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The Nakagami–Weibull distribution in modeling real-life data

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Abstract

In this article, a four-parameter Nakagami Weibull distributions (NW) is proposed. We study a few statistical properties such as quantile function, moments, moment generating function, entropy, and order statistics have been derived. The maximum likelihood estimate is used to estimate the parameter of the NW distribution. We fit the proposed NW distribution to a real-life data set to examine its potential and flexibility. Our findings showed that the NW distribution performs much better than its competitors, with favorable comparisons to existing distributions in terms of goodness-of-fit.

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1. Introduction

The continuous probabilities distribution has some essential problems and limitations in modeling real-life data set, has led statistician by adding at least one shape parameter to the baseline distribution to developed new flexible distributions. Methods for generating new families of distributions have been developed by many mathematical statisticians. The beta-generalized family of distribution was developed by [1], the exponentiated generalized class of distributions by [2], Exponentiated Weibull distribution: statistical properties and applications by [3], Beta-Nakagami distribution by [4], Weibull generalized family of distributions by [5], On the exponentiated generalized inverse exponential distribution by [6], Beta generated Kumaraswamy and many compound families of distribution by [7], Exponentiated half-logistic family of distributions by [8], additive Weibull generated distributions by [9], Kummer beta generalized family of distributions by [10], The generalized odd inverted exponential-G family of distributions by [11], the Marshall-Olkin odd Burr III-G family of distributions by [12] and the generalized odd Gamma-G family of distributions by [13].

The Nakagami distribution is a continuous probability distribution with applications in measuring alternation of wireless signal traversing multiple paths, and Weibull distribution is one of the continuous probability distributions used to model a variety of life behaviors.

2. Theoretical Framework of Nakagami Weibull (NW) Distribution

If X is a continuous random variable from the Nakagami distribution with two parameter λ and β, then the *cdf* Eq. (1) and *pdf* Eq. (2) of the Nakagami generalized family of distribution (OGNak-G) due to [14] is given by:

$$F(x.\lambda,\beta,\eta)=\frac{1}{\Gamma(\lambda)}\gamma\left\{\lambda,\frac{\lambda}{\beta}\left[\frac{G(x;\eta)}{\left(1-G(x;\eta)\right)}\right]^2\right\} \quad (1)$$

The probability density function of the OGNak-G is given by:

$$f(x;\lambda,\beta,\eta)=\frac{2\lambda^{\lambda}}{\Gamma(\lambda)\beta^{\lambda}}g(x;\eta)\frac{\left[G(x;\eta)\right]^{2\lambda-1}}{\left[1-G(x;\eta)\right]^{2\lambda+1}}e^{-\frac{\lambda}{\beta}\left[\frac{G(x;\eta)}{1-G(x;\eta)}\right]^2};\ x\in\Re, \quad (2)$$

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2.1. The proposed NW Distribution

The Weibull distribution is our parent distribution, with two parameters. δ is the scale parameter and α is the shape parameter that has its cdf and pdf given by:

$$g(x;\delta,\alpha)=\delta\alpha x^{\alpha-1}e^{-\delta x^{\alpha}};x\geq 0,\delta,\alpha>0 \quad (3)$$

$$G(x;\delta)=1-e^{-\delta x^{\alpha}} \quad (4)$$

Using the generator propose by [14] in Eq. (1), the *cdf* of the proposed Nakagami Weibull distribution is given by:

$$F(x.\lambda,\beta,\eta)=\gamma_{*}\left\{\lambda,\frac{\lambda}{\beta}\left[\frac{1-e^{-\delta x^{\alpha}}}{e^{-\delta x^{\alpha}}}\right]^{2}\right\} \quad (5)$$

where $\eta=(\delta,\alpha)$

and its corresponding pdf is given by:

$$f(x;\lambda,\beta,\eta)=\frac{2\lambda^{\lambda}}{\Gamma(\lambda)\beta^{\lambda}}\delta\alpha x^{\alpha-1}e^{-\delta x^{\alpha}}\frac{\left(1-e^{-\delta x^{\alpha}}\right)^{2\lambda-1}}{\left(e^{-\delta x^{\alpha}}\right)^{2\lambda+1}}e^{-\frac{\lambda}{\beta}\left(e^{\delta x^{\alpha}}-1\right)^{2}};x\in\Re \quad (6)$$

2.2. Investigation of the proposed NW distribution for a PDF

$$\int_{0}^{\infty}f(x)dx=1$$

To demonstrate that the NW distribution is a pdf, we proceed as follows:

$$\int_{0}^{\infty}\frac{2\lambda^{\lambda}}{\Gamma(\lambda)\beta^{\lambda}}\delta\alpha x^{\alpha-1}e^{-\delta x^{\alpha}}\frac{\left(1-e^{-\delta x^{\alpha}}\right)^{2\lambda-1}}{\left(e^{-\delta x^{\alpha}}\right)^{2\lambda+1}}e^{-\frac{\lambda}{\beta}\left(e^{\delta x^{\alpha}}-1\right)^{2}}dx=1 \quad (7)$$

