m_2}(a_1,\ a_2);\ l\} = \int_0^\infty \zeta^{l-1}\left\{\int_0^1 t^{a_1-1}(1-t)^{a_2-1} E_{\alpha_1,\alpha_2}\left(-\frac{\zeta}{t^{m_1}(1-t)^{m_2}}\right)dt\right\}d\zeta. \tag{48}$$

Interchanging the order of integrations in equation (48), yield

$$M\{B_{\zeta,\ \alpha_1}^{\alpha_2;\ m_1,\ m_2}(a_1,\ a_2);\ l\} = \int_0^1 t^{a_1-1}(1-t)^{a_2-1}\left\{\int_0^\infty \zeta^{l-1}\ E_{\alpha_1,\alpha_2}\left(-\frac{\zeta}{t^{m_1}(1-t)^{m_2}}\right)d\zeta\right\}dt. \tag{49}$$

On setting $\zeta = ut^{m_1}(1-t)^{m_2}$ in equation (49), we obtain

$$M\{B_{\zeta,\ \alpha_1}^{\alpha_2;\ m_1,\ m_2}(a_1,a_2);\ l\} = \int_0^1 t^{a_1+m_1 l-1}(1-t)^{a_2+m_2 l-1}\ \left\{\int_0^\infty u^{l-1} E_{\alpha_1,\alpha_2}(-u)du\right\}dt. \tag{50}$$

On applying equations (4), (5) and (17) to (50), the desired result can be obtained.

Corollory 8. The inverse Mellin transform:

$$B_{\zeta,\ \alpha_1}^{\alpha_2;\ m_1,\ m_2}(a_1,\ a_2)=\frac{1}{2\pi i}\int_{\gamma'-i\infty}^{\gamma'+i\infty}\frac{\Gamma(a_1+m_1 l)\,\Gamma(a_2+m_2 l)}{\Gamma(\alpha_1-\alpha_2 l)\,\Gamma(a_1+a_2+m_1 l+m_2 l)}\zeta^{-s}dl,$$

where

$Re(m_1)>0,\ Re(m_2)>0,\ Re(a_1)>0, Re(a_2)>0, Re(\alpha_1-s\alpha_2)>0, Re(a_1+m_1 l)>0,$ $Re(a_2+m_2 l)>0,\ Re(\zeta)\geq 0,\ \gamma'>0$.

6. Beta Distribution

The beta distribution of the new extended beta function is

$$f(t)=\begin{cases}\frac{1}{B_{\zeta,\ \alpha_1}^{\alpha_2;\ m_1,\ m_2}(a_1,\ a_2)}t^{a_1-1}(1-t)^{a_2-1}E_{\alpha_1,\alpha_2}\left(-\frac{\zeta}{t^{m_1}(1-t)^{m_2}}\right), & 0<t<1,\\ 0, & \text{elsewhere},\end{cases} \tag{51}$$

$$(a_1, a_2\in\mathbb{R}, \zeta, \alpha_1, \alpha_1\in\mathbb{R}^+).$$

The moment of X, is given by:

$$E(X^r)=\frac{B_{\zeta,\ \alpha_1}^{\alpha_2;\ m_1,m_2}(a_1+r,\ a_2)}{B_{\zeta,\ \alpha_1}^{\alpha_2;\ m_1,\ m_2}(a_1,\ a_2)},\ (a_1,\ a_2,\ r\in\mathbb{R};\ \zeta,\ \alpha_1,\ \alpha_1\in\mathbb{R}^+). \tag{52}$$

On setting $r=1$ in (52), we obtained the mean of the distribution as

$$E(X^r)=\frac{B_{\zeta,\ \alpha_1}^{\alpha_2;\ m_1,\ m_2}(a_1+r,\ a_2)}{B_{\zeta,\ \alpha_1}^{\alpha_2;\ m_1,\ m_2}(a_1, a_2)}.$$

The variance of the distribution given in equation (51) is

$$\delta=E(X^2)-\{E(X)\}^2=\frac{B_{\zeta,\ \alpha_1}^{\alpha_2;\ m_1,\ m_2}(a_1,a_2)B_{\zeta,\ \alpha_1}^{\alpha_2;\ m_1,\ m_2}(a_1+2,\ a_2)-\{B_{\zeta,\ \alpha_1}^{\alpha_2;\ m_1,\ m_2}(a_1+1,\ a_2)\}^2}{\{B_{\zeta,\ \alpha_1}^{\alpha_2;\ m_1,\ m_2}(a_1, a_2)\}^2}.$$

