On the solution of a Sturm-Liouville problem by using Laplace transform on time scales

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Abstract

In this study, we solve a Sturm-Liouville problem on time scales with constant graininess by using Laplace transform which is one of the finest representatives of integral transformation used in applied mathematics. Eigenfunctions on the time scale were obtained in different cases with the Laplace transform. Thus, it was seen that the Laplace transform is an effective method on time scales. The results that will contribute to the spectral theory were obtained on the time scale with the examples discussed. It is very interesting that the results obtained differ as the time scale changes and this transformation can be applied to other types of problems. The problems that were established and solved enabled the subject to be understood on the time scale.

1. Introduction

Laplace transform is a valuable tool to solve linear differential equations which include constant coefficients and integral equations. It plays a crucial role in mathematics and engineering. Laplace transform from time domain to frequency domain converts differential equations into algebraic equations and convolution into product. Detailed information on the general structure of the Laplace transform in the classical situation can be found in Schiff's study [1].

Discrete version of Laplace transform is known as Z-transform. It is convenient for linear recurrence relations and summation equations. Laplace transform on time scale was firstly considered by Hilger [2] to unify continuous Laplace transform and discrete Z-transform in one theory in 1999. For arbitrary time scales, Laplace transform was investigated by Bohner and Peterson [3] in 2002. The various forms of Laplace transform on time scale were studied in detail by many authors in literature [4, 5-9, 10-14].

For a better understanding for readers, we provide some principle notions related to delta calculus on time scales. By the time scale $\mathbb{T}$, we understand any non-empty, closed, arbitrary subset of $\mathbb{R}$ with ordering inherited from reals. This theory was first put forward by Hilger [15, 16] in 1988, and in the following years, numerous studies were conducted on this subject in various fields. Since a time scale is not necessarily connected, forward and backward jump operators $\sigma, \rho: \mathbb{T} \to \mathbb{T}$ are defined as

$$\sigma(t) = \inf\{s \in \mathbb{T}: s > t\} \text{ and } \rho(t) = \sup\{s \in \mathbb{T}: s < t\},$$

respectively for $t \in \mathbb{T}$ such that $a < t < b, t < \sup\mathbb{T}, \inf\phi = \sup\mathbb{T}, \sup\phi = \inf\mathbb{T}$ where ϕ is empty set; $a = \inf\mathbb{T}$ and $b = \sup\mathbb{T}$. Corresponding forward-step function μ is defined by

$$\mu: \mathbb{T}^{\kappa} \to [0, \infty), \mu(t) = \sigma(t) - t.$$

However, $t \in \mathbb{T}$ is left dense, left scattered, right dense, right scattered, isolated and dense iff $\rho(t) = t, \rho(t) < t, \sigma(t) = t, \sigma(t) > t, \rho(t) < t < \sigma(t)$ and $\rho(t) = t = \sigma(t)$, respectively. We also should remind delta differentiability region $\mathbb{T}^{\kappa}$ along with $\mathbb{T}$ to define delta derivative of any function. $\mathbb{T}^{\kappa} = \mathbb{T} - \{b\}$ if $\mathbb{T}$ is bounded above and b is left-scattered; otherwise $\mathbb{T}^{\kappa} = \mathbb{T}$. $f: \mathbb{T} \to \mathbb{R}$ is right side continuous at $t \in \mathbb{T}$ if there is some $\delta > 0$ such that $|f(t) - f(s)| < \varepsilon$ for all $s \in [t, t + \delta)$ and $\varepsilon > 0$. $C_{rd}(\mathbb{T})$ indicates the set of all right continuous functions on $\mathbb{T}$. One can define $f^{\Delta}(t)$ to be the value for $t \in \mathbb{T}^{\kappa}$, if one exists, such that for all $\varepsilon > 0$, there is a neighborhood U of t such that for all $s \in U$

$$|[f^{\sigma}(t) - f(s)] - f^{\Delta}(t)[\sigma(t) - s]| < \varepsilon |\sigma(t) - s|.$$