let $y=\frac{\lambda}{\beta}\left(e^{\delta x^{\alpha}}-1\right)^{2}$

Therefore, $\frac{1}{\Gamma(\lambda)}\int_{0}^{\infty}y^{\lambda-1}e^{-y}dy=1$

Nakagami Weibull Distribution is a pdf

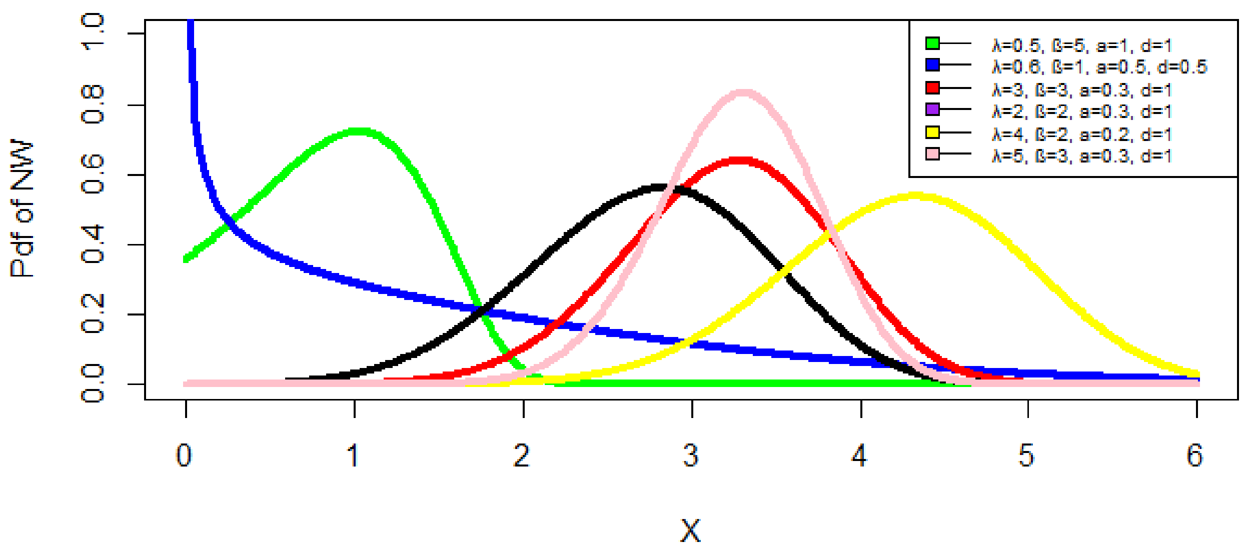

Figure 1. The pdf of the NW distribution for different set of values of the parameters.

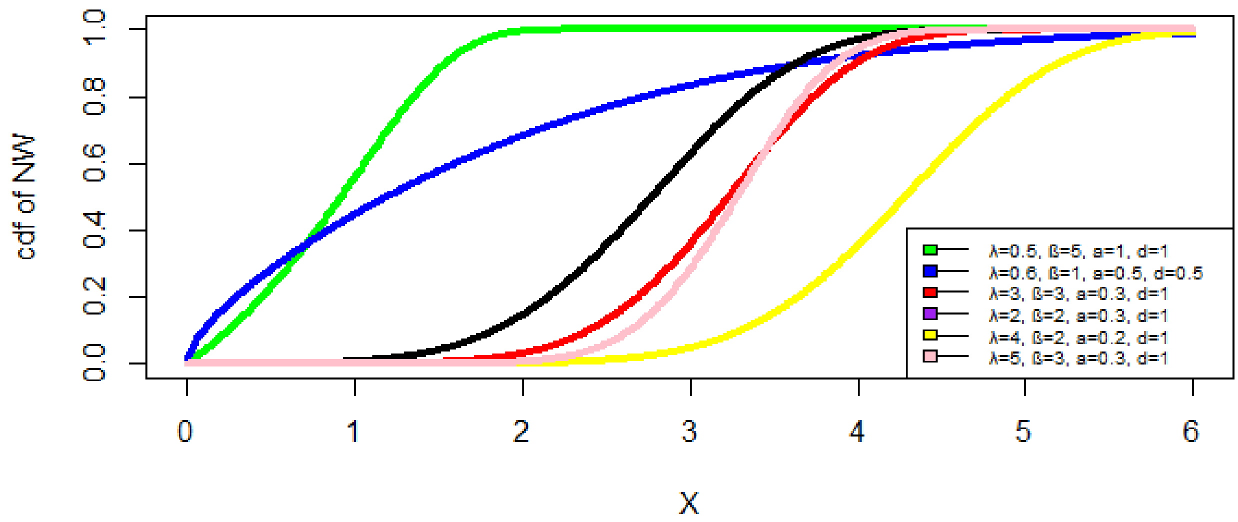

Figure 2. The cdf of the NW distribution for different set of values of the parameters.

2.3. Linear representation

Using generalized binomial and Taylor expansion in Eq. (6) one can obtain

$$f\left(x;\lambda,\beta,\eta\right)=\frac{2}{\Gamma(\lambda)}\delta\alpha\sum_{i,k,j=0}^{\infty}\left(\frac{\lambda}{\beta}\right)^{\lambda+i}\omega_{i,k,j}x^{\alpha-1}e^{-j\delta x^{\alpha}} \tag{8}$$

where $\omega_{i,k,j}=\frac{(-1)^{i+j}}{i!}\binom{2(\lambda+i)+k-1}{k}\binom{2(\lambda+i)+k-1}{j}$

2.4. Reliability analysis for the new Nakagami Weibull Distribution

We proposed new survival function and the hazard function of the Nakagami Weibull distribution are provided as follows:
Survival function is given by:

$$S\left(x\right)=1-\gamma_{*}\left\{\lambda,\frac{\lambda}{\beta}\left[\frac{1-e^{-\delta x^{\alpha}}}{e^{-\delta x^{\alpha}}}\right]^{2}\right\} \tag{9}$$