Cumulative distribution is

$$F(x)=\frac{B_{\zeta,\alpha_1,x}^{\alpha_2;\ m_1,\ m_2}(a_1+1, a_2)}{B_{\zeta,\ \alpha_1}^{\alpha_2;\ m_1,\ m_2}(a_1, a_2)},$$

where $B_{\zeta,\ \alpha_1,\ x}^{\alpha_2;\ m_1,\ m_2}(a_1,\ a_2)$ is the new extended incomplete beta function defined by:

$$B_{\zeta,\ \alpha_1,\ x}^{\alpha_2;\, m_1,\, m_2}(a_1,\ a_2) = \int_0^x t^{a_1-1}(1-t)^{a_2-1}E_{\alpha_1,\,\alpha_2}\left(-\frac{\zeta}{t^{m_1}(1-t)^{m_2}}\right)dt,$$

$$(a_1, a_2 \in \mathbb{R};\ \zeta, \alpha_1, \alpha_1 \in \mathbb{R}^+).$$

7. Gauss and Confluent Hypergeometric Function

Theorem 9.

$$F_{\zeta,\ \alpha_1}^{\alpha_2;\, m_1, m_2}(a_1,\ a_2;\ a_3;\ z) = \frac{1}{B(a_2,\ a_3-a_2)}\int_0^1 t^{a_2-1}(1-t)^{a_3-a_2-1}$$

$$\times\ (1-zt)^{-a_1}E_{\alpha_1,\ \alpha_2}\left(-\frac{\zeta}{t^{m_1}(1-t)^{m_2}}\right)dt. \qquad (53)$$

Proof. Applying equation (18) to (19), gives

$$F_{\zeta,\ \alpha_1}^{\alpha_2;\, m_1, m_2}(a_1,\ a_2;\ a_3;\ z) = \frac{1}{B(a_2,\ a_3-a_2)}\sum_{\varkappa=0}^{\infty}(a_1)_\varkappa \int_0^1 t^{a_2+\varkappa-1}(1-t)^{a_3-a_2-1}$$

$$\times\ E_{\alpha_1,\,\alpha_2}\left(-\frac{\zeta}{t^{m_1}(1-t)^{m_2}}\right)dt\ \ \frac{z^\varkappa}{\varkappa!}. \qquad (54)$$

Interchanging the order of summation and integration in equation (54), we have

$$F_{\zeta,\ \alpha_1}^{\alpha_2;\, m_1, m_2}(a_1,\ a_2;\ a_3;\ z) = \frac{1}{B(a_2,\ a_3-a_2)}\int_0^1 t^{a_2-1}(1-t)^{a_3-a_2-1}$$

$$\times\ \sum_{\varkappa=0}^{\infty}(a_1)_\varkappa \frac{(zt)^\varkappa}{\varkappa!}E_{\alpha_1,\,\alpha_2}\left(-\frac{\zeta}{t^{m_1}(1-t)^{m_2}}\right)dt. \qquad (55)$$

Applying equation (9) to (55), give the desired result in (53).

Theorem 10.

$$F_{\zeta,\ \alpha_1}^{\alpha_2;\, m_1, m_2}(a_1,\ a_2;\ a_3;\ z) = \frac{1}{B(a_2,\ a_3-a_2)}\int_0^\infty \frac{t^{a_2-1}}{(1+t)^{a_3-a_1}}\{1+t(1-z)\}^{a_1}\ E_{\alpha_1,\ \alpha_2}\left(-\frac{\zeta(1+t)^{m_2+m_2}}{t^{m_1}}\right)dt, \qquad (56)$$

$$F_{\zeta,\ \alpha_1}^{\alpha_2;\, m_1,\ m_2}(a_1, a_2;\ a_3;\ z) = \frac{2}{B(a_2,\ a_3-a_2)}\int_0^{\frac{\pi}{2}} \frac{sin^{2a_2-1}\varphi\ cos^{2a_3-2a_2-1}\varphi}{(1-zsin^2\varphi)^{a_1}}$$

$$\times\ E_{\alpha_1,\,\alpha_2}(-\zeta sec^{2m_1}\varphi\ csc^{2m_2}\varphi)d\varphi, \qquad (57)$$

$$F_{\zeta,\ \alpha_1}^{\alpha_2;\, m_1, m_2}(a_1, a_2;\ a_3;\ z) = \frac{2}{B(a_2,\ a_3-a_2)}\int_0^\infty \frac{sinh^{2a_2-1}\varphi\ cosh^{2a_3-2a_2-1}\varphi}{(cosh^2\varphi - zsinh^2\varphi)^{a_1}}$$

$$\times\ E_{\alpha_1,\,\alpha_2}(-\zeta cosh^{2m_1}\varphi\ coth^{2m_2}\varphi)d\varphi, \qquad (58)$$

which are the new extended hypergeometric function integral representations.

