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Stability analysis of a discrete-time prey-predator population model with immigration

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Abstract

In this paper, a discrete-time prey-predator population model with immigration which is obtained by implementing forward Euler's scheme has been considered. The existence of fixed points of the presented model has been investigated. Moreover, the stability analysis of the fixed points of the population model has been examined and the topological classification of the fixed points of the model has been made. Moreover, the OGY feedback control method is to implement to controlchaos caused by the Flip bifurcation. Finally, Flip bifurcation,chaos control strategy, and asymptotic stability of the only positive fixed point are verifiedwith the help of numerical simulations.

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1. Introduction

Some interdisciplinary studies have been carried out to understand and explain natural events recently. These studies led to the development of new fields such as mathematical biology and biophysics. First of all, to understand the events in nature, a mathematical model that reflects these events is needed.

The species in nature interact with each other. The population density of these interacting species also affects the population density of other living species. Therefore, population models are among the most striking issues for many ecologists, mathematicians, and biologists recently.

Prey-predator models are among the most common population models that involve the interaction of the two species. The predator feeds on prey. The prey also feeds on other foods. Fox and rabbits, sharks, and fish are examples of the prey-predator species.

Lotka [1] and Volterra [2] introduced a predator-prey model firstly. In this model, the prey consumption rate by a predator is consideredto be directly proportional to the abundance of the prey. This indicates thatthe predator was fed to a limited extent by the amount of prey in the environment. While this is realistic in environments with low hunting density, it is anabsolutely unrealistic assumption in high hunting densities. In later processes,Lotka-Volterra model was arranged in different ways.

The immigration factor is an effect that makes the predator-prey population model more realistic [3-7]. So manyresearches studied the role of immigration and its impact on population dynamics [8-12]. It wasinvestigated the existence and uniqueness of limitcycles in predator-prey models [4], alsothe local and global stability of fixed-rate migration-effective, delayed predator-prey system [3]. They showed that the existence of the global Hopf bifurcation. Thara at allanalyzed the asymptotic stability of prey-predator systems, which was formedby adding individual immigration factors to the prey and predator population inthe classical Lotka-Volterra system [13]. Furthermore, many ecological concepts such as diffusion, functional responses, time delays and Allee effect have beenadded to the predator-prey model to gain a more accurate description and beter understanding [14-19].

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In the last few centuries, chaos and unusual behavior of non-linear discrete dynamical system attracted the attention of scientists. Chaotic behavior examined in almost every field, such as chemistry, physics, ecology, biology, chemical engineering, telecommunications etc.Moreover, the practical methods related to chaos control can be implemented in various areas such as communications, physics laboratories, biochemistry, turbulence, and cardiology.Chaos is the general name for non-linear dynamical systems that behave noise-like. Chaos is indecomposable, is highly dependent on the initial condition, and consists of a large number of periodic points and orbits. Because of this, the solution of a chaotic system is difficult to predict, which calls for a way to control it. The control algorithm of Ott, Grebogi, and Yorke (OGY, [20]) manages to do this.The proposed methodology is known as the OGY method.OGY is a discrete control algorithm, perturbing the system at discrete moments in time.It is well known that existence or non-existence of chaotic solutions for a dynamical system is determined by calculating Lyapunov exponent. Generally, a positive lyapunov exponentis considered to be one of the characteristics which imply the existence of chaos. That is, when the system has a positive largest Lyapunov exponent, then the system exhibits chaotic dynamics [21].

In [19], the author has considered the following continuous-time model with Allee effect on prey population:

$$\begin{cases} \frac{dx(t)}{dt} = x(t)\left(b_1 - a_{11}x(t)\right)\frac{x(t)}{\beta + x(t)} + a_{12}x(t)y(t) \\ \frac{dy(t)}{dt} = y(t)(b_2 - a_{22}y(t)) \end{cases} \tag{1}$$

where $x(t)$and $y(t)$ represent population densities of prey and predator at time *t*, respectively; $b_i, i = 1,2$ are the intrinsic growth rate of the prey x and predator y; $\frac{b_i}{a_{ii}}, i = 1,2$ is the carrying capacity of prey and predator, respectively; a_{12} reflects the efficiency of every single population y that can contribute to population x. The term $\frac{x(t)}{\beta + x(t)}$ is Allee effect here. The author investigated the local and global property of the fixed point of the system (1) with the Allee effect on prey population [1].