Here, f is delta differentiable on $\mathbb{T}^{\kappa}$ if $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}^{\kappa}$. Let $f \in C_{rd}(\mathbb{T})$, then there exists a function F such that $F^{\Delta}(t) = f(t)$, and delta integral is constructed by $\int_a^b f(t)\Delta t = F(b) - F(a)$. Additionally, if $a \in \mathbb{T}$, $\sup \mathbb{T} = \infty$ and f is *rd*-continuous on $[0, \infty)$, improper integral is defined by

$$\int_a^{\infty} f(t)\Delta t = \lim_{b \to \infty} \int_a^b f(t)\Delta t,$$

when right side limit exists and finite [17, 18].

As far as we concerned, the application of Laplace transform to spectral theory on time scales has not been studied. In this study, the eigenfunction of a Sturm-Liouville problem will be constructed by using Laplace transform on time scale with a constant forward-step function.

When Sturm and Liouville investigated the heat conduction problem by the method of separating them into variables, the problem of searching for the solutions of ordinary differential equations containing eigenvalue parameters that meet some boundary

conditions arose. This new equation is known as the Sturm-Liouville equation in literature. The problems in which this equation is handled with various boundary conditions have been studied by many mathematicians ([19-25]). The spectral properties of the Sturm-Liouville equation are given by Levitan et. al [26] from different angles for usual case.

Because of its iportance, the main goal of this study is to solve below Sturm- Liouville problem on $\mathbb{T}$ by using Laplace transform:

$$-y^{\Delta\Delta}(t) + cy^{\sigma}(t) = \lambda y^{\sigma}(t), t \in (0, \infty)_{\mathbb{T}}, \qquad (1)$$

$$y(0, \lambda) = 0, y^{\Delta}(0, \lambda) = 1, \qquad (2)$$

where $\mu(t) \equiv h \geq 0$, $c \in \mathbb{R}$, λ is a spectral parameter. Here, $y \colon \mathbb{T} \to \mathbb{C}$ is the solution (eigenfunction) of (1)-(2) where y is second-order delta differentiable on $(0, \infty)_{\mathbb{T}}$.

Our study will be organized as follows: In section 2, we recall some fundamental notions and theorems related to Laplace transform on $\mathbb{T}$. We construct a Sturm-Liouville problem on $\mathbb{T}$ to solve it by using Laplace transform in section 3. Proof of the main theorem was made in three cases.

2. Preliminaries

Here, we remind some principle notions and theorems related to Laplace transform on $\mathbb{T}$.

Definition 1 [17]. $f \colon \mathbb{T} \to \mathbb{R}$ is a regulated function if its right-sided limits exist (finite) at all right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$.

Definition 2 [17]. $p \colon \mathbb{T} \to \mathbb{R}$ is a regressive function if $1 + \mu(t)p(t) \neq 0$ holds for all $t \in \mathbb{T}^{\kappa}$. $\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R})$ indicates the set of all regressive and *rd*-continuous functions on $\mathbb{T}$. $\mathcal{R}$ forms an Abelian group with the addition operation $\oplus$ defined by $(p \oplus q)(t) = p(t) + q(t) + \mu(t)p(t)q(t)$ for all $t \in \mathbb{T}^{\kappa}$, $p, q \in \mathcal{R}$. In addition, the additive inverse of p for this group is denoted by

$$(\ominus p)(t) = -\frac{p(t)}{1 + \mu(t)p(t)},$$

for all $t \in \mathbb{T}^{\kappa}$, $p \in \mathcal{R}$.

Definition 3 [17]. Exponential function on $\mathbb{T}$ is defined by

$$e_p(t, s) = \exp\left(\int_s^t \xi_{\mu(\tau)}\big(p(\tau)\big) \Delta\tau \right),$$

for $s, t \in \mathbb{T}$, $p \in \mathcal{R}$. Here, $\xi_{\mu(\tau)}(z)$ is cylinder transformation where $\xi_h(z) = \frac{1}{h} Log(1 + hz)$ where $h > 0$. If $h = 0, \xi_0(z) = z$ for all $z \in \mathbb{C}$. For details on exponential function, we refer to the books [17, 18].

