The fundamental frequencies of longitudinally vibrating rods carrying tip mass and transversally vibrating beams carrying tip mass by using several methods

Aydın DEMİR [1, *] (iD)

[1]*Sivas Cumhuriyet University, Engineering Faculty, Department of Mechanical Engineering, TR-58140 Sivas/ TURKEY*

Abstract

The present paper is concerned with the determination of the frequency equation and sensitivity of the eigenfrequencies of a fixed-free longitudinally vibrating rod and transversally vibrating beam carrying a tip mass by using several methods. First, the exact frequency equations of the such systems are established, and then approximate formulas are given for the fundamental frequency using several methods which contain the equivalent system, Rayleigh quotient, Dunkerley's formula and continuous system model. The applicability and proximity of these methods versus exact solutions reviewed. The results are compared in a wide range of relevant parameters to give a clear idea about the validity of the proposed formulas. These new derived equations can be very useful for a design engineer who is interested in the eigencharacteristics of similar systems and their sensitivity.

1. Introduction

Rods and beams especially carrying tip mass systems are often used as approximation models for a variety of structural and machine elements. Hence, we frequently face the task of determining the natural frequencies of such systems. Naturally, the governing equations of longitudinal vibrations are simpler than that for axial vibrations. However, axial vibrations are of greater importance in practice, because the natural frequencies in flexure of a particular beam tend to be considerably lower than those in extension and torsion.

The eigenanalysis problem of rods with tip mass is a common subject of interest, also treated in textbooks [1, 2]. Flexural vibrations of uniform beams for different boundary conditions are studied in textbooks [3, 4]. There are many publications in the literature on vibrations of rods and beams for various boundary conditions in different configurations. The dynamics of longitudinal and transversal vibrations has been a subject of many research reports for many years. Examination of the existing literature shows that the solution of the frequency equations of rods and beams carrying point or heavy masses has attracted the interest of many investigators. Gürgöze [5] investigated the frequency equation and sensitivity of the eigenfrequencies of a fixed-free longitudinally

vibrating rod carrying a tip mass. Turhan [6] studied on the effect of a cross-section discontinuity on the eigencharacteristics of longitudinally vibrating rods. Gürgöze and Erol [7] reviewed the establishment of two methods for computing the eigencharacteristics of a continuous rod, carrying a tip mass, consisting of several parts having different physical parameters and subjected to external viscous damping. Lin and Chang [8] examined the longitudinal free vibrations of a system in which two rods are coupled by multi-spring-mass devices. Gu and Cheng [9] studied the dynamic response of a high-speed spindle subject to a moving mass. There are a number of studies [10-18] dealing with the problem of transverse vibrations of beams carrying a tip mass or point masses or concentrated masses using analytical and various numerical approaches. Chang [10] presented a comprehensive study on the lateral vibration of a simply supported beam carrying a concentrated mass at the center of beam. In the case of selecting the appropriate parameters, Turhan [11] proposed Rayleigh approximations versus exact solutions for finding the fundamental frequency of beams carrying a point mass. Low [12] used a modified Dunkerley formula for eigenfrequencies of Euler-Bernoulli beams carrying concentrated masses. Li [13] proposed a new exact approach for free vibration analysis of a multi-step beam with an arbitrary number of crack and

concentrated masses. Kirk and Wiedemann [14] investigated the natural frequencies and mode shapes of a free-free beam with large end masses. Özkaya [15] studied non-linear transverse vibrations of an Euler–Bernoulli beam carrying concentrated masses. Low [16] presented an eigenanalysis and the Rayleigh's estimation for a frequency analysis of a beam carrying a concentrated mass at an arbitrary locations. Banerjee [17] used the dynamic stiffness approach method for exact free vibration analysis of beams carrying spring-mass systems. Li et all. [18] studied free bending vibration of a Rayleigh cantilever with arbitrary axial loading and tip mass.

In recent years, studies on the longitudinal and transversal vibrations of the beams carrying tip mass have also been encountered. Matt and Frederico [19] proposed a new simulation of the transverse vibrations of a cantilever beam with an eccentric tip mass in the axial direction using integral transforms. Labędzki, Pawlikowski and Radowicz [20] used fractional rheological model for transverse vibration of a cantilever beam under base excitation. Şakar [21] examined the effect of axial force on the free vibration of an Euler-Bernoulli beam carrying a number of various concentrated elements. Transverse vibration of Euler beam for different situations are studied in textbook [22].

