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On the Euler method of summability and concerning Tauberian theorems

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Abstract

For any two regular summability methods (U) and (V), the condition under which $V - \lim x_n = \lambda$ implies $U - \lim x_n = \lambda$ is called a Tauberian condition and the corresponding theorem is called a Tauberian theorem. Usually in the theory of summability, the case in which the method U is equivalent to the ordinary convergence is taken into consideration. In this paper, we give new Tauberian conditions under which ordinary convergence or Cesàro summability of a sequence follows from its Euler summability by means of the product theorem of Knopp for the Euler and Cesàro summability methods.

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1. Introduction

We consider throughout complex sequences $x = \{x_n\}$ and discuss the relations of Euler and Cesàro summability methods. We say that a sequence $\{x_n\}$ is summable to λ by the

1. Cesàro method C_1, briefly $C_1 - \lim x_n = \lambda$, if

$$x_n^{(1)} := \frac{1}{n+1} \sum_{k=0}^{n} x_k \to \lambda \quad \text{as } n \to \infty;$$

2. Euler method E_p of order p, briefly $E_p - \lim x_n = \lambda$, if

$$\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} x_k \to \lambda \quad \text{as } n \to \infty.$$

Cesàro method and Euler method of order $p \in (0,1)$ are regular (see [1]). In other words, they sum a convergent sequence to its limit.

For any sequence $\{u_n\}$, the symbols $u_n = O(n^\alpha)$ and $u_n = o(n^\alpha)$ denote, as usual, that $\limsup |n^{-\alpha} u_n| < \infty$ and $\lim n^{-\alpha} u_n = 0$, respectively. The backward difference of $\{u_n\}$ is defined for all $n \geq 0$ by $\Delta u_0 = u_0$ and $\Delta u_n = u_n - u_{n-1}$.

The difference of a sequence and its arithmetic mean is given with the Kronecker identity (see [2])

$$x_n - x_n^{(1)} = \delta_n \tag{1}$$

where

$$\delta_n := \frac{1}{n+1} \sum_{k=0}^{n} k \Delta x_k = n \Delta x_n^{(1)}.$$

The $r-$times iterated arithmetic mean of sequences $\{x_n\}$ and $\{\delta_n\}$ are defined respectively as

$$x_n^{(r)} := \frac{1}{n+1} \sum_{k=0}^{n} x_k^{(r-1)}$$

and

$$\delta_n^{(r)} := \frac{1}{n+1} \sum_{k=0}^{n} \delta_k^{(r-1)}$$

where $x_n^{(0)} = x_n$ and $\delta_n^{(0)} = \delta_n$.

A sequence $\{x_n\}$ is called slowly oscillating, if

$$x_m - x_n = o(1)$$

as $n \to \infty, m > n$ and $m/n \to 1$.

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Historically, the concept 'slow oscillation' goes back to Schmidt [3].

For any two regular summability methods (U) and (V), the condition under which $V - \lim x_n = \lambda$ implies $U - \lim x_n = \lambda$ is called a Tauberian condition and the corresponding theorem is called a Tauberian theorem. Usually in the theory of summability, the case in which the method U is equivalent to the ordinary convergence is taken into consideration.

Tauberian theorems for various methods of summation have a long history; see the classical books [4,5] and they found new attention recently in (see e.g., [6-8]).

In the present paper, we consider Tauberian conditions on $\{x_n\}$ under which $E_p - \lim x_n = \lambda$ implies $C_1 - \lim x_n = \lambda$ or $\lim x_n = \lambda$.

The major Tauberian results for Euler method of summation were proved by Knopp [9]. We use these theorems as a stepping stone to obtain stronger results.

Theorem 1.1 *If* $E_p - \lim x_n = \lambda$ *for some* $0 < p < 1$ *and* $\Delta x_n = O(n^{-1/2})$, *then* $\lim x_n = \lambda$.

Theorem 1.2 *If* $E_p - \lim x_n = \lambda$ *for some* $0 < p < 1$ *and* $\Delta x_n = o(n^{-1/2})$, *then* $\lim x_n = \lambda$.