Let us consider a 2-nd order linear dynamic homogeneous equation with constant coefficients on $\mathbb{T}$

$$y^{\Delta\Delta}(t) + \alpha y^{\Delta\Delta}(t) + \beta y(t) = 0,$$

where $\alpha, \beta \in \mathbb{R}$. From the last equation, the hyperbolic functions (when $\alpha = 0, \beta < 0$) and the trigonometric functions (when $\alpha = 0, \beta > 0$) are defined as follows:

Definition 4 [17]. Let $p \in C_{rd}$. If $-\mu p^2 \in \mathcal{R}$, the hyperbolic functions $\cosh_p$ and $\sinh_p$ are defined by

$$\cosh_p = \frac{e_p + e_{-p}}{2} \text{ and } \sinh_p = \frac{e_p - e_{-p}}{2}.$$

If $\mu p^2 \in \mathcal{R}$, the trigonometric functions $\cos_p$ and $\sin_p$ are defined by

$$\cos_p = \frac{e_{ip} + e_{-ip}}{2} \text{ and } \sin_p = \frac{e_{ip} - e_{-ip}}{2i}.$$

For a constant $\alpha \in \mathbb{R}$, the functions $e_\alpha(t,0), \sin_\alpha(t,0)$ and $\sinh_\alpha(t,0)$ have the below forms for common time scales $\mathbb{T} = \mathbb{R}, \mathbb{T} = \mathbb{Z}$ and $\mathbb{T} = h\mathbb{Z}, (h > 0)$, respectively.

Table 1. Representations of $e_\alpha(t,0), \sin_\alpha(t,0)$ and $\sinh_\alpha(t,0)$ on $\mathbb{T} = \mathbb{R}, \mathbb{T} = \mathbb{Z}$ and $\mathbb{T} = h\mathbb{Z}$

$\mathbb{T}$	$e_\alpha(t,0)$	$\sin_\alpha(t,0)$	$\sinh_\alpha(t,0)$
$\mathbb{R}$	$e^{\alpha t}$	$\sin(\alpha t)$	$\sinh(\alpha t)$
$\mathbb{Z}$	$(1+\alpha)^t$	$\dfrac{(1+i\alpha)^t - (1-i\alpha)^t}{2i}$	$\dfrac{(1+\alpha)^t - (1-\alpha)^t}{2}$
$h\mathbb{Z}$	$(1+\alpha h)^{\frac{t}{h}}$	$\dfrac{(1+i\alpha h)^{\frac{t}{h}} - (1-i\alpha h)^{\frac{t}{h}}}{2i}$	$\dfrac{(1+\alpha h)^{\frac{t}{h}} - (1-\alpha h)^{\frac{t}{h}}}{2}$

Definition 5 [17]. Suppose that $y: \mathbb{T}_0 \to \mathbb{R}$ regulated. Then, Laplace transform of y is defined by

$$L\{y\}(z) = \int_0^\infty y(t) e_{\ominus z}^\sigma(t,0)\,\Delta t,$$

for $z \in D\{y\}$, where $\mathbb{T}_0$ is a time scale, $0 \in \mathbb{T}_0$ and $\sup\mathbb{T}_0 = \infty$; $D\{y\}$ consists of all complex numbers $z \in \mathcal{R}$ when the improper integral exists.

It was easily seen from the Definition 5 that L is linear as follows:

Theorem 6 [17]. Let x and y be regulated on $\mathbb{T}_0$ and α, β be constants. Then,

$$L\{\alpha x + \beta y\}(z) = \alpha L\{x\}(z) + \beta L\{y\}(z),$$

for $z \in D\{x\} \cap D\{y\}$.

