Cumhuriyet Science Journal
e-ISSN: 2587-246X
ISSN: 2587-2680
Cumhuriyet Sci. J., 42(1) (2021) 123-131
http://dx.doi.org/10.17776/csj.831339

Statistical relative uniform convergence of a double sequence of functions at a point and applications to approximation theory

Sevda YILDIZ *

Sinop University, Department of Mathematics, Sinop/ TURKEY

Abstract

In the present paper, we introduce a new kind of convergence, called the statistical relative uniform convergence, for a double sequence of functions at a point, where the relative uniform convergence of the set of the neighborhoods of the given point is considered. By the use of the statistical relative uniform convergence, we investigate a Korovkin type approximation theorem which makes the proposed method stronger than the ones studied before. After that, we give an example using this new type of convergence. We also study the rate of convergence of the proposed convergence.

Article info

History:
Received: 25.11.2020
Accepted: 03.03.2021

Keywords:
Double sequence,
Korovkin theorem,
Rate of convergence,
Statistical relative uniform convergence.

1. Introduction

It is well-known that the first definition of statistical convergence was given by Fast [1] and Steinhaus [2], independently. It is more general than the ordinary convergence. The statistical convergence in approximation theory was first used by Gadjiev and Orhan [3] to prove the Korovkin-type approximation theorem [4]. The studies and related results can be found in [5-8].

Moore [9] was the first who introduced the notion of relative uniform convergence of a sequence of functions. Later on, Chittenden [10] gave a detailed definition of this convergence (which is equivalent to Moore's definition). Recently, Demirci and Orhan [11] defined a new type of statistical convergence by using this convergence and they presented its applications to Korovkin type approximation. For more details on these types of convergences and their applications, we refer to [12-17].

Another interesting type of convergence is the uniform convergence of a sequence of functions at a point [18]. More recently, Demirci et al. [19] extended this type of convergence to relative uniform convergence of a sequence of functions at a point where the set of the neighborhoods of the point at which relative uniform convergence is considered (see, e.g., [20, 21]).

Our focus of the present work is to generalize the concept of statistical convergence using relative uniform convergence at a point. For this purpose, we first define the concept of statistical relative uniform convergence of double sequences of functions at a point and we give some examples, showing that our results are strict generalization of the corresponding classical ones. We also establish some important approximation results.

2. Preliminaries

The idea of uniform convergence of a sequence of functions at a point was defined by J. Klippert and G. Williams in [18]. This type of convergence is a stronger method than the well known uniform convergence. Recently, Demirci et al. [19] gave the notion of relative uniform convergence of a sequence of functions at a point and they proved the Korovkin type approximation theorems. Now we recall these interesting types of convergences:

Definition 2.1 [18] Suppose that (f_j) is a sequence of real functions defined on $G \subset \mathbb{R}$. Let $u_0 \in G$. We say that (f_j) converges uniformly at the point u_0 to $f: G \to \mathbb{R}$ provided that for every $\varepsilon > 0$, there exist $\delta > 0$ and $J \in \mathbb{N}$ such that for every $j \geq J$, if $|u - u_0| < \delta$, then $|f_j(u) - f(u)| < \varepsilon$.

*Corresponding author. *email address: sevdaorhan@sinop.edu.tr*

Definition 2.2 [19] Let (f_j) be a sequence of real functions defined on G and $u_0 \in G$. We say that (f_j) converges relatively uniformly at the point u_0 to $f: G \to \mathbb{R}$ with respect to the scale function $\sigma(u)$, $|\sigma(u)| \neq 0$, if for every $\varepsilon > 0$, there exist $\delta > 0$ and $J \in \mathbb{N}$ such that for every $j \geq J$, if $|u - u_0| < \delta$, then

$$\left|f_j(u) - f(u)\right| < \varepsilon|\sigma(u)|.$$

More recently, Dirik et al. [21] introduced the uniform convergence of a sequence of functions at a point for double sequences. Before this definition, we first give the following:

A double sequence $u = (u_{ij})$ is said to be convergent in Pringsheim's sense iff for every $\varepsilon > 0$, there exists $J = J(\varepsilon) \in \mathbb{N}$ such that $\left|u_{ij} - L\right| < \varepsilon$ whenever $i, j > J$. Then, L is called the Pringsheim limit of u and is denoted by $P - \lim_{i,j\to\infty} u_{ij} = L$ (see [22]). In this case, we say that u is " P −convergent to L ". Also, if there exists a positive number I such that $\left|u_{ij}\right| \leq I$ for all $(i,j) \in \mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$, then u is said to be bounded. It is important to say that a convergent double sequence need not be bounded but it is necessary to be bounded for a convergent single sequence.