Proof. Equations (56), (57) and (58) can be obtained by substituting $t = u(1+u)^{-1}, t = sin^2\varphi$ and $t = tanh^2\varphi$, respectively in to (53).

Theorem 11.

$$\Phi_{\zeta,\ \alpha_1}^{\alpha_2;\, m_1, m_2}(\ a_2;\ a_3;\ z) = \frac{1}{B(a_2,\ a_3-a_2)} \int_0^1 t^{a_2-1}(1-t)^{a_3-a_2-1} \exp(zt)\ E_{\alpha_1, \alpha_2}\left(-\frac{\zeta}{t^{m_1}(1-t)^{m_2}}\right) dt, \qquad (59)$$

which is the new extended confluent hypergeometric function integral representation.

Proof. Applying equation (18) to (20), gives

$$\Phi_{\zeta,\ \alpha_1}^{\alpha_2;\, m_1,\ m_2}(\ a_2;\ a_3;\ z) = \frac{1}{B(a_2,\ a_3-a_2)} \sum_{\varkappa=0}^{\infty} \int_0^1 t^{a_2+\varkappa-1}(1-t)^{a_3-a_2-1} E_{\alpha_1,\alpha_2}\left(-\frac{\zeta}{t^{m_1}(1-t)^{m_2}}\right) dt\ \frac{z^{\varkappa}}{\varkappa!}. \qquad (60)$$

Interchanging the order of summation and integration in equation (60), we have

$$\Phi_{\zeta,\ \alpha_1}^{\alpha_2;\, m_1,\ m_2}(\ a_2;\ a_3;\ z) = \frac{1}{B(a_2,\ a_3-a_2)} \int_0^1 t^{a_2-1}(1-t)^{a_3-a_2-1}$$

$$\times\ \sum_{\varkappa=0}^{\infty} \frac{(zt)^{\varkappa}}{\varkappa!} E_{\alpha_1,\alpha_2}\left(-\frac{\zeta}{t^{m_1}(1-t)^{m_2}}\right) dt\ . \qquad (61)$$

Corollary 12. For the new extended confluent hypergeometric function, the following formula hold.

$$\Phi_{\zeta,\ \alpha_1}^{\alpha_2;\, m_1, m_2}(\ a_2;\ a_3;\ z) = \frac{\exp(z)}{B(a_2,\ a_3-a_2)} \int_0^1 t^{a_2-1}(1-t)^{a_3-a_2-1} \exp(-zt) E_{\alpha_1,\ \alpha_2}\left(-\frac{\zeta}{t^{m_1}(1-t)^{m_2}}\right) dt.$$

Theorem 13.

$$\frac{d}{dz} F_{\zeta,\ \alpha_1}^{\alpha_2;\, m_1,\ m_2}(a_1,\ a_2;\ a_3;\ z) = \frac{a_1 a_2}{a_3} F_{\zeta,\ \alpha_1}^{\alpha_2;\, m_1, m_2}(a_1+1,\ a_2+1;\ a_3+1;\ z), \qquad (62)$$

$$\frac{d^{\varkappa}}{dz^{\varkappa}} F_{\zeta,\ \alpha_1}^{\alpha_2;\, m_1, m_2}(a_1,\ a_2;\ a_3;\ z) = \frac{(a_1)_{\varkappa}(a_2)_{\varkappa}}{(a_3)_{\varkappa}} F_{\zeta,\ \alpha_1}^{\alpha_2;\, m_1,\ m_2}(a_1+\varkappa,\ a_2+\varkappa;\ a_3+\varkappa;\ z), \qquad (63)$$

which are differential formulas.

Proof. Using equation (19), we have

$$\frac{d}{dz} F_{\zeta,\ \alpha_1}^{\alpha_2;\, m_1, m_2}(a_1,\ a_2;\ a_3;\ z) = \sum_{\varkappa=1}^{\infty} \frac{B_{\zeta,\ \alpha_1,\ x}^{\alpha_2;\, m_1, m_2}(a_2+\varkappa,\ a_3-a_2)}{B(a_2,\ a_3-a_2)}\ (a)_{\varkappa} \frac{z^{\varkappa-1}}{(\varkappa-1)!}. \qquad (64)$$

Setting $\varkappa \to \varkappa+1$ in equation (64), we get

$$\frac{d}{dz}F_{\zeta,\ \alpha_1}^{\alpha_2;\ m_1,\ m_2}(a_1,\ a_2;\ a_3;\ z) = a\sum_{\varkappa=0}^{\infty}\frac{B_{\zeta,\ \alpha_1,\ \varkappa}^{\alpha_2;\ m_1,\ m_2}(a_2+\varkappa+1,\ a_3-a_2)}{B(a_2,\ a_3-a_2)}(a+1)_\varkappa\frac{z^\varkappa}{\varkappa!}. \quad (65)$$

Applying equation (7) to (65), the desired result in equation (62) is obtained. On successive differentiation of equation (62), also the required result in (63) is obtained.