In this paper, the validity and applicability of several methods applied to uniform rods and beams carrying point mass has been investigated. The obtained results have been compared to with each other and the literature. The corresponding exact frequency equation is also given for each case and the results are compared in a broad range of the relevant parameters so that a clear idea on the presented methods. It is found that the methods such as Rayleigh quotient, Dunkerley's formula and continuous system model can generally yield good approximation and high accuracy if compared with the results associated to the eigenanalysis. These proposed methods are computationally efficient and give very close to exact results. For this reason, these methods are highly

recommended for uniform rod and beams carrying point mass. At the same time, these methods can be extended to complex structural systems and the obtained results can be also very useful for the design engineers who are working in the dynamical behaviour of such systems. There is not enough study in the literature about the use of these approximate solution methods in such systems. Therefore, this study also gives an idea about the superiority and reliability of these approximate methods used.

2. Theory and Formulation

2.1. Longitudinal vibrations of rods carrying tip mass

The uniform rod carrying tip mass is shown in Figure 1. It is essentially an longitudinally vibrating fixed-free rod of axial rigidity EA and mass per unit length m carrying a tip mass M. The exact frequency equation of the system described above must derive in order to determine eigenfrequencies. It is well known that the longitudinal vibrations of a uniform elastic rod are governed by the partial differential equation [2]

$$EA\, u'' = \rho\, A\, \ddot{u} \tag{1}$$

where ρ mass density, overdots and primes denote partial derivatives with respect to time t and x, u(x,t) denotes the longitudinal displacement at point x and time t.
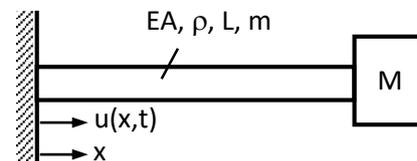

Figure 1. Rod carrying tip mass system.

Assuming harmonic motion of the form

$$u(x,\, t) = y(x)\, \cos(\omega t - \varepsilon) \tag{2}$$

and obtains the general solution

$$u(x,\, t) = (A_1 \sin(\omega t) + A_2 \cos(\omega t))\ (B_1 \sin(\frac{\omega}{c} x) + B_2 \cos(\frac{\omega}{c} x)) \tag{3}$$

where A_i and B_i i=1,2 are arbitrary integration constants to be evaluated from the boundary conditions, constant $c = \sqrt{\dfrac{E}{\rho}}$ expression is also known as the wave propagation velocity and ω eigenfrequency is defined by

$$\omega = \lambda\, \frac{c}{L} \tag{4}$$

Given the shape of the system shown in the Figure 1, boundary conditions becomes

$$u(0, t) = 0$$

$$EA\left. u' \right|_{x=L} = - M\left. \ddot{u} \right|_{x=L} \tag{5}$$

It can be shown after same algebraic manipulations, the eigenequation results in the following simple form

$$\tan\lambda = \frac{1}{\mu\,\lambda} \tag{6}$$

where the following non-dimensional parameters are introduced:

$$\lambda = \frac{\omega\,L}{c} \quad, \quad \mu = \frac{M}{m} \tag{7}$$

μ and λ non-dimensional parameters refer to ratio of masses and dimensionless frequency, respectively. The roots of transandantal equation (6) give dimensionless frequency parameter λ and hence by considering equation (4) the eigenfrequencies of the system shown in Figure 1. The approximate solution methods of the frequency equations of uniform rods and beams carrying tip mass will be handled in detail later.