2. Auxilary Results

We shall make use of the following four lemmas.

Lemma 2.1 ([10]) *If* $\{x_n\}$ *is slowly oscillating, then* $\delta_n = O(1)$ *and* $\{\delta_n\}$ *is slowly oscillating.*

Lemma 2.2 ([3]) *If* $C_1 - \lim x_n = \lambda$ *and* $\{x_n\}$ *is slowly oscillating, then* $\lim x_n = \lambda$.

Lemma 2.3 ([9]) *Let* $0 < p < 1$. *Then* $E_p \subset E_p C_1$; *that is, if* $\{x_n\}$ *is Euler summable to* λ, *then so is* $\{x_n^{(1)}\}$.

The next lemma proposes a relation between Euler and Cesàro methods.

Lemma 2.4 ([9]) *If* $E_p - \lim x_n = \lambda$ *for some* $0 < p < 1$ *and* $\Delta x_n = o(1)$, *then* $C_1 - \lim x_n = \lambda$.

3. Main Results

In this section, we establish Tauberian conditions for an Euler summable sequence to be Cesàro summable or convergent.

Our first result is a $E_p \to C_1$ type theorem.

Theorem 3.1 *Let* $0 < p < 1$. *Then* $E_p - \lim x_n = \lambda$ *and*

$$\delta_n = O(n^{1/2}) \tag{2}$$

imply $C_1 - \lim x_n = \lambda$.

Proof. By the assumption and Lemma 2.3, we have

$$E_p - \lim x_n^{(1)} = \lambda. \tag{3}$$

Besides, since

$$\delta_n = n\Delta x_n^{(1)} = O(n^{1/2})$$

by (2), we obtain

$$\Delta x_n^{(1)} = O(n^{-1/2}). \tag{4}$$

Therefore, combining (3) and (4) together with Theorem 1.1 imply our result.

Remark 3.1 Note that condition (2) may be replaced with the weaker condition $\delta_n = O(1)$.

Corollary 3.1 ([9]) *Let* $0 < p < 1$. *Then* $E_p - \lim x_n = \lambda$ *and*

$$x_n = O(n^{1/2}) \tag{5}$$

imply $C_1 - \lim x_n = \lambda$.

Proof. It is enough to prove $\delta_n = n\Delta x_n^{(1)} = O(n^{1/2})$ or equivalently

$$\psi_n := n^{1/2}\Delta x_n^{(1)} = O(1).$$

In view of (5), we observe

$$\begin{aligned}\psi_n &= n^{1/2}\Big[\frac{1}{n+1}\sum_{k=0}^{n} x_k - \frac{1}{n}\sum_{k=0}^{n-1} x_k\Big]\\ &= n^{1/2}\Big[\frac{1}{n+1}x_n - \frac{1}{n+1}\frac{1}{n}\sum_{k=0}^{n-1} x_k\Big]\\ &= n^{1/2}\Big[\frac{1}{n+1}O(n^{1/2}) - \frac{1}{n+1}O(n^{1/2})\Big]\\ &= O(1),\end{aligned}$$

which completes the proof.

Now, we prove some $E_p \to c$ type theorems.

Theorem 3.2 *Let* $0 < p < 1$. *Then* $E_p - \lim x_n = \lambda$ *and*

$$\Delta\delta_n = O(n^{-1/2}) \tag{6}$$

imply $\lim x_n = \lambda$.

Proof. Plainly, we have $E_p - \lim x_n^{(1)} = \lambda$ from Lemma 2.3. We observe using (1) that

$$E_p - \lim \delta_n = 0. \tag{7}$$

Combining (6) and (7) with Theorem 1.1, we get

$$\delta_n = n\Delta x_n^{(1)} = o(1),$$

that necessiates

$\Delta x_n^{(1)} = o(n^{-1/2})$.

Further, applying Theorem 1.2 to $\{x_n^{(1)}\}$, we conclude

$\lim x_n^{(1)} = \lambda$.

Therefore, the proof follows from (1).