Definition 2.3 [21] Suppose that (f_{ij}) is a double sequence of real functions defined on $G^2 \subset \mathbb{R}^2$. Let $(u_0, v_0) \in G^2$. We say that (f_{ij}) converges uniformly at the point (u_0, v_0) to $f: G^2 \to \mathbb{R}$ iff for every $\varepsilon > 0$, there are $\delta > 0$ and $J \in \mathbb{N}$ such that for every $i, j \geq J$, if $\sqrt{(u-u_0)^2 + (v-v_0)^2} \leq \delta$ (or $|u - u_0| \leq \delta$ and $|v - v_0| \leq \delta$), then

$$\left|f_{ij}(u,v) - f(u,v)\right| < \varepsilon.$$

Example 2.1 Define $g_{ij}: [0,1]^2 \to \mathbb{R}$ by

$$g_{ij}(u,v) = \begin{cases} u^2 + v^2, & j \text{ is a square}, \\ 0, & \text{otherwise}. \end{cases}$$

We claim that (g_{ij}) converges uniformly to $g = 0$ at $(u_0, v_0) = (0,0)$. Indeed, let $\varepsilon > 0$ be given and choose $\delta = \sqrt{\frac{\varepsilon}{2}}$ and $J = 1$. Let $i, j \geq J$ and $u, v \in [0,1]$ with $|u| \leq \delta$, $|v| \leq \delta$. Then,

$$\left|g_{ij}(u,v) - g(u,v)\right| = \left|g_{ij}(u,v)\right| \leq u^2 + v^2 \leq 2\delta^2 = \varepsilon.$$

However, (g_{ij}) does not converge uniformly to $g = 0$ on $[0,1]^2$.

3. Statistical Relative Uniform Convergence At a Point

The statistical convergence for single sequences was given in 1951 and this concept was extended to the double sequences by Moricz [23] in 2004 as follows:

Let $A \subseteq \mathbb{N}^2$ be a two-dimensional subset of positive integers, then A_{ij} denotes the set $\{(m,n) \in A: m \leq i, n \leq j\}$ and $\left|A_{ij}\right|$ denotes the cardinality of A_{ij}. The double natural density of A is given by

$$\delta_2(A) := P - \lim_{i,j\to\infty} \frac{1}{ij}\left|A_{ij}\right|,$$

if it exists. The number sequence $u = (u_{ij})$ is said to be statistically convergent to L provided that for every $\varepsilon > 0$, the set

$$A = A_{mn}(\varepsilon) := \left\{i \leq m, j \leq n: \left|u_{ij} - L\right| \geq \varepsilon\right\}$$

has natural density zero; in that case, we write $st_2 - \lim_{i,j\to\infty} u_{ij} = L$ (see [23]).

Now, we can give the following definition which is our new type of convergence:

Definition 3.1 Let $G^2 \subset \mathbb{R}^2$ and suppose that (f_{ij}) is a double sequence of real functions defined on G^2. Let $(u_0, v_0) \in G^2$. We say that (f_{ij}) converges statistically relatively uniformly at the point (u_0, v_0) to $f: G^2 \to \mathbb{R}$ with respect to the scale function σ iff for each $\varepsilon > 0$ there are $\eta > 0$ and $A \subseteq \mathbb{N}^2$ with $\delta_2(A) = 0$ such that for every $\varepsilon > 0$ and $(i,j) \in \mathbb{N}^2 \backslash A$, if $\sqrt{(u-u_0)^2 + (v-v_0)^2} \leq \eta$ (or $|u-u_0| \leq \eta$ and $|v-v_0| \leq \eta$), then

$|f_{ij}(u,v) - f(u,v)| < \varepsilon|\sigma(u,v)|$.

Now we give the following remark that gives the relations between types of convergences.

Remark 3.1 It can be immediately seen that if (f_{ij}) is statistically relatively uniformly convergent to a function f on G^2, then (f_{ij}) converges statistically relatively uniformly at each point in G^2. Also, observe that the statistical uniform convergence of a double sequence of functions at a point is the special case of statistical relative uniform convergence of a double sequence of functions at a point in which the scale function is a non-zero constant. If $\sigma(u,v)$ is bounded, then the statistical relative uniform convergence at a point implies the statistical uniform convergence at a point. However, the statistical relative uniform convergence at a point does not imply the statistical uniform convergence at a point, when $\sigma(u,v)$ is unbounded.