2.2 Transverse vibrations of beams carrying tip mass

Consider an Euler-Bernouilli beam carrying a tip mass M *(*Figure 2). Transverse vibration of the beam is represented by the partial differential equation given below.
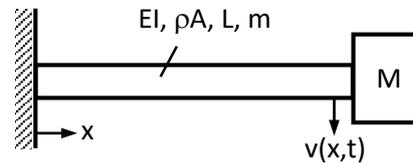
Figure 2. Beam carrying tip mass system.

$$EIv^{IV} + \rho A\ddot{v} = 0 \tag{8}$$

where EI flexural rigiditiy, ρ mass density, A sectional area, overdots and primes denote partial derivatives with respect to time t and x, v(x,t) denotes the transverse displacement at point x and time t. Assuming harmonic motion of the form

$$v(x, t) = y(x)\cos(\omega\,t - \varepsilon) \tag{9}$$

one obtains the general solution of the motion

$$v(x, t) = [A\cos(\frac{\lambda}{L}x) + B\sin(\frac{\lambda}{L}x) + C\cosh(\frac{\lambda}{L}x) + D\sinh(\frac{\lambda}{L}x)]\cos(\omega\,t - \varepsilon) \tag{10}$$

where λ and ξ are defined by

$$\xi = \frac{x}{L} \quad, \quad \lambda^4 = \frac{\rho\,A\,L^4}{E\,I}\omega^2 \tag{11}$$

In this case,

$$y(\xi) = A\cos(\lambda\,\xi) + B\sin(\lambda\,\xi) + C\cosh(\lambda\,\xi) + D\sinh(\lambda\,\xi) \tag{12}$$

where A, B, C and D are arbitrary integration constants that can be found in the boundary conditions. Boundary conditions for transverse vibration of the beam shown in the Figure 2 becomes

$$v(0, t) = 0$$
$$v'(0, t) = 0$$
$$M\ddot{v}(L, t) = EI\,v'''(L, t)$$
$$v''(L, t) = 0 \tag{13}$$

By applying boundary conditions, frequency equation is obtained as follows

$$1 + \cos\lambda\cosh\lambda + \mu\lambda(\cos\lambda\sinh\lambda - \sin\lambda\cosh\lambda) = 0 \tag{14}$$

where $\mu = \dfrac{M}{m}$ non-dimensional parameters refer to ratio of masses and m is mass per unit length ($m = \rho\, A\, L$) In the same way, the roots of transandantal equation (14) give dimensionless frequency parameter λ and hence by considering equation (11) the eigenfrequencies of the system.

3. Case study and numerical solutions

In this section, exact frequency equations and approximate frequency equations obtained by several different methods will be derived for rods/beams carrying tip mass systems and their results be compared. The results are presented in accordance with

some appropriate parameters and it is possible to get an idea about the validity of approximate solutions.

3.1. The fundamental frequencies of longitudinal vibrating of rods carrying tip mass

For the system shown in Figure 1, equation (6) yields dimensionless frequency parameter λ. Approximate frequency equations of the same system were obtained by using such as equivalent system, Rayleigh quotient, Dunkerley's formula and continuous system model. As a result of some corrections and mathematical operations, frequency equations of the longitudinal vibrations of the system given in Figure 1 are obtained by these methods. The results are given by the following equations.
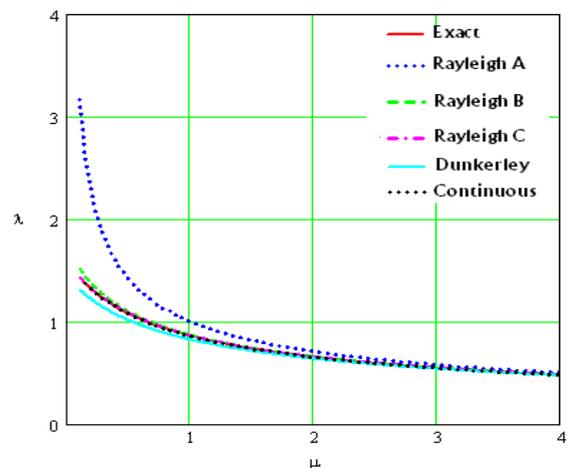
Rayleigh quotient A　　　　:　　$\lambda = \sqrt{\dfrac{1}{\mu}}$　　　　　　　　　　　　(15)

Rayleigh quotient B　　　　:　　$\lambda = \sqrt{\dfrac{1}{\mu + \dfrac{1}{3}}}$　　　　　　　　　(16)

Rayleigh quotient C　　　　:　　$\lambda = \sqrt{5}\,\sqrt{\dfrac{3\mu^2 + 3\mu + 1}{15\mu^3 + 20\mu^2 + 10\mu + 2}}$　　　(17)