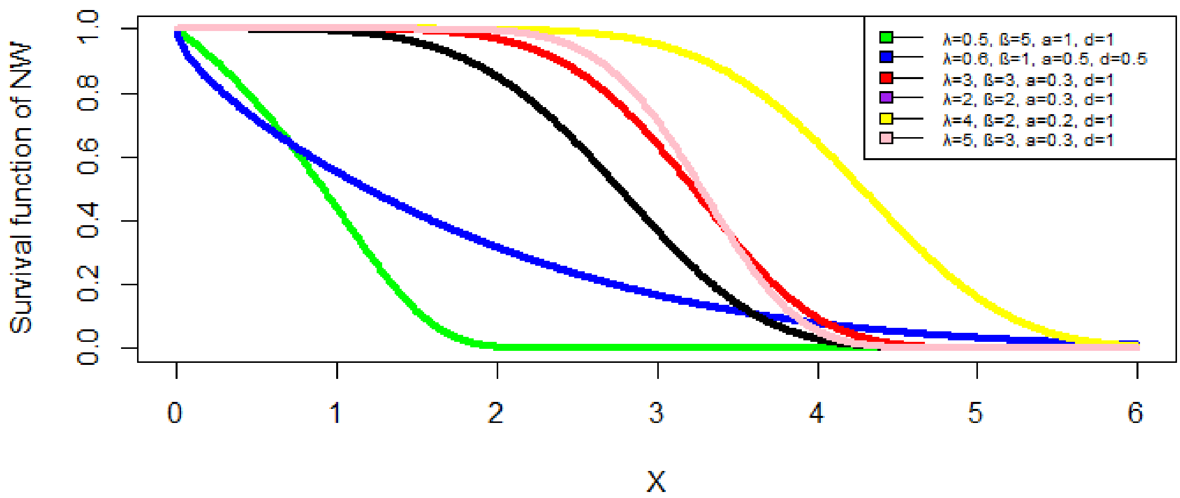

Figure 3. The survival function of the NW distribution for different set of values of the parameters.

Hazard function is given by:

$$h(x)=\frac{\dfrac{2\lambda^{\lambda}}{\Gamma(\lambda)\beta^{\lambda}}\delta\alpha x^{\alpha-1}e^{-\delta x^{\alpha}}\dfrac{\left(1-e^{-\delta x^{\alpha}}\right)^{2\lambda-1}}{\left(e^{-\delta x^{\alpha}}\right)^{2\lambda+1}}e^{-\frac{\lambda}{\beta}\left(e^{\delta x^{\alpha}}-1\right)^{2}}}{1-\gamma_{*}\left\{\lambda,\dfrac{\lambda}{\beta}\left[\dfrac{1-e^{-\delta x^{\alpha}}}{e^{-\delta x^{\alpha}}}\right]^{2}\right\}} \tag{10}$$

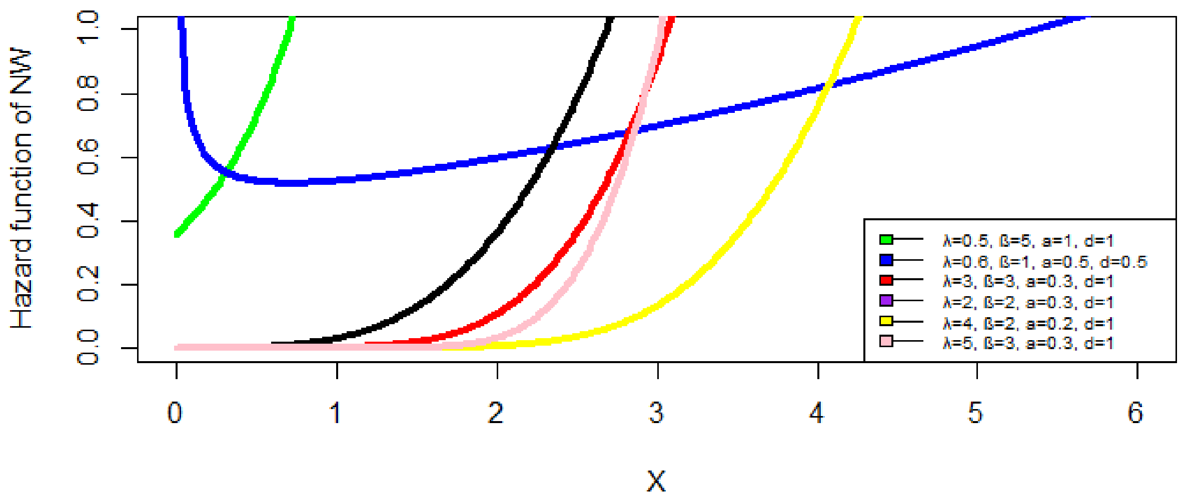

Figure 4. The hazard function of the NW distribution for different set of values of the parameters.

2.5. Mathematical and statistical properties

Moment play an important role in statistical analysis when a new probability distribution is developed. Using Eq. (8), we obtain the following:

$$E\left(X^{r}\right)=\int_{0}^{\infty}\frac{2}{\Gamma(\lambda)}\delta\alpha\sum_{i,k,j=0}^{\infty}\left(\frac{\lambda}{\beta}\right)^{\lambda+i}\omega_{i,k,j}x^{r+\alpha-1}e^{-j\delta x^{\alpha}}dx \tag{11}$$

$$\mu_r' = \frac{2}{\Gamma(\lambda)} \sum_{i,k,j=0}^{\infty} \left(\frac{\lambda}{\beta}\right)^{\lambda+i} \omega_{i,k,j} \frac{\Gamma\left(\frac{r}{\alpha}+1\right)}{\delta^{\frac{r}{\alpha}} j^{\frac{r}{\alpha}+1}}; \quad r = 1,2,3\ldots \tag{12}$$

The mean and variance of NW distribution are obtained, respectively as follows

$$E(X) = \frac{2}{\Gamma(\lambda)} \sum_{i,k,j=0}^{\infty} \left(\frac{\lambda}{\beta}\right)^{\lambda+i} \omega_{i,k,j} \frac{\Gamma\left(\frac{1}{\alpha}+1\right)}{\delta^{\frac{1}{\alpha}} j^{\frac{1}{\alpha}+1}} \tag{13}$$

$$Var(X) = \frac{2}{\Gamma(\lambda)} \sum_{i,k,j=0}^{\infty} \left(\frac{\lambda}{\beta}\right)^{\lambda+i} \omega_{i,k,j} \frac{\Gamma\left(\frac{2}{\alpha}+1\right)}{\delta^{\frac{2}{\alpha}} j^{\frac{2}{\alpha}+1}} - \left\{ \frac{2}{\Gamma(\lambda)} \sum_{i,k,j=0}^{\infty} \left(\frac{\lambda}{\beta}\right)^{\lambda+i} \omega_{i,k,j} \frac{\Gamma\left(\frac{1}{\alpha}+1\right)}{\delta^{\frac{1}{\alpha}} j^{\frac{1}{\alpha}+1}} \right\}^2 \tag{14}$$