Theorem 7 [3]. If $y: \mathbb{T}_0 \to \mathbb{C}$ is a function whose first order delta derivative is regulated, then

$$L\{y^\Delta\}(z) = zL\{y\}(z) - y(0), \tag{3}$$

for all regressive $z \in \mathbb{C}$ when $\lim_{t\to\infty}\{y(t)e_{\ominus z}(t,0)\} = 0$.

One of the consequences of this theorem is as follows:

Corollary 8 [3]. If $y: \mathbb{T}_0 \to \mathbb{C}$ is a function where $y^{\Delta\Delta}$ is regulated, then

$$L\{y^{\Delta\Delta}\}(z) = z^2 L\{y\}(z) - zy(0) - y^\Delta(0), \tag{4}$$

for all regressive $z \in \mathbb{C}$ when $\lim_{t\to\infty}\{y(t)e_{\ominus z}(t,0)\} = \lim_{t\to\infty}\{y^\Delta(t)e_{\ominus z}(t,0)\} = 0$.

Lemma 9 [17]. If $\mathbb{T}_0$ has constant forward-step function $\mu(t) \equiv h \geq 0$, then

$$L\{y^\sigma\}(z) = (1 + hz)L\{y\}(z) - hy(0). \tag{5}$$

Proof. Considering $y^\sigma(t) = y(t) + hy^\Delta(t)$ (see [17, Theorem 1.16]), we get

$$L\{y^\sigma\}(z) = \int_0^\infty y^\sigma(t)e^\sigma_{\ominus z}(t,0)\,\Delta t = \int_0^\infty y(t)e^\sigma_{\ominus z}(t,0)\,\Delta t + h\int_0^\infty y^\Delta(t)e^\sigma_{\ominus z}(t,0)\,\Delta t$$
$$= L\{y\}(z) + hL\{y^\Delta\}(z)$$

If the formula (3) is taken into consideration in the last equation, the proof is completed.

Theorem 10 [9]. Suppose that $\mathbb{T}$ has constant forward-step function and $y \in C_{rd}([s,\infty)_\mathbb{T}, \mathbb{C})$ is a function of exponential order η. Then,

$$L\{y\}(z \ominus w; s) = L\{ye^\sigma_w(.,s)\}(z; s)$$

for all $z \in \mathbb{C}_h(\eta \oplus |w|)$, where $w \in \mathbb{R}([s,\infty)_\mathbb{T}, \mathbb{C})$. This theorem is called "First Translation Theorem" in literature.

The following table gives Laplace transforms of some basic functions for usage in Section 3.

Table 2. Laplace transforms of Some Common Functions on $\mathbb{T}$

$y(t)$	1	t	$e_\alpha(t,0)$			$\sin_\alpha(t,0)$	$\sinh_\alpha(t,0)$	$e_\alpha(t,0)\sin_{\frac{\beta}{1+\mu\alpha}}(t,0)$	$e_\alpha(t,0)\sinh_{\frac{\beta}{1+\mu\alpha}}(t,0)$
$L\{y\}(z)$	$\dfrac{1}{z}$	$\dfrac{1}{z^2}$	$\dfrac{1}{z-\alpha}$			$\dfrac{\alpha}{z^2+\alpha^2}$	$\dfrac{\alpha}{z^2-\alpha^2}$	$\dfrac{\beta}{(z-\alpha)^2+\beta^2}$	$\dfrac{\beta}{(z-\alpha)^2-\beta^2}$

First, we solve Sturm-Liouville problem for classical situation $\mathbb{T} = \mathbb{R}$ by Laplace transform.