Corollary 14.

$$\frac{d}{dz}\Phi_{\zeta,\ \alpha_1}^{\alpha_2;\ m_1,\ m_2}(a_2;\ a_3;\ z) = \frac{a_2}{a_3}F_{\zeta,\ \alpha_1}^{\alpha_2;\ m_1,\ m_2}(a_2+1;\ a_3+1;\ z),$$

$$\frac{d^\varkappa}{dz^\varkappa}\Phi_{\zeta,\ \alpha_1}^{\alpha_2;\ m_1,\ m_2}(a_2;\ a_3;\ z) = \frac{(a_2)_\varkappa}{(a_3)_\varkappa}F_{\zeta,\alpha_1}^{\alpha_2;\ m_1,\ m_2}(a_2+\varkappa;\ a_3+\varkappa;\ z).$$

8. Mellin Transform

Theorem 15.

$$M\{F_{\zeta,\ \alpha_1}^{\alpha_2;\ m_1,\ m_2}(a_1,\ a_2;\ a_3;\ z);\ l\} = \frac{\pi B(a_2+m_1 l,\ a_3+m_1 l-a_2)}{\sin\pi l\ \Gamma(\alpha_1-\alpha_2)B(a_2,\ a_3-a_2)}F(a_1,\ a_2+m_1 l,\ a_3+m_1 l+m_2 l;\ z). \quad (66)$$

Proof. Using definition of the Mellin transform in equation (10), we have

$$M\left\{F_{\zeta,\ \alpha_1}^{\alpha_2;\ m_1,\ m_2}(a_1,\ a_2;\ a_3;\ z);\ l\right\} = \int_0^\infty \zeta^{l-1}F_{\zeta,\ \alpha_1}^{\alpha_2;\ m_1,\ m_2}(a_1,\ a_2;\ a_3;\ z)d\zeta. \quad (67)$$

Substituting equation (19) into (67), we have

$$M\{F_{\zeta,\ \alpha_1}^{\alpha_2;\ m_1,\ m_2}(a_1,\ a_2;\ a_3;\ z);\ l\} = \int_0^\infty \zeta^{l-1}\{\frac{1}{B(a_2,\ a_3-a_2)}\int_0^1 t^{a_2-1}(1-t)^{a_3-a_2-1}$$

$$\times(1-zt)^{-a_1}E_{\alpha_1,\alpha_2}\left(-\frac{\zeta}{t^{m_1}(1-t)^{m_2}}\right)dt\}d\zeta. \quad (68)$$

Interchanging the order of integrations in equation (68), yields

$$M\{F_{\zeta,\ \alpha_1}^{\alpha_2;\ m_1,\ m_2}(a_1,\ a_2;\ a_3;\ z);\ l\} = \frac{1}{B(a_2,\ a_3-a_2)}\int_0^1 t^{a_2-1}(1-t)^{a_3-a_2-1}(1-zt)^{-a_1}$$

$$\times\left\{\int_0^\infty \zeta^{l-1}E_{\alpha_1,\alpha_2}\left(-\frac{\zeta}{t^{m_1}(1-t)^{m_2}}\right)d\zeta\right\}dt. \quad (69)$$

On setting $\zeta = ut^{m_1}(1-t)^{m_2}$ in (69), we obtain

$$M\{F_{\zeta,\ \alpha_1}^{\alpha_2;\ m_1,\ m_2}(a_1,\ a_2;\ a_3;\ z);\ l\} = \frac{1}{B(a_2,\ a_3-a_2)}\int_0^1 t^{a_2-1}(1-t)^{a_3-a_2-1}(1-zt)^{-a_1}$$

$$\times\left\{\int_0^\infty u^{l-1}E_{\alpha_1,\alpha_2}(-u)du\right\}dt. \quad (70)$$

On applying equations (4), (5) and (17) to (70), the desired result obtained.