Dunkerley's Formula　　　　:　　$\lambda = \sqrt{\dfrac{1}{\dfrac{1}{(\frac{\pi}{2})^2} + \dfrac{1}{(\frac{3\pi}{2})^2} + \dfrac{1}{(\frac{5\pi}{2})^2} + \dfrac{1}{(\frac{7\pi}{2})^2} + \dfrac{1}{(\frac{9\pi}{2})^2} + \mu}}$　　　(18)

Continuous system model　　:　　$\dfrac{\beta}{\mu} + (\dfrac{\beta}{\mu} - 1)\sin\lambda\left[\cos\lambda\,\tan((1+\beta)\lambda) - \sin\lambda_1\right]$　　　(19)

In equation (19), $\beta = \dfrac{L_2}{L}$ (length of mass M / length of the beam). Figure 3 shows dimensionless frequency parameter λ according to ratio of masses by theset methods for longitudinal vibrating of rods carrying tip mass. If Fig. 3 is examined carefully, it is seen that the dimensionless frequency parameter λ decreases with the ratio of masses. However, the dimensionless frequency parameter obtained by the equation (15) don't give very good results according to the exact solution given by the equation (6). Therefore, other methods except this method are considered in the calculation of the relative error. The relative error is the % error calculated according to the exact solution given by the equation (6).

Figure 3. Dimensionless frequency parameter λ according to ratio of masses by different methods for longitudinal vibrating of rods carrying tip mass.

Figure 4 shows relative errors in dimensionless frequency parameter λ according to ratio of masses by different methods for longitudinal vibrating of rods carrying tip mass. When Fig. 4 is examined, it is seen that the relative error in dimensionless frequency parameter of Dunkerley's Formula is slightly high, Rayleigh quotient B is slightly less error and Rayleigh quotient C and continuous system model give very close results to the exact solution. Especially, since continuous system model is quite close to the exact solution, it is seen as the least error method. Therefore, continuous system model is discussed in more detail. Table 1 and Fig. 5 give the results of this model.
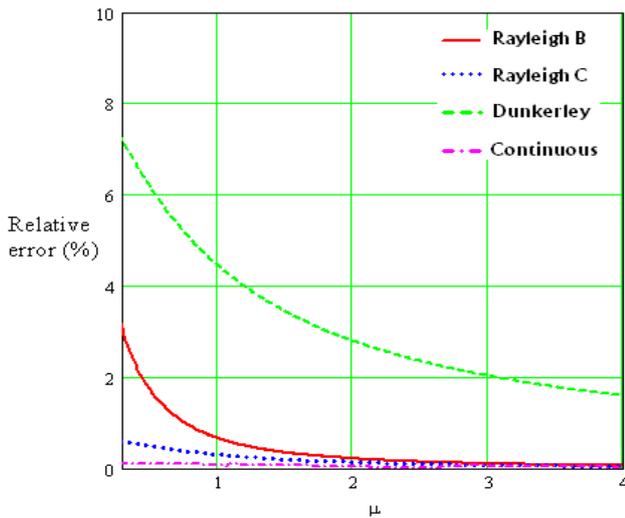

Figure 4. Relative errors in dimensionless frequency parameter λ according to ratio of masses by different methods for longitudinal vibrating of rods carrying tip mass.

In the continuous system model, dimensionless frequency parameter λ by given by the equation (19) varies according to ratio of masses and ratio of lengths. Table 1 shows some values of dimensionless frequency parameter λ by continuous system model for longitudinal vibrating of rods carrying tip mass. Similarly, Fig. 5. displays dimensionless frequency parameter λ according to ratio of masses and ratio of length in continuous system model for longitudinal vibrating of rods carrying tip mass. An inspection of Table 1 and Fig. 5 show that as the ratio of masses and the ratio of lengths increases, the dimensionless frequency parameters decrease. With the increase in the ratio of length, there is less decrease in dimensionless frequency parameters. The higher the ratio of masses, the greater the decrease in dimensionless frequency parameters.

Table 1. Dimensionless frequency parameter λ by continuous system model for longitudinal ibrating of rods carrying tip mass.