In [22],Kangalgil has considered discrete- time version of the system (1) with an Allee effect on predator species by applying the forward Euler scheme as follows:

$$\begin{cases} x_{t+1} = x_t + \delta(x_t(b_1 - a_{11}x_t) + a_{12}x(t)y(t) \\ y_{t+1} = y_t + \delta\left(y_t\left(b_2\frac{y(t)}{m + y(t)} - a_{22}y(t)\right)\right) \end{cases} \tag{2}$$

where $\delta > 0$ is the step size, $x(t)$ and $y(t)$ represent population densities of prey and predator at time *t*, respectively. All parameters are positive constants. The term $f(\mathrm{y}) = \frac{y}{m + y}$ is called the Allee effect where m is Allee constant [17-19].The author investigated dynamical behavior of the system (2) with Allee effect on predator population at the coexistence fixed point and showed that the system (2) undergoes Flip bifurcation.

In this study discrete-time version of the system (1) with immigration instead of Allee effect on predator has been investigated

$$\begin{cases} \frac{dx(t)}{dt} = x(t)\left(b_1 - a_{11}x(t)\right) + a_{12}x(t)y(t)) \\ \frac{dy(t)}{dt} = y(t)\left(b_2 - a_{22}y(t)\right) + s \end{cases} \tag{3}$$

Where$s > 0$is the immigration parameter. If we apply the following Euler scheme:

$$x'(t_0) = x(t_0)\big(b_1 - a_{11}x(t_0)\big) + a_{12}x(t_0)y(t_0)$$

we can write

$$x'(t_0) \approx \frac{x(t_1) - x(t_0)}{t_1 - t_0}.$$

Then at t_1 step,

$$x(t_1) = \delta\left(x(t_0)\big(b_1 - a_{11}x(t_0)\big) + a_{12}x(t_0)y(t_0)\right) + x(t_0)$$

is obtained. Therefore we get

$$x \to x + \delta(x(b_1 - a_{11}x) + a_{12}xy).$$

Similarly

$$y(t_1) \approx \delta\big(y(t_0)(b_2 - a_{22}y(t_0) + s)\big) + y(t_0),$$

$$y \to y + \delta\big(y(b_2 - a_{22}y) + s)\big).$$

So we havethe following system:

$$\begin{cases} x_{t+1} = x_t + \delta(x_t(b_1 - a_{11}x_t) + a_{12}x_t y_t) \\ y_{t+1} = y_t + \delta(y_t(b_2 - a_{22}y_t) + s) \end{cases} \tag{4}$$

In [23], authors have investigated the Flip bifurcation analysis of the system (4) by choosing as a δ bifurcation parameter. They showed that the step size $\delta$for Euler's scheme has strong stability effect on positive-steady state or vice versa. We have seen that there is no chance of Neimark-Sacker bifurcation to occur in the system (4). We present the Flip bifurcation diagrams and Maximum Lyapunov exponent for the system (4) by choosing bifurcation parameter as immigration instead of the step size δ.Also, to control the chaos in the system (4), we study the OGY Feedback control method [24]. Numerical simulations are presented to support obtained theoretical results and to show the complex dynamical behaviors.

2. Existence of The Fixed Points

In this section, we investigate positive fixed points of the system (4) and analyze stability of these fixed points.

Definition2.1A point $(\bar{x}, \bar{y})$is called fixed point of system (4), when it satisfies the following system:

$$\bar{x} = \bar{x} + \delta(\bar{x}(b_1 - a_{11}\bar{x}) + a_{12}\bar{x}\bar{y})$$

$$\bar{y} = \bar{y} + \delta(\bar{y}(b_2 - a_{22}\bar{y}) + s). \tag{5}$$

Definition2.2A matrix

$$J(x,y) = \begin{pmatrix} f_{1x}(x,y) & f_{1y}(x,y) \\ f_{2x}(x,y) & f_{2y}(x,y) \end{pmatrix}$$

is called Jacobian matrix of system (4) at fixed point (x, y), where

$$f_1(x,y) = x + \delta(x(b_1 - a_{11}x) + a_{12}xy)$$

$$f_2(x,y) = y + \delta(y(b_2 - a_{22}y) + s). \tag{6}$$

Definition2.3 An equation

$F(\lambda) = \lambda^2 - izJ(x,y)\lambda + detJ(x,y) = 0$

is called characteristic equation of fixed point (x, y) and$F(\lambda)$ is called characteristic polynomial as well.