Corollary 16.

$$F_{\zeta,\ \alpha_1}^{\alpha_2;\, m_1, m_2}(a_1,\ a_2;\ a_3;\ z)=\frac{1}{2iB(a_2,\ a_3-a_2)}\int_{\gamma'-i\infty}^{\gamma'+i\infty}\frac{\Gamma(a_2+m_1l)\Gamma(a_3+m_2l-a_2)}{\Gamma(\alpha_1-\alpha_2 l)\Gamma(a_1+a_2+m_1l+m_2l)}$$

$$\times\ F(a_1,\ a_2+m_1l,\ a_3+m_1l+m_2l;\ z)\zeta^{-l}ds,$$

where

$Re(m_1)>0, Re(m_2)>0, Re(a_1)>0, Re(a_2)>0, Re(\alpha_1-sl)>0, Re(a_1+m_1l)>0, Re(a_2+m_2l)>0, Re(\zeta)\geq 0, \gamma'>0.$

Corollary 17.

$$M\{\Phi_{\zeta,\ \alpha_1}^{\alpha_2;\ m_1,\ m_2}(a_1,\ a_2;\ a_3;\ z);\ l\}=\frac{\pi\ B(a_2+m_1l,\ a_3+m_1l-a_2)}{sin\pi l\ \Gamma(\alpha_1-\alpha_2)\ B(a_2,\ a_3-a_2)}$$

$$\times\ \Phi(a_2+m_1l,\ a_3+m_1l+m_2l;\ z),$$

$$\Phi_{\zeta,\ \alpha_1}^{\alpha_2;\, m_1, m_2}(a_2;\ a_3;\ z)=\frac{1}{2i\ B(a_2,\ a_3-a_2)}\int_{\gamma'-i\infty}^{\gamma'+i\infty}\frac{\Gamma(a_2+m_1s)\Gamma(a_3+m_2s-a_2)}{\Gamma(\alpha_1-\alpha_2 l)\Gamma(a_1+a_2+m_1l+m_2l)}$$

$$\times\ \Phi(a_2+m_1l,\ a_3+m_1l+m_2l;\ z)\zeta^{-l}dl,$$

Where

$Re(m_1)>0, Re(m_2)>0, Re(a_1)>0, Re(a_2)>0, Re(\alpha_1-l\alpha_2)>0,\ Re(a_1+m_1l)>0, Re(a_2+m_2l)>0, Re(\zeta)\geq 0, \gamma'>0$

.

Theorem 18.

$$F_{\zeta,\ \alpha_1}^{\alpha_2;\, m_1, m_2}(a_1,\ a_2;\ a_3;\ z)=(1-z)^{-a_1}F_{\zeta,\ \alpha_1}^{\alpha_2;\, m_1, m_2}\left(a_1,\ a_2;\ a_3;\ \frac{z}{z-1}\right), \tag{71}$$

$$\Phi_{\zeta,\ \alpha_1}^{\alpha_2;\, m_1, m_2}(a_2;\ a_3;\ z)=\exp(z)\ \ \Phi_{\zeta,\ \alpha_1}^{\alpha_2;\ m_1,\ m_2}(a_3-a_2;\ a_3;\ z), \tag{72}$$

which are the transformation formulas for the extended Gauss hypergeometric and Kumar confluent hypergeometric functions.

Proof. Setting $t \to 1-t$ in equations (53) and (59), we obtained the required results in (71) and (72), respectively.

Theorem 19.

$$F_{\zeta,\ \alpha_1}^{\alpha_2;\, m_1, m_2}(a_1,\ a_2;\, a_3;\ 1) = \frac{B_{\zeta,\ \alpha_1}^{\alpha_2;\, m_1, m_2}(a_2,\ a_3 - a_2)}{B(a_2,\ a_3 - a_2)}, \tag{73}$$

is the extended Gauss summation formula.

Proof. Taking $z = 1$, in equation (53), the required result in (73) is obtained.

9. Conclusions

The new extension of the extended beta function $B_{\zeta,\ \alpha_1}^{\alpha_2;\, m_1, m_2}(a_1,\ a_2)$, Gauss hypergeometric function $F_{\zeta,\ \alpha_1}^{\alpha_2;\, m_1, m_2}(a_1,\ a_2;\ a_3;\ z)$ and confluent hypergeometric function $\Phi_{\zeta,\ \alpha_1}^{\alpha_2;\, m_1, m_2}(a_2;\ a_3;\ z)$ were obtained and presented with their important properties. The extended beta, Gauss and confluent hypergeometric functions and their special cases proposed in [21, 33-37] can be regained from the newly proposed functions. It is hoped that it will be useful in Science and Technology [38-40].

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Conflict of interest

The authors state that did not have conflict of interests.

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