Now, we give the following example to show the effectiveness of newly proposed method:

Example 3.1 Define $h_{ij}: [0,1]^2 \to \mathbb{R}$ by

$$h_{ij}(u,v) = \begin{cases} i^2 j u^2 v, & i = k^2 \text{ and } j = l^2, \\ \frac{2i^2 j u^2 v}{5 + i^2 j u^2 v}, & \text{otherwise.} \end{cases} \tag{1}$$

$k, l = 1,2, \ldots$. We claim that (h_{ij}) converges statistically relatively uniformly to $h = 0$ at $(u_0, v_0) = (0,0)$ to the scale function

$$\sigma(u,v) = \begin{cases} \frac{1}{uv}, & (u,v) \in]0,1]^2, \\ 1, & u = 0 \text{ or } v = 0. \end{cases} \tag{2}$$

Indeed, let $\varepsilon > 0$ be given and choose $\eta = \sqrt{\frac{\varepsilon}{2}}$ and $A = \{(i,j): i = k^2 \text{ and } j = l^2, k, l = 1,2, \ldots\}$. Then $\delta_2(A) = 0$. Let $(i,j) \in \mathbb{N}^2 \backslash A$ and $(u,v) \in [0,1]^2$ with $|u| \leq \eta$ and $|v| \leq \eta$. Then,

$$\left|\frac{h_{ij}(u,v)}{\sigma(u,v)}\right| \leq \left|\frac{2i^2 j u^3 v^2}{5 + i^2 j u^2 v}\right| \leq 2|u||v| < 2\eta^2 = \varepsilon.$$

However, (h_{ij}) does not converge statistically uniformly at $(0,0)$. Indeed, for $\varepsilon = \frac{1}{5}$, $(u,v) = \left(\frac{1}{i}, \frac{1}{j}\right) \in]0,1]^2$ with $\frac{1}{i} < \eta$, $\frac{1}{j} < \eta$ and $(i,j) \in \mathbb{N}^2 \backslash A$, we get

$$\frac{2i^2 j u^2 v}{5 + i^2 j u^2 v} = \frac{1}{3} > \frac{1}{5}.$$

Also, it is neither uniformly nor relatively uniformly convergent to $h = 0$.

4. Korovkin Type Approximation

In this section, we apply the notion of statistical relative uniform convergence of double sequences of functions at a point to prove a Korovkin type approximation theorem. Suppose that $C(G^2)$ is the space of all functions f continuous on G^2. We know that $C(G^2)$ is a Banach space with norm $\|f\| = \sup_{(u,v) \in G^2} |f(u,v)|$.

We denote the value of $L_{ij}(f)$ at a point $(u,v) \in G^2$ by $L_{ij}(f(s,t);u,v)$ or briefly, $L_{ij}(f;u,v)$ and we use the test functions

$$e_0(u,v) = 1, e_1(u,v) = u, e_2(u,v) = v \; and \; e_3(u,v) = u^2 + v^2.$$

Theorem 4.1 [7] Suppose that (L_{ij}) is a double sequence of positive linear operators acting from $C(G^2)$ into itself. Then, for all $f \in C(G^2)$,

$$st_2 - \lim_{i,j\to\infty} \|L_{ij}(f) - f\| = 0 \text{ iff}$$

$$st_2 - \lim_{i,j\to\infty} \|L_{ij}(e_r) - e_r\| = 0, \; (r = 0,1,2,3).$$

Now, we give the following main theorem.

Theorem 4.2 Let (L_{ij}) be a double sequence of positive linear operators acting from $C(G^2)$ into itself. Then $\left(L_{ij}(e_r)\right)$ $(r = 0,1,2,3)$ converges statistically relatively uniformly at (u_0, v_0) to e_r with respect to the scale function σ_r iff for each $f \in C(G^2)$, $\left(L_{ij}(f)\right)$ converges statistically relatively uniformly at (u_0, v_0) to f with respect to the scale function σ where $\sigma(u,v) = \max\{|\sigma_r(u,v)| : r = 0,1,2,3\}$, $|\sigma_r(u,v)| > 0$ and $|\sigma_r(u,v)|$ is possibly unbounded, $r = 0,1,2,3$.