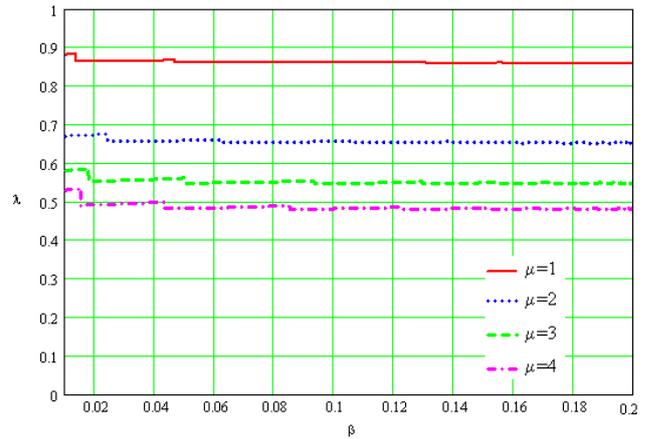| μ \ β	0.05	0.10	0.15	0.20
1.0	0.861	0.860	0.859	0.857
1.5	0.738	0.737	0.735	0.734
2.0	0.659	0.657	0.653	0.653
2.5	0.603	0.593	0.594	0.595
3.0	0.561	0.548	0.549	0.546
3.5	0.512	0.511	0.514	0.510
4.0	0.483	0.482	0.485	0.480

Figure 5. Dimensionless frequency parameter λ according to ratio of masses and ratio of length in continuous system model for longitudinal vibrating of rods carrying tip mass.

3.2. The fundamental frequencies of transversal vibrating of beams carrying tip mass

For the system shown in Figure 2, equation (14) yields the exact solution of dimensionless frequency parameter λ. Approximate frequency equations of these system were obtained by using such as equivalent system, Dunkerley's formula and stepped beam model (continuous system model). The formulas calculated by the equivalent system and Dunkerley formula are given directly below, and the results of the stepped beam model are explained in detail below.

Equivalent System:
$$\lambda = \sqrt{\dfrac{3}{\sqrt[4]{\dfrac{33}{140} + \mu}}} \tag{20}$$

Dunkerley's Formula:

$$\frac{1}{\lambda^4} = \frac{1}{\lambda_{11}^{\,4}} + \frac{1}{\lambda_{22}^{\,4}} + \frac{1}{\lambda_{33}^{\,4}} + \frac{1}{\lambda_{44}^{\,4}} + \frac{\mu}{3} \tag{21}$$

where $\lambda_{11} = 1.8751, \quad \lambda_{22} = 4.6941,$

$\lambda_{33} = 7.8548, \quad \lambda_{44} = 10.9955$

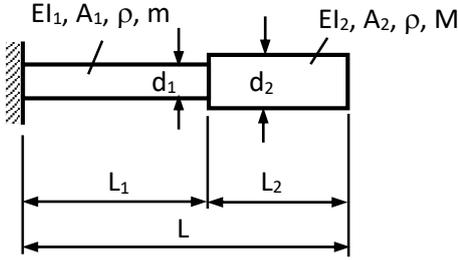### 3.2.1 Stepped beam model (continuous system model)

Figure 6. Stepped beam model of the beam carrying tip mass system.

Consider an Euler-Bernouilli beam carrying a tip mass M (Figure 6). Let the transversal motions of the beam points at the left and right of M be represented by y_1 (x,t) and y_2 (x,t). Both y_1 and y_2 have to obey the partial diferential equation (8). Using the dimensionless parameters given the equation (22) and with the assumption of harmonic motion,

$$\alpha = \frac{L_1}{L}, \quad \beta = \frac{L_2}{L}, \quad \gamma = \frac{A_2}{A_1} = \frac{d_2^2}{d_1^2}$$

$$\lambda_1^4 = \frac{\rho A_1 L_1^4}{E I_1} \omega^2, \quad \lambda_2^4 = \frac{\rho A_2 L_2^4}{E I_2} \omega^2,$$

$$\frac{\lambda_1^4}{\lambda_2^4} = \frac{A_1 L_1^4}{I_1} \cdot \frac{I_2}{A_2 L_2^4} = \gamma \left(\frac{\alpha}{\beta}\right)^4, \tag{22}$$

$$\frac{\lambda_1}{\lambda_2} = \sqrt[4]{\gamma} \ \frac{\alpha}{1-\alpha} = \eta$$

$$\mu = \frac{M}{m} = \frac{\rho A_2 L_2}{\rho A_1 L_1} = \gamma \frac{\beta}{\alpha}, \quad \frac{I_2}{I_1} = \frac{d_2^4}{d_1^4} = \gamma^2$$

one obtains the general solutions for the space dependence of the motion.