Theorem 11. The eigenfunction of the problem

$$-y''(t) + cy(t) = \lambda y(t), t \in (0,\infty)_\mathbb{R}, \tag{6}$$
$$y(0,\lambda) = 0, y'(0,\lambda) = 1, \tag{7}$$

has following form

$$y(t) = \begin{cases} \dfrac{1}{\sqrt{\lambda-c}}\sin(\sqrt{\lambda-c}\,t), & if\ c < \lambda \\ t, & if\ c = \lambda. \\ \dfrac{1}{\sqrt{c-\lambda}}\sinh(\sqrt{c-\lambda}\,t), & if\ c > \lambda \end{cases} \tag{8}$$

Proof. Let us reorganize the equation (6) as

$$y''(t) + (\lambda - c)y(t) = 0, \tag{9}$$

The "usual" Laplace transform is known as

$$L\{y\}(z) = \int_0^\infty y(t)e^{-zt}dt,$$

whenever the right side integral is convergent. It can be easily shown that,

$$L\{y'\}(z) = zL\{y\}(z) - y(0),$$

holds by integration by parts formula. Moreover,

$$L\{y''\}(z) = z^2L\{y\}(z) - zy(0) - y'(0),$$

is one consequence of the last formula.

Let y be the solution of (6)-(7). Then, applying Laplace transform to both sides of (9) yields

$$(z^2 + \lambda - c)L\{y\}(z) - zy(0) - y'(0) = 0.$$

Considering the initial conditions (7) on the last equation, we get

$$L\{y\}(z) = \frac{1}{z^2 + \lambda - c}. \tag{10}$$

After applying the "usual" inverse Laplace transform on the last equality, considering the known table of the "usual" Laplace transform, the solution (8) is obtained.

3. Main Results

Here, we obtain eigenfunction expansion of a Sturm-Liouville problem on $\mathbb{T}$ by using Laplace transform for different cases.

Theorem 12. The eigenfunction of the problem (1)-(2) has the below forms:

i.   If $h \geq 0$ and $c - \lambda = 0$, then
$$y(t) = t.$$
ii.   If $h = 0$, then
$$y(t) = \frac{1}{\sqrt{c - \lambda}} \sinh_{\sqrt{c-\lambda}}(t, 0),$$
and
$$y(t) = \frac{1}{\sqrt{\lambda - c}} \sin_{\sqrt{\lambda-c}}(t, 0),$$
for $c - \lambda > 0, c - \lambda < 0$, respectively.

iii.   If $h > 0$, then
$$y(t) = e^{\sigma}_{-\frac{2}{h}}(t, 0)t,$$

$$y(t) = \frac{1}{\sqrt{\frac{(c-\lambda)^2 h^2}{4} + (c - \lambda)}} e_{\frac{(c-\lambda)h}{2}}(t, 0)\sinh_{\frac{\sqrt{\frac{(c-\lambda)^2 h^2}{4}+(c-\lambda)}}{1+\frac{(c-\lambda)h^2}{2}}}(t, 0),$$

and

$$y(t) = \frac{1}{\sqrt{-\frac{(c-\lambda)^2 h^2}{4} - (c - \lambda)}} e_{\frac{(c-\lambda)h}{2}}(t, 0)\sin_{\frac{\sqrt{\frac{(c-\lambda)^2 h^2}{4}-(c-\lambda)}}{1+\frac{(c-\lambda)h^2}{2}}}(t, 0),$$

for $c - \lambda = -\frac{4}{h^2}, c - \lambda \in \mathbb{R} \backslash \left(-\frac{4}{h^2}, 0\right), c - \lambda \in \left(-\frac{4}{h^2}, 0\right)$, respectively.

Proof. Let us again reorganize the equation (1) as
$$y^{\Delta\Delta}(t) + (\lambda - c)y^{\sigma}(t) = 0. \tag{11}$$

Then, applying the Laplace transform to both sides of (11) gives
$$L\{y^{\Delta\Delta}\}(z) + (\lambda - c)L\{y^{\sigma}\}(z) = 0.$$

By the formulas (4) and (5) into account on the last equality, we have

$$z^2 L\{y\}(z) - zy(0) - y^{\Delta}(0) + (\lambda - c)\big((1 + hz)L\{y\}(z) - hy(0)\big) = 0.$$

Using the initial conditions (2) yields

$$L\{y\}(z) = \frac{1}{z^2 + (\lambda-c)hz + (\lambda-c)}. \tag{12}$$

i.   If $h \geq 0$ and $c - \lambda = 0$, then the formula (12) turns into the form

$$L\{y\}(z) = \frac{1}{z^2}.$$

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By the formula given in Table 2, it yields that $y(t) = t$.