Theorem 3.3 *Let* $0 < p < 1$. *Then* $E_p - \lim x_n = \lambda$ *and*

$$\Delta \delta_n^{(1)} = o(n^{-1}) \tag{8}$$

imply $\lim x_n = \lambda$.

Proof. From the hypothesis, it is clear that $E_p - \lim x_n^{(1)} = \lambda$ and $E_p - \lim x_n^{(2)} = \lambda$. We may write the identity

$$x_n^{(1)} - x_n^{(2)} = \delta_n^{(1)} \tag{9}$$

by taking Cesàro mean of both sides of the Kronecker identity (1). Then, it follows from (9) that

$$E_p - \lim \delta_n^{(1)} = 0. \tag{10}$$

Taking (8) and (10) into account together with Theorem 1.2, we observe

$$\delta_n^{(1)} = n\Delta x_n^{(2)} = o(1), \tag{11}$$

which also implies

$\Delta x_n^{(2)} = o(n^{-1/2})$.

Now, applying Theorem 1.2 to $\{x_n^{(2)}\}$, we conclude

$$\lim x_n^{(2)} = \lambda. \tag{12}$$

Using (11) and (12), we get via the identity (9) that

$\lim x_n^{(1)} = \lambda$.

Since

$\delta_n - \delta_n^{(1)} = n\Delta \delta_n^{(1)}$,

we find $\delta_n = o(1)$ from (8) and (11). Consequently, it is easy to obtain $\lim x_n = \lambda$ by using (1).

Corollary 3.2 *Let* $0 < p < 1$. *Then* $E_p - \lim x_n = \lambda$ *and*

$$\delta_n = o(1) \tag{13}$$

imply $\lim x_n = \lambda$.

Proof. Assuming (13), we have $\delta_n^{(1)} = o(1)$. Hence, by the identity $\delta_n - \delta_n^{(1)} = n\Delta \delta_n^{(1)}$, it follows $\Delta \delta_n^{(1)} = o(n^{-1})$. Thus, the proof follows from Theorem 3.3.

Remark 3.2 In (8) and (13) o-type condition can not be replaced with O-type condition.

The following theorem is first proved by Tam [11]. Here, we give an alternative proof.

Theorem 3.4 *Let* $0 < p < 1$. *If* $E_p - \lim x_n = \lambda$ *and* $\{x_n\}$ *is slowly oscillating, then* $\lim x_n = \lambda$.

Proof. Taking Lemma 2.1 and the slow oscillation of $\{x_n\}$ into account, we clearly have $\delta_n = O(n^{1/2})$ and the slow oscillation of $\{x_n^{(1)}\}$. Hence, we obtain

$C_1 - \lim x_n = \lambda$

from Theorem 3.1. Thus, the proof is completed via Lemma 2.2.

Corollary 3.3 *Let* $0 < p < 1$. *Then* $E_p - \lim x_n = \lambda$ *and*

$$\Delta x_n = O(n^{-1}) \tag{14}$$

imply $\lim x_n = \lambda$.

Proof. The proof is completed from the fact that (14) implies the slow oscillation of $\{x_n\}$.

Theorem 3.5 *Let* $0 < p < 1$. *If* $E_p - \lim x_n = \lambda$ *and* $\{\delta_n\}$ *is slowly oscillating, then* $\lim x_n = \lambda$.

Proof. By the definition of slow oscillation, obviously $\Delta \delta_n = o(1)$. Further, since $E_p - \lim x_n = \lambda$ we have $E_p - \lim \delta_n = 0$. Then, from Lemma 2.4, we find $C_1 - \lim \delta_n = 0$. Now, by Lemma 2.2, we obtain $\lim \delta_n = 0$ which leads us to

$\Delta x_n^{(1)} = o(n^{-1/2})$.

By applying Theorem 1.2 to $\{x_n^{(1)}\}$, we have $C_1 - \lim x_n = \lambda$. Therefore, using (1) we conclude $\lim x_n = \lambda$.

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Conflicts of interest

The authors declare that there is no conflict of interest.

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