Moment generating function of NW will take this form:

$$M_X(t) = E\left(e^{tX}\right) = \sum_{r=0}^{\infty} \frac{t^r \mu_r'}{r!}$$

$$M_X(t) = \frac{2}{\Gamma(\lambda)} \sum_{i,k,j,r=0}^{\infty} \frac{t^r}{r!} \left(\frac{\lambda}{\beta}\right)^{\lambda+i} \omega_{i,k,j} \frac{\Gamma\left(\frac{r}{\alpha}+1\right)}{\delta^{\frac{r}{\alpha}} j^{\frac{r}{\alpha}+1}} \tag{15}$$

2.5.1. Incomplete moments of NW distribution

The main applications of the first incomplete moment refer to the mean deviations and the Bonferroni and Lorenz curves. These curves are very useful in reliability, medicine, economics, insurance and demography (see [15]). Considering the NW distribution discussed in Eq. (8) the r^{th} incomplete moment for NW is derived by using Eq. (16) as follows:

$$M_r^q = \int_0^t x^r f(x) dx \tag{16}$$

$$= \frac{2}{\Gamma(\lambda)} \delta\alpha \sum_{i,k,j=0}^{\infty} \left(\frac{\lambda}{\beta}\right)^{\lambda+i} \omega_{i,k,j} \int_0^t x^{r+\alpha-1} e^{-j\delta x^{\alpha}} dx$$

$$M_r^q = \frac{2}{\Gamma(\lambda)} \sum_{i,k,j=0}^{\infty} \left(\frac{\lambda}{\beta}\right)^{\lambda+i} \omega_{i,k,j} \frac{1}{j^{\frac{r}{\alpha}+1} \delta^{\frac{r}{\alpha}}} \gamma\left(\frac{r}{\alpha}+1, j\delta x^{\alpha}\right) \tag{17}$$

2.5.2. Entropy

The entropy of a random variable X is a measure of variation of the uncertainly. The Renyi entropy is defined as

$I_R(\sigma)=\frac{1}{1-\sigma}\log\left[\int_0^\infty f^\sigma(x)dx\right]$, Where $\sigma>0$ and $\sigma\neq 1$. Based on *f(x)* of any distribution from Eq. (6).

$$f^\sigma(x)=\left(\frac{2\delta\alpha}{\Gamma(\lambda)}\right)^\sigma\left(\frac{\lambda}{\beta}\right)^{\lambda\sigma}x^{\sigma(\alpha-1)}e^{-\sigma\delta x^\alpha}\frac{\left(1-e^{-\delta x^\alpha}\right)^{\sigma(2\lambda-1)}}{\left(e^{-\delta x^\alpha}\right)^{\sigma(2\lambda+1)}}e^{-\frac{\sigma\lambda}{\beta}\left(e^{\delta x^\alpha}-1\right)^2} \tag{18}$$

Using generalized binomial and taylor expansion in Eq. (18), one can obtain

$$f^\sigma(x)=\sum_{q,i,j=0}^\infty \pi_{q,i,j}x^{\sigma(\alpha-1)}e^{-j\delta x^\alpha} \tag{19}$$

where

$$\pi_{q,i,j}=\frac{(-1)^{q+j}}{q!}\binom{2\lambda\sigma+2q-1+i}{i}\binom{\sigma(2\lambda-1)+2q+i}{j}\sigma^q\left(\frac{\lambda}{\beta}\right)^{\lambda\sigma+q}\left(\frac{2\delta\alpha}{\Gamma(\lambda)}\right)^\sigma$$

$$\int_0^\infty f^\sigma(x)dx=\int_0^\infty\sum_{q,i,j=0}^\infty \pi_{q,i,j}x^{\sigma(\alpha-1)}e^{-j\delta x^\alpha}dx \tag{20}$$

$$I_R(\sigma)=\frac{1}{1-\sigma}\log\left[\int_0^\infty\sum_{q,i,j=0}^\infty \pi_{q,i,j}x^{\sigma(\alpha-1)}e^{-j\delta x^\alpha}dx\right] \tag{21}$$

2.5.3. The mode

The mode of the NW density function can be derived by differentiating the natural logarithm of Eq. (6) with respect to *x* as follows:

$$\frac{\alpha-1}{x}-\delta\alpha x^{\alpha-1}+\frac{(2\lambda-1)\delta\alpha x^{\alpha-1}e^{-\delta x^\alpha}}{\left(1-e^{-\delta x^\alpha}\right)}+(2\lambda+1)\delta\alpha x^{\alpha-1}-\frac{2\lambda}{\beta}\left(e^{\delta x^\alpha}-1\right)\delta\alpha x^{\alpha-1}e^{\delta x^\alpha}=0 \tag{22}$$

Solving the Eq. (22) numerically we can find the mode(s).

If x_0 is a root of the Eq.(22), then it must be $\frac{d^2}{dx^2}\log f(x_0)<0,$

$$=-\frac{\alpha-1}{x}-\delta\alpha(\alpha-1)x^{\alpha-2}-\frac{(2\lambda-1)\delta\alpha x^{\alpha-2}e^{-\delta x^\alpha}\left[(\alpha-1)e^{-\delta x^\alpha}+\delta\alpha x^\alpha-\alpha+1\right]}{\left(1-e^{-\delta x^\alpha}\right)^2}+$$
$$(2\lambda+1)\delta\alpha(\alpha-1)x^{\alpha-2}-\frac{2\lambda}{\beta}\delta\alpha x^{\alpha-2}e^{\delta x^\alpha}\left(2\delta\alpha x^\alpha e^{\delta x^\alpha}-\delta\alpha x^\alpha+\alpha e^{\delta x^\alpha}-e^{\delta x^\alpha}-\alpha+1\right) \tag{23}$$

The Eq. (23) is a nonlinear and does not have an analytic solution with respect to *x* therefore we have to solve it numerically.