Definition2.4 Let $\lambda_1$and λ_2 are the roots of the characteristic polynomial

$$F(\lambda)=\lambda^2 - B\lambda + CB\ , C \epsilon N.$$

Then the fixedpoint of the system (4) is called

1) sink if $|\lambda_1| < 1$ and $|\lambda_2| < 1$,
2) source if $|\lambda_1| > 1$ and $|\lambda_2| > 1$,
3) saddle if $[|\lambda_1| < 1$ and $|\lambda_2| > 1]$ or $[|\lambda_1| > 1$ and $|\lambda_2| < 1]$,
4) non-hyperbolic if $|\lambda_1| = 1$ or $|\lambda_2| = 1$.

Lemma2.1 For all parameter values, the system (4) has four fixed points as follows:

$E_1 = (0, 0)$

$E_2 = (\frac{b_1}{a_{11}}, 0)$

$E_3 = (0, \frac{b_2 + s}{a_{22}})$

$E_4 = \left(\frac{b_1 a_{22} + a_{12} b_2 + a_{12} s}{a_{11} a_{22}}, \frac{b_2 + s}{a_{22}}\right)$.

3. Topological Classification for Fixed Points

In this section we make topological classification for the fixed points of the system (4).

3.1. The fixed point $E_1(0, 0)$:

The Jacobian matrix of the system (4) is obtained as follow:

$$J = \begin{pmatrix} 1 + \delta(b_1 - 2a_{11}x + a_{12}y) & \delta a_{12}x \\ 0 & 1 + \delta b_2 - 2\delta a_{22}y + \delta s \end{pmatrix} \quad (7)$$

If we write $x = 0, y = 0$ in Jacobian matrix (7), then we obtain corresponding matrix and characteristic polynomial for the fixed point E_1 like that:

$$J = \begin{pmatrix} 1 + \delta b_1 & 0 \\ 0 & 1 + \delta b_2 + \delta s \end{pmatrix} \quad (8)$$

$F(\lambda) = \lambda^2 - izJ(0,0)\lambda + detJ(0,0)$

or

$F(\lambda)=\lambda^2 - (1 + \delta b_1 + 1 + \delta b_2 + \delta s)\lambda + (1 + \delta b_2 + \delta s)(1 + \delta b_1)$.

Therefore characteristic values of the system (4) are obtained as follows:

$\lambda_1=1+\delta b_1$,

$\lambda_2 = 1 + \delta b_2 + \delta s$.

Because of $\delta, b_1, b_2$and $s$arepositive then,$|\lambda_1| > 1$and$|\lambda_2| > 1$. According to Definition 2.4,E_1 is the source fixed point of the system (4). So we proved the following theorem:

Theorem 3.1 For all parameter values, the fixed point $E_1$is source and unstable fixed point of the system (4).

3.2. The fixedpoint$E_2\left(\frac{b_1}{a_{11}}, 0\right)$:

Similarly, in Section 3.1, we get Jacobian matrix and characteristic polynomial for the fixed point E_2 like that:

$$J\left(\frac{b_1}{a_{11}}, 0\right) = \begin{pmatrix} 1-\delta b_1 & \frac{\delta a_{12} b_1}{a_{11}} \\ 0 & 1+\delta b_2+\delta s \end{pmatrix} \quad (9)$$

$F(\lambda)= \lambda^2 - izJ(\frac{b_1}{a_{11}}, 0)\lambda+detJ(\frac{b_1}{a_{11}},0)$

or

$F(\lambda)=\lambda^2 - (1-\delta b_1 + 1 + \delta b_2 + \delta s)\lambda+(1+\delta b_2+\delta s)(1-\delta b_1)$.

Therefore characteristic values of the system (4) as follows:

λ_1=1-δb_1,

$\lambda_2 = 1 + \delta b_2 + \delta s$.

Theorem 3.2 For the fixed point $E_2$of the system (4).

- **a)** if $\delta < \frac{2}{b_1}$, then it is source,
- **b)** if$\delta > \frac{2}{b_1}$, then it is saddle.
- **c)** if$\delta = \frac{2}{b_1}$,then it isnon-hyperbolic.