Proof We begin the proof of the "if" part. Our hypothesis is that $\left(L_{ij}(f)\right)$ converges statistically relatively uniformly at (u_0, v_0) to f for each $f \in C(G^2)$ with respect to the scale function σ, which means that $\forall f \in C(G^2)$, $\forall \varepsilon > 0, \exists \eta > 0$ and $A \subseteq \mathbb{N}^2$ with $\delta_2(A) = 0$ such that $\left|L_{ij}(f;u,v) - f(u,v)\right| \leq \varepsilon|\sigma(u,v)|$, $\forall (i,j) \in \mathbb{N}^2 \backslash A$ and $\sqrt{(u-u_0)^2 + (v-v_0)^2} \leq \eta$. Since $e_r \in C(G^2)$, $r = 0,1,2,3$, if we choose $\varepsilon = \frac{\varepsilon^*|\sigma_r(u,v)|}{\sigma(u,v)}$, then we get $\forall \varepsilon^* > 0, \exists \eta > 0$ and $A \subseteq \mathbb{N}^2$ such that $\left|L_{ij}(e_r;u,v) - e_r(u,v)\right| \leq \varepsilon^*|\sigma_r(u,v)|$, $r = 0,1,2,3$, $\forall (i,j) \in \mathbb{N}^2 \backslash A$ and $\sqrt{(u-u_0)^2 + (v-v_0)^2} \leq \eta$. Now, we turn to the "only if" part. Let $f \in C(G^2)$ and $(u,v) \in G^2$ be fixed. Let $E = \max\{|u|, |u|^2\}$, $F = \max\{|v|, |v|^2\}$ and $H = \max\{E, F\}$. Also, by the continuity of f on G^2, we can write $|f(u,v)| \leq M$. Hence,

$$|f(s,t) - f(u,v)| \leq |f(s,t)| + |f(u,v)| \leq 2M.$$

Moreover, since f is uniformly continuous on G^2, we write that for every $\varepsilon > 0$, there exists a number $\delta > 0$ such that $|f(s,t) - f(u,v)| < \varepsilon$ holds for all $(s,t) \in G^2$ satisfying $\sqrt{(s-u)^2 + (t-v)^2} < \delta$. Hence, we get

$$|f(s,t) - f(u,v)| < \frac{\varepsilon}{4} + \frac{2M}{\delta^2}\{(s-u)^2 + (t-v)^2\}. \tag{3}$$

This means

$$-\frac{\varepsilon}{4} - \frac{2M}{\delta^2}\{(s-u)^2 + (t-v)^2\} < f(s,t) - f(u,v) < \frac{\varepsilon}{4} + \frac{2M}{\delta^2}\{(s-u)^2 + (t-v)^2\}.$$

Without loss of generality, ε can be chosen such that $0 < \varepsilon \leq 1$. By the hypothesis, for every $r = 0,1,2,3$ there are $\eta_r > 0$ and $A_r \subseteq \mathbb{N}^2$ with $\delta_2(A_r) = 0$ such that

$$\left|L_{ij}(e_r;u,v) - e_r(u,v)\right| \leq \min\left\{\frac{\varepsilon}{4}, \frac{\varepsilon}{4M}, \frac{\varepsilon\delta^2}{56M}, \frac{\varepsilon\delta^2}{56MH}\right\}|\sigma_r(u,v)|$$

whenever $(i,j) \in \mathbb{N}^2 \backslash A_r$, $|u-u_0| \le \eta_r$ and $|v-v_0| \le \eta_r$. Then, we get

$$\left|L_{ij}(e_r;u,v)-e_r(u,v)\right| \le \min\left\{\frac{\varepsilon}{4},\frac{\varepsilon}{4M},\frac{\varepsilon\delta^2}{56M},\frac{\varepsilon\delta^2}{56MH}\right\}\sigma(u,v)$$

whenever $(i,j) \in \mathbb{N}^2 \backslash A$, $|u-u_0| \le \eta$ and $|v-v_0| \le \eta$ where

$A = \bigcup_{r=0}^{3} A_r$ with $\delta_2(A)=0$ and $\eta = \min\{\eta_r : r = 0,1,2,3\}$.