$$y_1(x) = A_1 \cos(\lambda_1 \frac{x}{L}) + B_1 \sin(\lambda_1 \frac{x}{L}) + C_1 \cosh(\lambda_1 \frac{x}{L}) + D_1 \sinh(\lambda_1 \frac{x}{L}) \tag{23}$$

$$y_2(x) = A_2 \cos(\lambda_2 \frac{x}{L}) + B_2 \sin(\lambda_2 \frac{x}{L}) + C_2 \cosh(\lambda_2 \frac{x}{L}) + D_2 \sinh(\lambda_2 \frac{x}{L}) \tag{24}$$

$$y_1(0) = 0$$

$$y_1'(0) = 0$$

$$y_1(L_1) = y_2(L_1)$$

$$y_1'(L_1) = y_2'(L_1) \tag{25}$$

$$I_1 y_1''(L_1) = I_2 y_2''(L_1)$$

$$I_1 y_1'''(L_1) = I_2 y_2'''(L_1)$$

$$y_2''(L) = 0$$

$$y_2'''(L) = 0$$

The following matrix was obtained by applying the boundary conditions given equation (22). The roots of the determinant of this matrix give dimensionless frequency parameter λ of the problem. Because this calculation is very difficult, the solutions have been found with numerical methods. Table 2 gives dimensionless frequency parameter λ calculated by the four methods for transversal vibrating of beams carrying tip mass according to the ratio of masses.

$$\left(\begin{array}{cccccccc}
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
\cos(\lambda 1 \cdot \alpha) & \sin(\lambda 1 \cdot \alpha) & \cosh(\lambda 1 \cdot \alpha) & \sin(\lambda 1 \cdot \alpha) & -\cos(\lambda 1 \cdot \alpha) & -\sin(\lambda 1 \cdot \alpha) & -\cosh(\lambda 1 \cdot \alpha) & -\sinh(\lambda 1 \cdot \alpha) \\
-\eta \cdot \sin(\lambda 1 \cdot \alpha) & \eta \cdot \cos(\lambda 1 \cdot \alpha) & \eta \cdot \sinh(\lambda 1 \cdot \alpha) & \eta \cdot \cosh(\lambda 1 \cdot \alpha) & \sin(\lambda 2 \cdot \alpha) & -\cos(\lambda 2 \cdot \alpha) & -\sinh(\lambda 2 \cdot \alpha) & -\cosh(\lambda 2 \cdot \alpha) \\
-\dfrac{\eta^2}{\gamma^2} \cdot \cos(\lambda 1 \cdot \alpha) & -\dfrac{\eta^2}{\gamma^2} \cdot \sin(\lambda 1 \cdot \alpha) & \dfrac{\eta^2}{\gamma^2} \cdot \cosh(\lambda 1 \cdot \alpha) & \dfrac{\eta^2}{\gamma^2} \cdot \sinh(\lambda 1 \cdot \alpha) & \cos(\lambda 2 \cdot \alpha) & \sin(\lambda 2 \cdot \alpha) & -\cosh(\lambda 2 \cdot \alpha) & -\sinh(\lambda 2 \cdot \alpha) \\
\dfrac{\eta^3}{\gamma^2} \cdot \sin(\lambda 1 \cdot \alpha) & -\dfrac{\eta^3}{\gamma^2} \cdot \cos(\lambda 1 \cdot \alpha) & \dfrac{\eta^3}{\gamma^2} \cdot \sinh(\lambda 1 \cdot \alpha) & \dfrac{\eta^3}{\gamma^2} \cdot \cosh(\lambda 1 \cdot \alpha) & -\sin(\lambda 2 \cdot \alpha) & \cos(\lambda 2 \cdot \alpha) & -\sinh(\lambda 2 \cdot \alpha) & -\cosh(\lambda 2 \cdot \alpha) \\
0 & 0 & 0 & 0 & -\cos(\lambda 2) & -\sin(\lambda 2) & \cosh(\lambda 2) & \sinh(\lambda 2) \\
0 & 0 & 0 & 0 & \sin(\lambda 2) & -\cos(\lambda 2) & \sinh(\lambda 2) & \cosh(\lambda 2)
\end{array}\right) \tag{26}$$

Table 2. Dimensionless frequency parameter λ with the various methods for transversal vibrating of beams carrying tip mass according to the ratio of masses.