2.5.4. Order statistics

Let $X_{(1)},X_{(2)},\ldots,X_{(n)}$. represent random sample for the order statistic, $X_1,X_2,\ldots,X_n$ from a continuous population with *cdf* $F_X(x)$ and *pdf* $f_X(x)$. Then the pdf of $X_{(j)}$ is

$$f_{X_{(j)}}(x)=\frac{n!}{(j-1)!(n-j)!}f_X(x)[F_X(x)]^{j-1}[1-F_X(x)]^{n-j},\ for\ -\infty<x<\infty$$

By using binomial expansion

$$[1-F_X(x)]^{n-j}=\sum_{z=0}^{n-j}{}^{n-j}C_z[F_X(x)]^z$$

Therefore,

$$f_{X_{(j)}}(x)=\sum_{z=0}^{n-j}{}^{n-j}C_z f_X(x)[F_X(x)]^{z+j-1},\ for\ -\infty<x<\infty \tag{24}$$

(See [16]).

Hence, the j^{th} order statistic for the NW distribution is given by using Eq. (5) and Eq. (6), we obtain

$$f_{X_{(j)}}(x)=\sum_{z=0}^{n-j}{}^{n-j}C_z\frac{2\lambda^{\lambda}}{\Gamma(\lambda)\beta^{\lambda}}\delta\alpha x^{\alpha-1}e^{-\delta x^{\alpha}}\frac{\left(1-e^{-\delta x^{\alpha}}\right)^{2\lambda-1}}{\left(e^{-\delta x^{\alpha}}\right)^{2\lambda+1}}e^{-\frac{\lambda}{\beta}\left(e^{\delta x^{\alpha}}-1\right)^2}\gamma_*\left\{\lambda,\frac{\lambda}{\beta}\left(e^{\delta x^{\alpha}}-1\right)^2\right\}^{z+j-1} \tag{25}$$

2.5.5. Quantile Function

The quantile function of the NW distribution is obtained by inverting the distribution function defined in Eq. (5) as follows:

$$u=\gamma_*\left\{\lambda,\frac{\lambda}{\beta}\left[\frac{G(x;\eta)}{(1-G(x;\eta))}\right]^2\right\}$$

$$x=\left\langle\frac{-1}{\delta}\ln\left\langle 1-\left\{\frac{\left[\frac{\beta}{\lambda}\gamma^{-1}(\lambda,u\Gamma(\lambda))\right]}{1+\left[\frac{\beta}{\lambda}\gamma^{-1}(\lambda,u\Gamma(\lambda))\right]}\right\}^{1/2}\right\rangle\right\rangle^{\frac{1}{\alpha}} \tag{26}$$

2.6. Maximum likelihood estimates of the parameters of Nakagami Weibull Distribution

Many approaches of estimating parameter were introduced in the literature. In this section, we deal with the estimation of the unknown parameters for the NW distributions based on complete samples only by maximum likelihood. Let $X_1, X_{2,\cdots,}X_n$ be observed values from the NW distribution with set of parameters $\Theta=(\lambda,\beta,\delta,\alpha)^{\mathrm{T}}$. The log-likelihood function for parameter vector $\Theta=(\lambda,\beta,\delta,\alpha)^{\mathrm{T}}$ is obtained from (6) as follows:

$$L(x;\Theta)=\prod_{i=1}^{n}f(x_i;\Theta) \tag{27}$$

$$=\prod_{i=1}^{n}\frac{2\lambda^{\lambda}}{\Gamma(\lambda)\beta^{\lambda}}\delta\alpha x^{\alpha-1}e^{-\delta x^{\alpha}}\frac{\left(1-e^{-\delta x^{\alpha}}\right)^{2\lambda-1}}{\left(e^{-\delta x^{\alpha}}\right)^{2\lambda+1}}e^{-\frac{\lambda}{\beta}\left(e^{\delta x^{\alpha}}-1\right)^2} \tag{28}$$

$$\ell(x;\Theta)=n\log 2+n*\lambda\log\lambda-n\log\Gamma(\lambda)-n*\lambda\log\beta+n\log\delta+n\log\alpha-$$

$$\delta\sum x^{\alpha}+(2\lambda-1)\log\sum\left(1-e^{-\delta x^{\alpha}}\right)-(2\lambda+1)\log\sum\left(e^{-\delta x^{\alpha}}\right)-\frac{\lambda}{\beta}\sum\left(e^{\delta x^{\alpha}}-1\right)^2$$

$$\frac{\partial\ell(x;\Theta)}{\partial\lambda}=n\log\lambda n+-n\Psi(\lambda)-n\log\beta-\frac{1}{\beta}\left(e^{\delta x^{\alpha}}-1\right) \quad (29)$$

$$\frac{\partial\ell(x;\Theta)}{\partial\beta}=\frac{\lambda}{\beta^{2}}\left(e^{\delta x^{\alpha}}-1\right)^{2}-\frac{n\lambda}{\beta} \quad (30)$$

$$\frac{\partial\ell(x;\Theta)}{\partial\delta}=\frac{n}{\delta}-\sum x^{\alpha}+(2\lambda-1)\frac{\sum x^{\alpha}e^{-\delta x^{\alpha}}}{\sum\left(1-e^{-\delta x^{\alpha}}\right)}+(2\lambda+1)\frac{\sum x^{\alpha}e^{-\delta x^{\alpha}}}{\sum e^{-\delta x^{\alpha}}}-\frac{2\lambda}{\beta}\sum\left(e^{\delta x^{\alpha}}-1\right)x^{\alpha}e^{-\delta x^{\alpha}} \quad (31)$$

$$\frac{\partial\ell(x;\Theta)}{\partial\alpha}=\frac{n}{\alpha}-\delta\alpha\sum x^{\alpha-1}+(2\lambda-1)\frac{\sum\delta\alpha x^{\alpha-1}e^{-\delta x^{\alpha}}}{\sum\left(1-e^{-\delta x^{\alpha}}\right)}+(2\lambda+1)\frac{\sum\delta\alpha x^{\alpha-1}e^{-\delta x^{\alpha}}}{\sum\left(e^{-\delta x^{\alpha}}\right)}-\frac{2\lambda}{\beta}\sum\delta\alpha x^{\alpha-1}\left(1-e^{\delta x^{\alpha}}\right) \quad (32)$$

These estimates can't be be solved algebraically, and statistical software can be used to solve them numerically via iterative technique in the AdequacyModel package available in the R.