We write

$$\begin{aligned} L_{ij}&((.-u)^2+(.-v)^2;u,v) \\ &\le \left|L_{ij}(e_3;u,v)-e_3(u,v)\right| + 2|u|\left|L_{ij}(e_1;u,v)-e_1(u,v)\right| + 2|v|\left|L_{ij}(e_2;u,v)-e_2(u,v)\right| \\ &\quad + |u^2+v^2|\left|L_{ij}(e_0;u,v)-e_0(u,v)\right| \\ &\le \frac{\varepsilon\delta^2}{8M}\sigma(u,v) \end{aligned}$$

for $(i,j) \in \mathbb{N}^2 \backslash A$ and $(u,v) \in G^2$ with $|u-u_0| \le \eta$ and $|v-v_0| \le \eta$. Using the linearity and the positivity of the operators L_{ij} and (3), we have

$$\begin{aligned} \left|L_{ij}(f;u,v)-f(u,v)\right| &\le \left|L_{ij}(f;u,v)-f(u,v)L_{ij}(e_0;u,v)\right| + |f(u,v)|\left|L_{ij}(e_0;u,v)-e_0(u,v)\right| \\ &\le L_{ij}\left(\frac{\varepsilon}{4}+\frac{2M}{\delta^2}\{(s-u)^2+(t-v)^2\};u,v\right) + M\left|L_{ij}(e_0;u,v)-e_0(u,v)\right| \\ &= \frac{\varepsilon}{4}L_{ij}(e_0;u,v) + \frac{2M}{\delta^2}L_{ij}((.-u)^2+(.-v)^2;u,v) + M\left|L_{ij}(e_0;u,v)-e_0(u,v)\right| \\ &\le \left\{\frac{\varepsilon}{4}+M\right\}\left|L_{ij}(e_0;u,v)-e_0(u,v)\right| + \frac{\varepsilon}{4} + \frac{2M}{\delta^2}L_{ij}((.-u)^2+(.-v)^2;u,v) \\ &\le \varepsilon\sigma(u,v) \end{aligned}$$

whenever $(i,j) \in \mathbb{N}^2 \backslash A$ and $(u,v) \in G^2$ with $|u-u_0| \le \eta$ and $|v-v_0| \le \eta$. Hence, we get the desired result.

If one replaces the scale function by a non-zero constant, then the next result immediately follows from our main Korovkin type approximation theorem.

Corollary 4.1 Let (L_{ij}) be a double sequence of positive linear operators acting from $C(G^2)$ into itself. Then $(L_{ij}(e_r))$ $(r = 0,1,2,3)$ converges statistically uniformly at (u_0,v_0) to e_r iff for each $f \in C(G^2)$, $(L_{ij}(f))$ converges statistically uniformly at (u_0,v_0) to f.

Also, if one replaces the statistical limit with Pringsheim limit and scale function by a non-zero constant, then the next result which was obtained in [21] follows from our main Korovkin type approximation theorem.

Corollary 4.2 [21] Let (L_{ij}) be a double sequence of positive linear operators acting from $C(G^2)$ into itself. Then $\left(L_{ij}(e_r)\right)$ $(r = 0,1,2,3)$ converges uniformly at (u_0, v_0) to e_r iff for each $f \in C(G^2)$, $\left(L_{ij}(f)\right)$ converges uniformly at (u_0, v_0) to f.

5. An application

In this section, we deal with an example that shows our main Korovkin type approximation theorem is stronger than the corresponding classical ones.

Example 5.1 Let $G^2 = [0,1]^2$. Consider the following Bernstein operators [24] given by

$$B_{ij}(f; u, v) = \sum_{k=0}^{i} \sum_{l=0}^{j} f\left(\frac{k}{i}, \frac{l}{j}\right) \binom{i}{k} \binom{j}{l} u^k (1-u)^{i-k} v^l (1-v)^{j-l} \tag{4}$$

where $(u, v) \in G^2$, $f \in C(G^2)$. Using these polynomials, we introduce the following positive linear operators on $C(G^2)$:

$$L_{ij}(f; u, v) = \left(1 + h_{ij}(u, v)\right) B_{ij}(f; u, v) \tag{5}$$

where $h_{ij}(u, v)$ is given by (3.1). Now, observe that

$$L_{ij}(e_0; u, v) = \left(1 + h_{ij}(u, v)\right) e_0(u, v),\ L_{ij}(e_1; u, v) = \left(1 + h_{ij}(u, v)\right) e_1(u, v),$$

$$L_{ij}(e_2; u, v) = \left(1 + h_{ij}(u, v)\right) e_2(u, v),\ L_{ij}(e_3; u, v) = \left(1 + h_{ij}(u, v)\right) \left[e_3(u, v) + \frac{u-u^2}{i} + \frac{v-v^2}{j}\right].$$