3.3. The fixed point $E_3\left(0, \frac{b_2+s}{a_{22}}\right)$:

Similarly $E_1$and E_2, we obtain Jacobian matrix and characteristic polynomial for the fix point E_3 as follows:

$$J\left(0, \frac{b_2+s}{a_{22}}\right) = \begin{pmatrix} 1+\delta\left(b_1+\frac{a_{12}(b_2+s)}{a_{22}}\right) & 0 \\ 0 & 1-\delta b_2-\delta s \end{pmatrix} \quad (10)$$

or

$$J\left(0, \frac{K}{a_{22}}\right) = \begin{pmatrix} 1+\delta b_1+\delta KR & 0 \\ 0 & 1-\delta K \end{pmatrix}$$

where

$b_2 + s = K, K > 0$

$\frac{a_{12}}{a_{22}} = R, R > 0,$

$F(\lambda)= \lambda^2 - izJ(0, \frac{K}{a_{22}})\lambda+detJ(0, \frac{K}{a_{22}})$

or

$F(\lambda)=\lambda^2 - (1+\delta b_1+\delta KR1-\delta K)\lambda+(1+\delta b_2+\delta KR)(1-\delta K)$.

Therefore characteristic values of the system (4) are obtained as follows:

$\lambda_1 = 1 + \delta b_1 + \delta KR$

$\lambda_2 = 1 - \delta K.$

Theorem 3.3 For the fixed point $E_3$of the system (4),

- **a)** if $\delta < \frac{2}{K}$, then it is saddle,
- **b)** if$\delta > \frac{2}{K}$, then it is source.
- **c)** if$\delta = \frac{2}{K}$,then it isnon-hyperbolic.

3.4. The coexistence fixed point $E_4 = (\frac{b_1 a_{22} + a_{12} b_2 + a_{12} s}{a_{11} a_{22}}, \frac{b_2 + s}{a_{22}})$:

Finally, we obtain Jacobian matrix and characteristic polynomial for the last coexistence fixed point E_4 as follows:

$$J = \begin{pmatrix} 1 + \delta\left(b_1 - \frac{2(b_1 a_{22} + a_{12} b_2 + a_{12} s)}{a_{22}} + \frac{a_{12}(b_2 + s)}{a_{22}}\right) & \frac{\delta a_{12}(b_1 a_{22} + a_{12} b_2 + a_{12} s)}{a_{11} a_{22}} \\ 0 & 1 - \delta b_2 - \delta s \end{pmatrix} \quad (11)$$

or

$$J = \begin{pmatrix} 1 - \delta b_1 - \delta KR & \frac{\delta R(b_1 a_{22} + a_{12} K)}{a_{11}} \\ 0 & 1 - \delta K \end{pmatrix}$$

where

$b_2 + s = K, \qquad K > 0,$

$\frac{a_{12}}{a_{22}} = R, \qquad R > 0,$

$$F(\lambda) = \lambda^2 - izJ\left(\frac{b_1 a_{22} + a_{12} b_2 + a_{12} s}{a_{11} a_{22}}, \frac{b_2 + s}{a_{22}}\right)\lambda + detJ\left(\frac{b_1 a_{22} + a_{12} b_2 + a_{12} s}{a_{11} a_{22}}, \frac{b_2 + s}{a_{22}}\right)$$

Therefore characteristic valuesof the system (4) are obtained as follows.

$\lambda_1 = 1 - \delta(b_1 + KR)$

$\lambda_2 = 1 - \delta K.$

Theorem 3.4 Suppose $\delta_1 = \frac{2}{b_1 + KR}$ and $\delta_2 = \frac{2}{K}$. Then the fixed point $E_4$of the system (4),

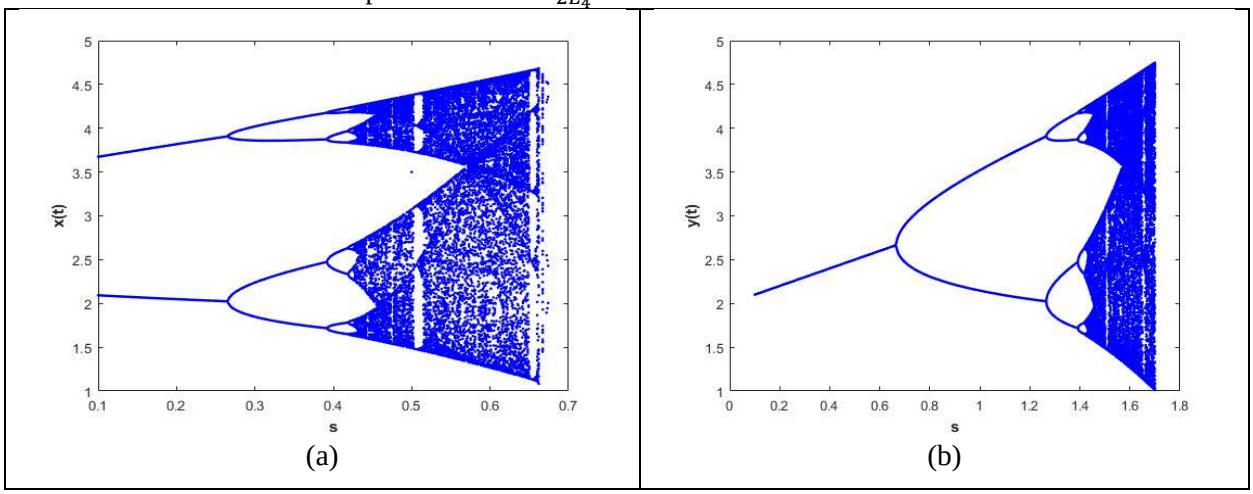
- **a)** if $\delta < min\{\delta_1, \delta_2\}$, then it islocal asymptotic stable [23],
- **b)** if $\delta < \delta_1$ and $\delta > \delta_2$, then it is saddle,
- **c)** if$\delta > max\{\delta_1, \delta_2\}$, then it is source,
- **d)** if $\delta = \delta_1$ or $\delta = \delta_2$, then it is non-hyperbolic.