3. Results and Discussion

In this section, fitting NW distribution. We provide two applications to real life data set to demonstrate the potentiality of the NW distribution and compare its performance, to other generated models. The Akaike information criterion (AIC), Consistent Akaikes Information Criterion (CAIC), Bayesian information criterion (BIC), Anderson-Darling (A), Kolmogorov Smirnov test (K.S), and the P-Value of K.S test, have been chosen for the comparison of the models. The distributions: Odd Generalize Gamma Frechet (OGGFr) [117], Odd Generalized Gamma Weibull (OGGW) [17], the generalized odd inverted exponential-G family of distributions [11], Exponentiated Weibull Weibull (EWW) [3], Weibull-Exponential (WE) [18] have been selected for comparison. The parameters of models have been estimated by the MLE method.

The first data, we used the breaking stress of carbon fibers of 50 mm length (GPa) from [19], [20] and [14]. The data is as follows: 0.39, 0.85, 1.08 ,1.25, 1.47, 1.57, 1.61, 1.61, 1.69, 1.80, 1.84, 1.87 ,1.89, 2.03, 2.03 ,2.05, 2.12 ,2.35 ,2.41, 2.43, 2.48, 2.50, 2.53 ,2.55, 2.55, 2.56, 2.59, 2.67, 2.73, 2.74, 2.79, 2.81, 2.82, 2.85, 2.87 ,2.88, 2.93, 2.95 ,2.96 ,2.97, 3.09, 3.11 ,3.11, 3.15, 3.15 ,3.19 ,3.22, 3.22, 3.27, 3.28, 3.31 ,3.31, 3.33 ,3.39, 3.39, 3.56 ,3.60 ,3.65, 3.68 ,3.70 ,3.75 ,4.20 ,4.38, 4.42, 4.70, 4.90.

Table 1 displays a summary of goodness-of-fit measures for the stress of carbon fibers of 50 mm length (GPa) and MLEs for this data with different models, respectively. The NW with the smallest AIC criteria is selected as the best model with all other criteria. As you see, the P-Value for NW is also more than all other distributions.

Table 1: Summary of MLEs and goodness-of-fit statistics for the first data set

Models	MLE	$-\ell$	AIC	CAIC	BIC	A.D	K.S	P Value
NW	λ =1.6103057 β =1.1390558 δ =0.2809454 α =0.8761429	86.190	180.381	181.037	189.140	0.528	0.109	0.413
OGGFr	λ =1.9147482 β =1.2807083 δ =1.7515636 α =0.9275479	91.388	190.776	191.432	199.534	1.249	0.166	0.052
OGGW	λ =1.5477753 β =1.5594958 δ =0.2641594 α =1.0745220	86.277	180.555	181.210	189.313	0.511	0.11583	0.339

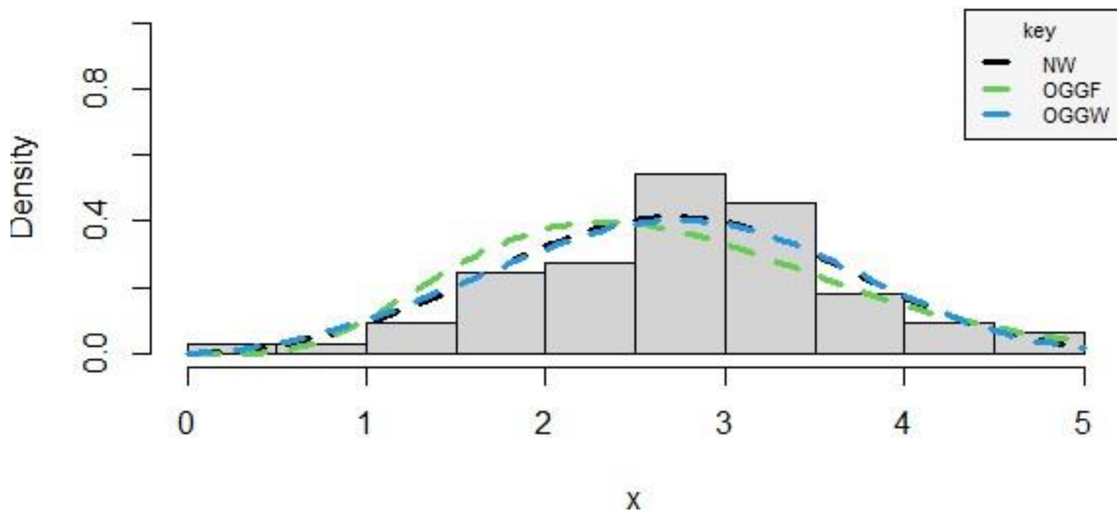

Figure 5. Histogram and estimated pdfs for data sets 1.