Now, we can rewrite the formula (12) as follows:

$$L\{y\}(z) = \frac{1}{z^2 - (c-\lambda)hz - (c-\lambda)} = \frac{1}{\left(z - \frac{(c-\lambda)h}{2}\right)^2 - \left(\frac{(c-\lambda)^2 h^2}{4} + (c-\lambda)\right)}. \tag{13}$$

After that, we will examine the solution of the problem (1)-(2) separately for $h = 0$ and $h > 0$.

ii.   If $h = 0$, then the formula (12) is written in the form (10). If this form is rearranged to

$$L\{y\}(z) = \frac{1}{\sqrt{c-\lambda}} \frac{\sqrt{c-\lambda}}{z^2 - \left(\sqrt{c-\lambda}\right)^2},$$

for $c - \lambda > 0$ and

$$L\{y\}(z) = \frac{1}{\sqrt{\lambda-c}} \frac{\sqrt{\lambda-c}}{z^2 + \left(\sqrt{\lambda-c}\right)^2},$$

for $c - \lambda < 0$, Theorem 12 is proved using Table 2. Note that, the solution of the problem (1)-(2) coincides with the solution (8) in this case.

iii.   If $h > 0$, then there are three cases relative to each other of λ and c.

Case 1: $c - \lambda = -\frac{4}{h^2}$. In this case, the formula (13) can be rewritten as:

$$L\{y\}(z) = \frac{1}{\left(z + \frac{2}{h}\right)^2}.$$

By Theorem 10 and the appropriate formula given in Table 2, we obtain

$$y(t) = e^{\sigma}_{-\frac{2}{h}}(t, 0)t,$$

as the solution of the problem (1)-(2) for Case 1.

Case 2: $c - \lambda \in \mathbb{R} \setminus \left(-\frac{4}{h^2}, 0\right)$. In this case, since $\frac{(c-\lambda)^2 h^2}{4} + (c-\lambda) > 0$, the formula (13) can be rewritten as:

$$L\{y\}(z) = \frac{1}{\sqrt{\frac{(c-\lambda)^2 h^2}{4} + (c-\lambda)}} \frac{\sqrt{\frac{(c-\lambda)^2 h^2}{4} + (c-\lambda)}}{\left(z - \frac{(c-\lambda)h}{2}\right)^2 - \left(\sqrt{\frac{(c-\lambda)^2 h^2}{4} + (c-\lambda)}\right)^2}.$$

By the appropriate formula given in Table 2, we obtain

$$y(t) = \frac{1}{\sqrt{\frac{(c-\lambda)^2 h^2}{4} + (c-\lambda)}} e_{\frac{(c-\lambda)h}{2}}(t, 0) \sinh_{\frac{\sqrt{\frac{(c-\lambda)^2 h^2}{4} + (c-\lambda)}}{1 + \frac{(c-\lambda)h^2}{2}}}(t, 0),$$

as the solution of the problem (1)-(2) for Case 2.