Now we claim that $\left(L_{ij}(e_r)\right)$ converges statistically uniformly to e_r at $(u_0, v_0) = (0,0)$ to the scale function $\sigma_r \coloneqq \sigma$ which is given by (3.2) $(r = 0,1,2,3)$. Let $\varepsilon > 0$ be given and $A_r \coloneqq A = \{(i, j): i = k^2 \ and\ j = l^2, k, l = 1,2, \dots\}$, then $\delta_2(A_r) = 0$ $(r = 0,1,2,3)$.

Now, let $(i, j) \in \mathbb{N}^2 \backslash A_0$ and $(u, v) \in [0,1]^2$ with $|u| \le \eta_0$ and $|v| \le \eta_0$ where $\eta_0 = \sqrt{\frac{\varepsilon}{2}}$. Then,

$$\left|\frac{L_{ij}(e_0; u, v) - e_0(u, v)}{\sigma_0(u, v)}\right| = \left|\frac{h_{ij}(u, v)}{\sigma(u, v)}\right| \le 2|u||v| < 2\eta_0^2 = \varepsilon$$

our claim is true for $r = 0$. Also,

$$\left|\frac{L_{ij}(e_1; u, v) - e_1(u, v)}{\sigma_1(u, v)}\right| = \left|\frac{e_1(u, v) h_{ij}(u, v)}{\sigma(u, v)}\right| \le 2|u|^2|v| < 2\eta_1^3 = \varepsilon$$

whenever $(i, j) \in \mathbb{N}^2 \backslash A_1$ and $|u| \le \eta_1$ and $|v| \le \eta_1$ where $\eta_1 = \sqrt[3]{\frac{\varepsilon}{2}}$. Similarly,

$$\left|\frac{L_{ij}(e_2; u, v) - e_2(u, v)}{\sigma_2(u, v)}\right| \le \left|\frac{e_2(u, v) h_{ij}(u, v)}{\sigma(u, v)}\right| \le 2|u||v|^2 < 2\eta_2^3 = \varepsilon$$

whenever $(i,j) \in \mathbb{N}^2 \backslash A_2$ and $|u| \leq \eta_2$ and $|v| \leq \eta_2$ where $\eta_2 = \sqrt[3]{\frac{\varepsilon}{2}}$. Hence, we get that our claim is true for $r = 1,2$. Finally,

$$\left|\frac{L_{ij}(e_3;u,v) - e_3(u,v)}{\sigma_3(u,v)}\right| = \left|\frac{1}{\sigma(u,v)}\left[\left(1 + h_{ij}(u,v)\right)\left[e_3(u,v) + \frac{u-u^2}{i} + \frac{v-v^2}{j}\right] - e_3(u,v)\right]\right|$$

$$\leq \left|\frac{h_{ij}(u,v)}{\sigma(u,v)}\left[e_3(u,v) + \frac{u-u^2}{i} + \frac{v-v^2}{j}\right]\right| + \left|\frac{u-u^2}{i\sigma(u,v)}\right| + \left|\frac{v-v^2}{j\sigma(u,v)}\right|$$

$$\leq 4\left|\frac{h_{ij}(u,v)}{\sigma(u,v)}\right| + 2|u||v|$$

$$\leq 10|u||v| \leq 10\eta_3^2 = \varepsilon$$

whenever $(i,j) \in \mathbb{N}^2 \backslash A_3$ and $|u| \leq \eta_3$ and $|v| \leq \eta_3$ where $\eta_3 = \sqrt{\frac{\varepsilon}{10}}$ and our claim is true for $r = 3$. Hence from our main Theorem 4.2, we get

$$\left|\frac{L_{ij}(f;u,v) - f(u,v)}{\sigma(u,v)}\right| \leq \varepsilon$$

whenever $(i,j) \in \mathbb{N}^2 \backslash A$, $|u| \leq \eta$ and $|v| \leq \eta$ where $\eta = \min\left\{\sqrt{\frac{\varepsilon}{2}}, \sqrt[3]{\frac{\varepsilon}{2}}\sqrt{\frac{\varepsilon}{10}}\right\}$. However, since $\left|L_{ij}(e_0;u,v) - e_0(u,v)\right| = h_{ij}(u,v)$, the sequence $\left(L_{ij}(e_0)\right)$ is not statistically uniformly convergent to e_0 at (0,0). Hence, we can say that Theorem 4.1 (statistical Korovkin type theorem) and Corollary 4.1 do not work for our operators defined by (5). Also, $\left(L_{ij}(e_0)\right)$ is not uniformly convergent to e_0 at (0,0) and Corollary 4.2 does not work for our operators, too.