Methods \ μ	0.5	1.0	1.5	2.0	2.5	3.0	3.5
Exact solution (Eq. (14))	1.420	1.248	1.146	1.076	1.023	0.981	0.947
Equivalent system (Eq. (20))	1.421	1.248	1.147	1.076	1.023	0.981	0.947
Dunkerley formula (Eq. (21))	1.414	1.245	1.144	1.075	1.022	0.980	0.946
Stepped beam model (Eq. (26))	1.418	1.249	1.145	1.079	1.026	0.981	0.948

In order to check the accuracy of recommended approximate solutions, values numerically calculated from these approximate solutions are compared in Table 2 with the exact values found by solving equation (14) for different values of the parameters μ. It is seen from Table 2 that the dimensionless frequency values obtained by the various methods are very close to each other. In addition, it is also seen from Table 2 that when the ratio of masses (μ) increase, dimensionless frequency parameter values (λ) decrease.
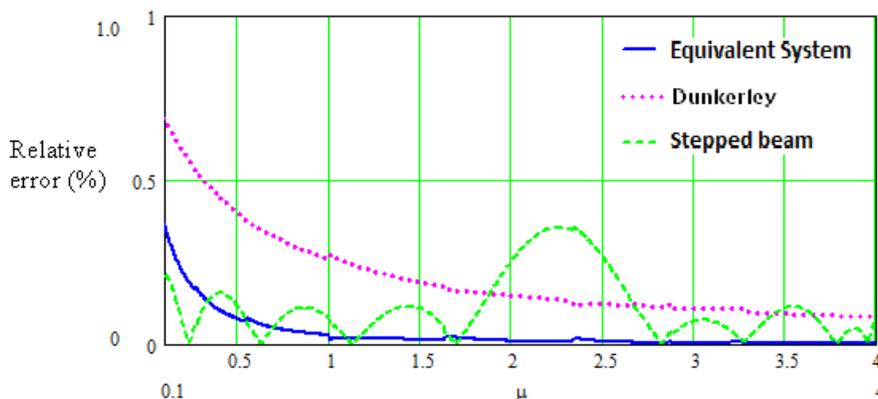

Figure 7. Relative errors in dimensionless frequency parameter λ according to ratio of masses by various methods for transversal vibrating of beams carrying tip mass.

Fig. 7 shows relative errors in dimensionless frequency parameter λ for transversal vibrating of beams carrying tip mass. From carefully review of Fig. 7, it is understood that the relative errors found by all three methods are less than 1%. The relative errors values obtained with the Dunkerley formula appear to be relatively little bit high. It can be said that the relative errors values obtained by the equivalent system are very stable and quite low.

4. Conclusions

In this study, several approximate solution methods have presented on longitudinal vibrations of uniform rods carrying tip mass and transverse vibrations of uniform beams carrying tip mass at the end. First, the exact frequency equations of the systems discussed were established and formulas for the dimemsionless frequency parameters of these systems were obtained by methods such as equivalent system, Rayleigh coefficient, Dunkerley formula and continuous system model. In addition to the approximate methods discussed, relative error percentages in case of using these methods are also discussed and presented in tables and graphs depending on the appropriate parameters.

It is shown that the approximate solution methods can reliably predict the fundamental frequencies for longitudinal vibrating of uniform rods carrying tip mass and transversal vibrating of uniform beams carrying tip mass, provided that they are used in proper parameter ranges. The Dunkerley formula and some Rayleigh approximations yielded relatively low errors in both longitudinal vibrating of uniform rods carrying tip mass and transversal vibrating of uniform beams carrying tip mass, however, the continuous system model and equivalent system yielded very close to almost complete solutions.