Case 3: $c - \lambda \in \left(-\frac{4}{h^2}, 0\right)$. In this case, since $\frac{(c-\lambda)^2 h^2}{4} + (c-\lambda) < 0$, the formula (13) can be rewritten as:

$$L\{y\}(z) = \frac{1}{\sqrt{-\frac{(c-\lambda)^2 h^2}{4} - (c-\lambda)}} \frac{\sqrt{-\frac{(c-\lambda)^2 h^2}{4} - (c-\lambda)}}{\left(z - \frac{(c-\lambda)h}{2}\right)^2 + \left(\sqrt{-\frac{(c-\lambda)^2 h^2}{4} - (c-\lambda)}\right)^2},$$

By the appropriate formula given in Table 2, we obtain

$$y(t) = \frac{1}{\sqrt{-\frac{(c-\lambda)^2 h^2}{4} - (c-\lambda)}} e_{\frac{(c-\lambda)h}{2}}(t,0) \sin_{\frac{\sqrt{\frac{-(c-\lambda)^2 h^2}{4} - (c-\lambda)}}{1 + \frac{(c-\lambda)h^2}{2}}}(t,0),$$

as the solution of the problem (1)-(2) for Case 3. It completes the proof.

Corollary 13. If $\mathbb{T} = \mathbb{R}$, then $\mu(t) = 0$ for all $t \in \mathbb{T}$. Therefore, the eigenfunctions of the problem (1)-(2) are the same as in the case (ii) of Theorem 12.

Indeed, the problem (1)-(2) is written in form of the problem (6)-(7) on $\mathbb{T} = \mathbb{R}$. It can be easily seen from Table 1 that this corollary is correct.

Corollary 14. If $\mathbb{T} = h\mathbb{Z} = \{hk : k \in \mathbb{Z}\}$, then $\mu(t) = h, h > 0$ for all $t \in \mathbb{T}$. Therefore, the eigenfunctions of the problem (1)-(2) are as in the cases (i) and (iii) of Theorem 12.

On $\mathbb{T} = h\mathbb{Z}$, it can be easily seen from Table 1 and Theorem 10 that the eigenfunctions of the problem (1)-(2) is in the following forms
If $c - \lambda = 0$,
$$y(t) = t,$$
If $c - \lambda = -\frac{4}{h^2}$,
$$y(t) = (-1)^{\frac{t+h}{h}} t,$$

If $c - \lambda \in \mathbb{R} \setminus \left(-\frac{4}{h^2}, 0\right)$,
$$y(t) = \frac{1}{2\sqrt{\frac{(c-\lambda)^2 h^2}{4} + (c-\lambda)}} \left\{ \left(1 + \frac{(c-\lambda)h^2}{2} + h\sqrt{\frac{(c-\lambda)^2 h^2}{4} + (c-\lambda)}\right)^{\frac{t}{h}} - \left(1 + \frac{(c-\lambda)h^2}{2} - h\sqrt{\frac{(c-\lambda)^2 h^2}{4} + (c-\lambda)}\right)^{\frac{t}{h}} \right\},$$

If $c - \lambda \in \left(-\frac{4}{h^2}, 0\right)$,
$$y(t) = \frac{1}{2i\sqrt{\frac{-(c-\lambda)^2 h^2}{4} - (c-\lambda)}} \left\{ \left(1 + \frac{(c-\lambda)h^2}{2} + ih\sqrt{\frac{-(c-\lambda)^2 h^2}{4} - (c-\lambda)}\right)^{\frac{t}{h}} - \left(1 + \frac{(c-\lambda)h^2}{2} - ih\sqrt{\frac{-(c-\lambda)^2 h^2}{4} - (c-\lambda)}\right)^{\frac{t}{h}} \right\}.$$

4. Conclusion

Models obtained by applying the laws of physics (Newton's law or Kirchoff's law) to systems are in the form of differential equations. However, by applying Laplace transformation to linear ordinary differential equations, the transfer function model of the system (frequency domain) can be obtained. With the transfer function representation, dynamic analysis of the system can be performed without the need of solving differential equations. Because of this importance of Laplace transform, the Sturm Liouville equation is solved with the help of Laplace transform on time scale when the potential function is constant. Although it is possible to find Laplace transformation of the basic functions in many tables in the classical case, it is more difficult to prepare this table on time scale. The results obtained in this study are

therefore very valuable and will bring a different perspective to spectral theory.

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Conflicts of interest

The authors state that there is no conflict of interests.

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