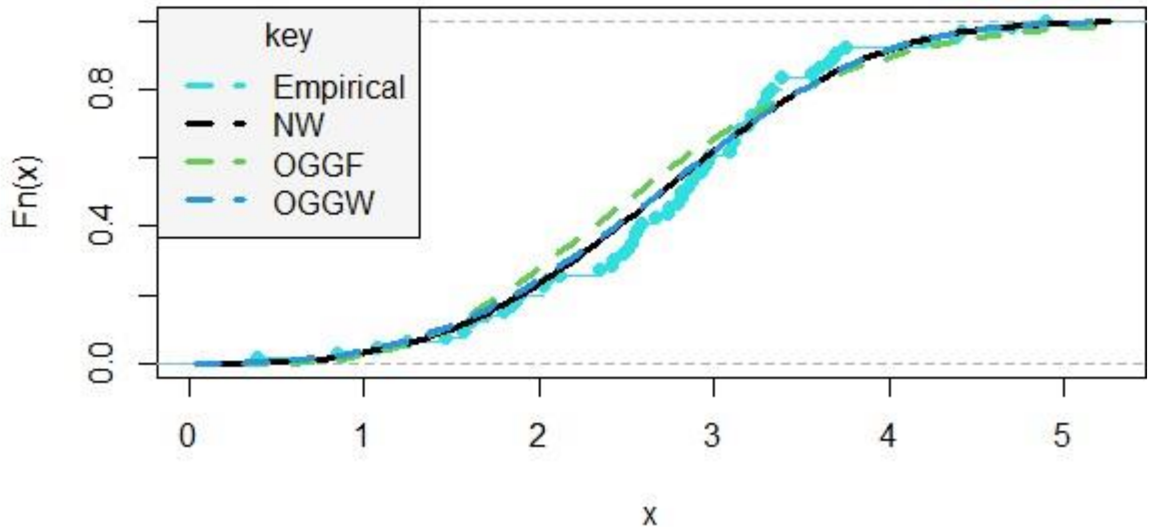

Figure 6. Plots of estimated cdf for data Sets 1.

The second data, we used thirty successive values of March precipitation in Minneapolis/St Paul from [21] and [22]. The data set are as follows:

0.77, 1.74, 0.81, 1.20, 1.95, 1.20, 0.47, 1.43, 3.37, 2.20, 3.00, 3.09, 1.51, 2.10, 0.52, 1.62, 1.31, 0.32, 0.59, 0.81, 2.81, 1.87, 1.18, 1.35, 4.75, 2.48, 0.96, 1.89, 0.90, 2.05.

Table 2 displays a summary of the goodness-of-fit measures for the March precipitation in Minneapolis/St Paul and MLEs for this data with different models, respectively. The NW with the smallest AIC criteria is selected as the best model with all other criteria. As you see, the P-Value for NW is also more than all other distributions.

Table 2: Summary of MLEs and goodness-of-fit statistics for the second data set

Models	MLE	$-\ell$	AIC	CAIC	BIC	A.D	K.S	P Value
NW	λ =1.5347926 β =0.5178431 δ =0.3999663 α =0.5250054	38.747	85.493	87.093	91.098	0.170	0.066	0.999
EWW	λ =1.5264454 β =0.6836198 δ =1.3895067 α =0.6091976 θ =0.6701214	39.206	88.412	90.912	95.418	0.235	0.1023	0.911
OGGW	λ =1.3057645 β =1.5374857 δ =0.5881135 α =0.6735570	39.335	86.670	88.270	92.275	0.237	0.081	0.989

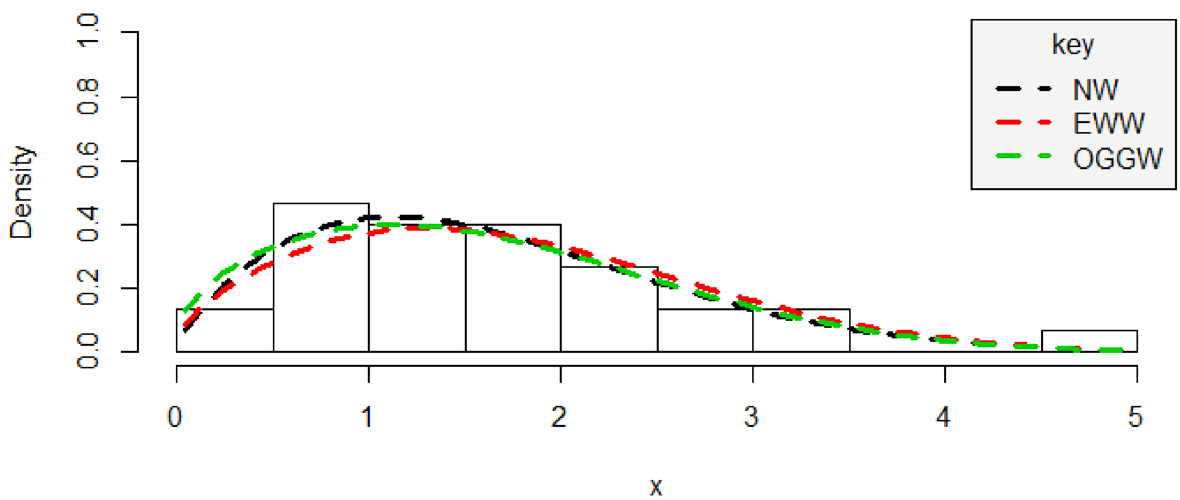

Figure 7. Histogram and estimated pdfs for data Sets 2.