6. Rate of Convergence

The main aim of this section is to study the rate of convergence with the aid of the modulus of continuity that is defined by

$$\omega_2(f;\delta) := \sup\left\{|f(s,t) - f(u,v)| : (s,t),(u,v) \in G^2, \sqrt{(s-u)^2 + (t-v)^2} \leq \delta\right\}$$

where $f \in C(G^2)$ and $\delta > 0$. In order to obtain our result, we will make use of the elementary inequality, for all $f \in C(G^2)$ and for $\lambda, \delta > 0$,

$$\omega_2(f;\lambda\delta) \leq (1 + [\lambda])\omega_2(f;\delta)$$

where $[\lambda]$ is defined to be the greatest integer less than or equal to λ (see also [25, 26]).

Theorem 6.1 Let $\left(L_{ij}\right)$ be a double sequence of positive linear operators acting from $C(G^2)$ into itself. Assume that the following conditions hold:

(i) $\left(L_{ij}(e_0)\right)$ converges statistically uniformly to e_0 at (u_0, v_0) with respect to the scale function σ_0,

(ii) $st_2 - \lim_{i,j\to\infty} \frac{\omega_2(f;\delta_{ij})}{|\sigma_1(u,v)|} = 0$ for each $(u,v) \in G^2$ where $\delta_{ij} := \sqrt{L_{ij}((.-u)^2 + (.-v)^2; u, v)}$.

Then we have, for all $f \in C(G^2)$, $\left(L_{ij}(f)\right)$ converges statistically uniformly to f at (u_0, v_0) with respect to the scale function σ, where $\sigma(u,v) = \max\{|\sigma_r(u,v)|: r = 0,1\}$.

Proof Let $(u,v) \in G^2$ and $f \in C(G^2)$ be fixed. Using the linearity and the positivity of the operators L_{ij}, for all $(i,j) \in \mathbb{N}^2$ and any $\delta > 0$, we have

$$\left|L_{ij}(f;u,v) - f(u,v)\right| \le \left|L_{ij}(f;u,v) - f(u,v)L_{ij}(e_0;u,v)\right| + |f(u,v)|\left|L_{ij}(e_0;u,v) - e_0(u,v)\right|$$

$$\le L_{ij}\left(\left(1 + \frac{(.-u)^2+(.-v)^2}{\delta^2}\right)\omega_2(f;\delta); u, v\right) + |f(u,v)|\left|L_{ij}(e_0;u,v) - e_0(u,v)\right|$$

$$= \omega_2(f;\delta)L_{ij}(e_0;u,v) + \frac{\omega_2(f;\delta)}{\delta^2}L_{ij}((.-u)^2 + (.-v)^2; u, v) + |f(u,v)|\left|L_{ij}(e_0;u,v) - e_0(u,v)\right|.$$

Put $\delta := \delta_{ij} = \sqrt{L_{ij}((.-u)^2 + (.-v)^2; u, v)}$. Hence, we get

$$\frac{\left|L_{ij}(f;u,v) - f(u,v)\right|}{\sigma(u,v)} \le \left[\omega_2\left(f;\delta_{ij}\right) + |f(u,v)|\right]\left|\frac{L_{ij}(e_0;u,v) - e_0(u,v)}{\sigma_0(u,v)}\right| + 2\frac{\omega_2(f;\delta_{ij})}{|\sigma_1(u,v)|}.$$

By using (i) and (ii) the proof is completed.

Acknowledgment

The author received no financial support for the research, authorship, and/or publication of this article.

Conflicts of interest

The author states that she did not have a conflict of interest.

References

[1] Fast H., Sur la Convergence Statistique, *Colloq. Math.*, 2 (1951) 241-244.

[2] Steinhaus H., Sur la Convergence Ordinaire et la Convergence Asymtotique, *Colloq. Math.*, 2 (1951) 73-74.

[3] Gadjiev A. D., Orhan C., Some Approximation Theorems via Statistical Convergence, *Rocky Mountain J. Math.*, 32 (2002) 129-138.