By using the frequency equations given in this study, the high frequencies of the related systems can be easily calculated if desired. Also, the user has the opportunity to make his own decision about the adequacy of the recommended methods. At the same times, these obtained results can be also very useful for the design engineers who are working in the dynamical behaviour of such systems. It can also be very useful for engineers who want to find the fundamental frequencies of such systems quickly in terms of showing which method will give close results to the exact solution.

Conflicts of interest

The authors state that there is no conflict of interests.

References

[1] Humar J. L., Dynamics of Structures, 3rd ed. NJ: Prentice-Hall, Englewood Cliffs, (1990).

[2] Meirovitch L., Analytical Methods in Vibrations 1st ed. London: Collier-Macmillan Limited, (1967).

[3] Timoshenko S., Young D. H., Weaver W. JR., Vibration Problems in Engineering, 4th ed. New York: Wiley, (1974).

[4] Tse F. S., Morse I. E., Hinkle R. T., Mechanical Vibrations, Theory and Applications, 2nd ed. Boston MA: Allyn and Bacon, (1978).

[5] Gürgöze M., On the eigenfrequencies of longitudinally vibrating rods carrying a tip mass and spring–mass in-span, *Journal of Sound and Vibration,* 216(2) (1998) 295-308.

[6] Turhan Ö., On the eigencharacteristics of longitudinally vibratıng rods with a cross-section discontinuity, *Journal of Sound and Vibration,* 248(1) (2001) 167-177.

[7] Gürgöze M., Erol H., On the eigencharacteristics of multi-step rods carrying a tip mass subjected to non-homogeneous external viscous damping, *Journal of Sound and Vibration,* 267 (2003) 355-365.

[8] Lin H.P., Chang S.C., Free vibrations of two rods connected by multi-spring–mass systems, *Journal of Sound and Vibration*, 330(11) (2011) 2509-2519.

[9] Gu U. C., Cheng C. C., Vibration analysis of a high-speed spindle under the action of a moving mass, *Journal of Sound and Vibration,* 278 (2004) 1131-1146.

[10] Chang C. H., Free vibration of a simply supported beam carrying a rigid mass at the middle, *Journal of Sound and Vibration,* 237(4) (2000) 733-744.

[11] Turhan Ö., On the fundamental frequency of beams carrying a point mass: Rayleigh approximations versus exact solutions, *Journal of Sound and Vibration,* 230(2) (2000) 449-459.

[12] Low K. H., A modified Dunkerley formula for eigenfrequencies of beams carrying concentrated masses, *International Journal of Mechanical Sciences,* 42 (2000) 1287-1305.

[13] Li Q. S., Vibratory characteristics of multi-step beams with an arbitrary number of crack and concentrated masses, *Applied Acoustics,* 62 (2001) 691-706.

[14] Kirk C. L., Wiedemann S. M., Natural frequencies and mode shapes of a free-free beam with large end masses, *Journal of Sound and Vibration,* 254(5) (2002) 939-949.

[15] Özkaya E., Non-linear transverse vibrations of a simply supported beam carrying concentrated masses, *Journal of Sound and Vibration,* 257(3) (2002) 413-424.

[16] Low K. H., Natural frequencies of a beam–mass system in transverse vibration: Rayleigh estimation versus eigenanalysis solutions, *International Journal of Mechanical Sciences,* 45 (6-7) (2003) 981–993.

[17] Banerjee J. R., Free vibration of beams carrying spring-mass systems -A dynamic stiffness approach, *Computers and Structures*, 104–105 (2012) 21–26.

[18] Li X. F., Tang A.Y., Xi L.Y., Vibration of a Rayleigh cantilever beam with axial force and tip mass, *Journal of Constructional Steel Research,* 80 (2013) 15-22.

[19] Matt Carlos Frederico T., Simulation of the transverse vibrations of a cantilever beam with an eccentric tip mass in the axial direction using integral transforms, *Applied Mathematical Modelling*, 37(22) (2013) 9338–9354.

[20] Labędzki P., Pawlikowski R., Radowicz A., Transverse vibration of a cantilever beam under base excitation using fractional rheological model, AIP Conference, (2018).

[21] Şakar G., The effect of axial force on the free vibration of an Euler-Bernoulli beam carrying a number of various concentrated elements, *Shock and Vibration*, 20(3) (2013) 357–367.

[22] Edwards P. , Transverse Vibration of Euler Beam, (2018) 247–266.