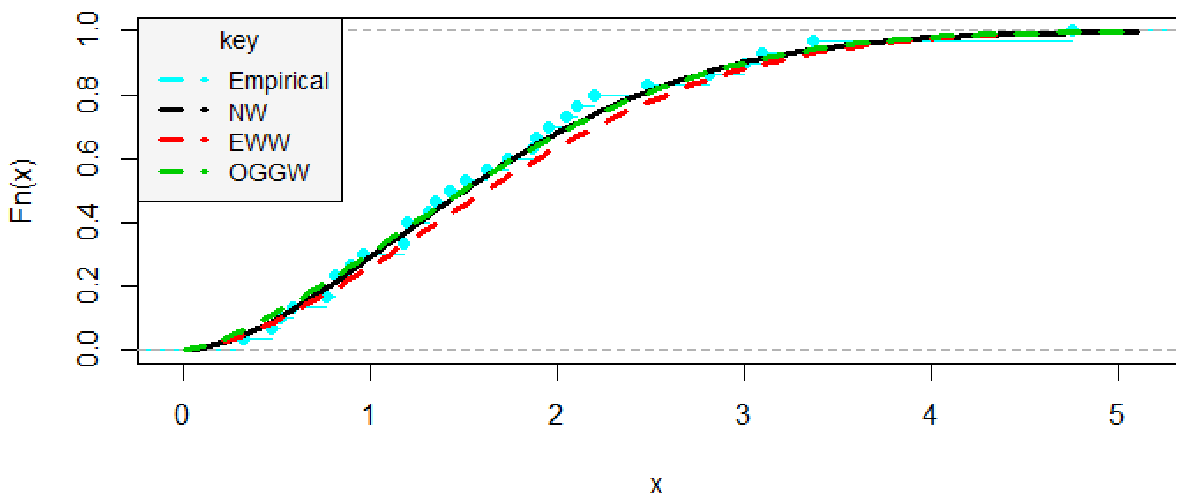

Figure 8. Plots of estimated cdf for data sets 2.

Conclusions

We introduce a new four-parameter, the so called Nakagami Weibull distribution. The main statistical properties are provided. The model parameters estimation is approached by maximum likelihood. We prove empirically the usefulness of the NW distribution is demonstrated in two applications to show its superiority compared to other competitive distributions in terms of minimum AIC criteria is selected as the best model. It seems that the result is consistent with other criteria. We hope that the NW may attract wider applications in many applied areas.

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Conflicts of interest

There are no conflicts of interest disclosed by the authors in relation to the publication of this work.

References

[1] Nicholas E, Carl L, and Felix F. Beta-normal distribution and its applications, *Communications in Statistics-Theory and methods,* 31(4) (2002) 497-512.

[2] Gauss MC, Edwin MMO, and Daniel CC da C. The exponentiated generalized class of distributions, *Journal of Data Science*, 11(1) (2013) 1-27.

[3] Mohammed E and Amal H. Exponentiated weibull weibull distribution: Statistical properties and applications, *Gazi University Journal of Science,* 32(2) (2019) 616-635.

[4] Olanrewaju IS and Kazeem AA. On the beta-nakagami distribution, *Progress in Applied Mathematics,* 5(1) (2013) 49-58.

[5] Marcelo B., Rodrigo B.S., Gauss M.C., The weibull-g family of probability distributions, *Journal of Data Science*, 12(1) (2014) 53-68.

[6] Oguntunde PE, Adejumo AO, and Owoloko EA. Exponential Inverse Exponential (EIE) Distribution with Applications to Lifetime Data, *Asian Journal of Scientific Research,* 10 (2017) 169-177.

[7] Laba H and Subrata C. Beta generated kumaraswamy-g and other new families of distributions. *arXiv preprint arXiv (*2016*) 1603.00634.*

[8] Gauss MC, Morad A, and Edwin MMO. The Exponentiated Half-Logistic Family of Distributions: Properties and Applications, *Journal of Probability and Statistics*, vol. 2014 (2014) ID 864396, 21. Available at: https://doi.org/10.1155/2014/864396.

[9] Amal SH and Saeed EH. A new family of additive weibull-generated distributions, *International Journal of Mathematics And its Applications,* 4(2) (2016) 151–164.

[10] Rodrigo RP, Gauss MC, Clarice GBD, Edwin MMO, and Saralees N. The new class of kummer beta generalized distributions, *SORT-Statistics and Operations Research Transactions,* 36(2) (2012) 153-180.

[11] Saliou D and Christophe C. The generalized odd inverted exponential-g family of distributions: properties and applications, *Eurasian Bulletin of Mathematics (ISSN: 2687-5632),* 2(3) (2019) 86-110.

[12] Ahmed Z.A., Gauss C., Farrukh J, Mohamed E, and Mohamed N. The marshall-olkin odd burr iii-g family of distributions: *Theory, estimation and applications*, Available at: https://hal.archives-ouvertes.fr/hal-02376067. Retrieved November 22, 2019.

[13] Bistoon H, Mahmoud A, and Morad A. The generalized odd gamma-g family of distributions: properties and applications, *Austrian Journal of Statistics*, 47(2) (2018) 69-89.

[14] Abdullahi, İ , Job, O . A new family of odd generalized Nakagami (Nak-G) distributions, *Turkish Journal of Science* , 5 (2) (2020) 85-101.

[15] Zenga, M. Inequality curve and inequality index based on the ratios between lower and upper arithmetic means, *Statistica e Applicazioni* 4, (2007) 3–27.

[16] George C., Roger L.B., Statistical inference, 2nd ed. Australia ; Pacific Grove, CA : Thomson Learning, (2002).

[17] Arslan MN, Muhammad HT, Christophe C, Farrukh J, and Akbar MAS. The odds generalized gamma-g family of distributions: Properties, regressions and applications, *Statistica*, 80(1) (2020) 3-38.

[18] Oguntunde PE, Balogun OS, Okagbue HI, and Bishop SA. The weibull exponential distribution: Its properties and applications, *Journal of Applied Sciences*, 15(11) (2015) 1305-1311.

[19] Michele DN and Padgett WJ. A bootstrap control chart for weibull percentiles, *Quality and reliability engineering international*, 22(2) (2006) 141-151.

[20] Gauss MC and Artur JL. The β-birnbaum-saunders distribution: an improved distribution for fatigue life modeling, *Computational Statistics & Data Analysis*, 55(3) (2011) 1445-1461.

[21] Amal HS, Mohammed AE, and Mohammed S. Type ii half logistic family of distributions with applications, *Pakistan Journal of Statistics and Operation Research, (*2017*)* 245-264.

[22] Amal SH,Abd-Elfattah AM, and Asmaa HM. The complementary exponentiated inverted weibull power series family of distributions and its applications, *Journal of Advances in Mathematics and Computer Science,* (2016) 1-20.

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