[4] Korovkin P. P., Linear Operators and Approximation Theory. Delhi: Hindustan Publ. Co., (1960).

[5] Belen C., Yıldırım M., Generalized *A*-Statistical Convergence and a Korovkin Type Approximation Theorem for Double Sequences, *Miskolc Mathematical Notes*, 14 (1) (2013) 31-39.

[6] Demirci K., Dirik F., Four-Dimensional Matrix Transformation and Rate of *A*-Statistical Convergence of Periodic Functions, *Math. Comput. Modelling*, 52 (9-10) (2010) 1858-1866.

[7] Dirik F., Demirci K., Korovkin Type Approximation Theorem for Functions of Two Variables in Statistical Sense, *Turk J Math*, 34 (2010) 73-83.

[8] Ünver M., Orhan C., Statistical Convergence with Respect to Power Series Methods and Applications to Approximation Theory, *Numerical Functional Analysis and Optimization*, 40 (5) (2019) 535-547.

[9] Moore E. H., An Introduction to a Form of General Analysis, The New Haven Mathematical Colloquium, Yale University Press, New Haven, (1910).

[10] Chittenden E. W., On the Limit Functions of Sequences of Continuous Functions Converging Relatively Uniformly, *Transactions of the AMS*, 20 (1919) 179-184.

[11] Demirci K., Orhan S., Statistically Relatively Uniform Convergence of Positive Linear Operators, *Results Math.*, 69 (2016) 359-367.

[12] Demirci K., Kolay B., *A*-Statistical Relative Modular Convergence of Positive Linear Operators, *Positivity*, 21 (3) (2017) 847-863.

[13] Demirci K., Orhan S., Kolay B., Relative Hemen Hemen Yakınsaklık ve Yaklaşım Teoremleri, *Sinop Üniversitesi Fen Bilimleri Dergisi*, 1 (2) (2016) 114-122.

[14] Demirci K., Yıldız S., Statistical Relative Uniform Convergence in Dually Residuated Lattice Totally Ordered Semigroups, *Sarajevo J. Math.*, 15 (2) (2019) 227-237.

[15] Sahin P. O., Dirik F., Statistical Relative Uniform Convergence of Double Sequences of Positive Linear Operators, *Appl. Math. E-Notes*, 17 (2017) 207-220.

[16] Sahin P. O., Dirik F., Statistical Relative Equal Convergence of Double Function Sequences and Korovkin-Type Approximation Theorem, *Applied Mathematics E-Notes*, 19 (2019) 101-115.

[17] Yılmaz B., Demirci K., Orhan S., Relative Modular Convergence of Positive Linear Operators, *Positivity*, 20 (3) (2016) 565-577.

[18] Klippert J., Williams G., Uniform Convergence of a Sequence of Functions at a Point, *Internat. J. Math. Ed. Sci. Tech.*, 33 (1) (2002) 51-58.

[19] Demirci K., Boccuto A., Yıldız S., Dirik F., Relative Uniform Convergence of a Sequence of Functions at a Point and Korovkin-Type Approximation Theorems, *Positivity*, 24 (1) (2020) 1-11.

[20] Boccuto A., Demirci K., Yıldız S., Abstract Korovkin-type theorems in the filter setting with respect to relative uniform convergence, *Turkish Journal of Mathematics*, 44 (4) (2020) 1238-1249.

[21] Dirik F., Demirci K., Yıldız S., Acu A. M., The Uniform Convergence of a Double Sequence of Functions at a Point and Korovkin-Type Approximation Theorems, *Georgian Mathematical Journal*, 1 (ahead-of-print) (2020)

[22] Pringsheim A., Zur Theorie der Zweifach Unendlichen Zahlenfolgen, *Math. Ann.*, 53 (1) (1900) 289-321.

[23] Moricz F., Statistical Convergence of Multiple Sequences, *Arch. Math., (Basel)* 81 (2004) 82-89.

[24] Stancu D. D., A Method for Obtaining Polynomials of Bernstein Type of Two Variables, *The American Mathematical Monthly*, 70 (3) (1963) 260-264.

[25] DeVore R.A., Lorentz G.G., Constructive Approximation (Grund. Math. Wiss. 303). Berlin: Springer Verlag, (1993).

[26] Altomare F., Campiti M., Korovkin-Type Approximation Theory and its Applications. New York: Walter de Gruyter, (1994).