In [23] researchers prove that discrete system (4) possesses the flip bifurcation at the fixed point $E_4$by choosing as a δ bifurcation parameter if parameters vary in a small neighborhood of FB_{1E_4} and FB_{2E_4} where

$FB_{1E_4} = \{(a_{11}, a_{12}, a_{22}, b_1, b_2, s) \in R^6 : \delta \neq \delta_1 \; and \; \delta \neq (\frac{\delta_1}{\delta} + 1)\frac{\delta_2 a_{22}}{2}\},$

$FB_{2E_4} = \{(a_{11}, a_{12}, a_{22}, b_1, b_2, s) \in R^6 : \delta \neq \delta_2 \; and \; \delta \neq (\frac{\delta_1}{\delta} + 1)\frac{\delta_2 a_{22}}{2}\}.$

We present the Flip bifurcation diagrams and Maximum Lyapunov exponent for the system (4) by choosing sinstead of δas a bifurcation parameter in FB_{2E_4} .



(a) (b)

Figure1. Bifurcation diagrams of system (4) for $a_{11} = 1, a_{12} = 1, a_{22} = 1, b_1 = 1, b_2 = 2, \delta = 0.75$ and initial value $(x_0, y_0) = (3.5,2.5)$.

Taking parameter values for $a_{11} = 1, a_{12} = 1, a_{22} = 1, b_1 = 1, b_2 = 2, \delta = 0.75$ and initial value $(x_0, y_0) = (3.5,2.5)$, flip bifurcation occurs from coexistence fixed point $E_4 = (3.66,2.66)$ at $s = 0.666666666$. From Figure (1b), It is seen that the coexistence fixed point$E_4 = (3.66,2.66)$ is stable $s < 0.666666666$, and loses its stability at the flip bifurcation parameter value $s > 0.666666666$. Also, It is observed that the flip bifurcation giving 2,4,8 periodic orbits occur.

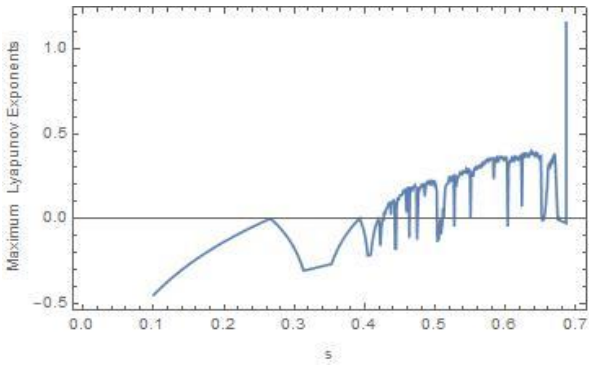


Figure2. Maximum Lyapunov Exponent

We know that it is determined existence or non-existence of the chaotic solutions for a dynamical system by calculating the Lyapunov exponent. If the system (4) has a positive largest exponent, we say that the system (4) shows chaotic dynamics. Some Lyapunov exponents are bigger than 0, some are smaller than 0. Therefore, there are stable fixed points or stable period windows in the chaotic region. In Figure 2, we calculate and plot the maximum Lyapunov exponent for the system (4). Figure 2 exhibits the existenceof the chaotic regions.

4. Chaos Control

We investigate a chaos control technique for the discrete-time system. Chaos control aims to make chaotic behavior more predictable and stable. We use the OGY method for the system (4). Therefore we consider the following controlled system:

$$\begin{cases} x_{t+1} = x_t + \delta(x_t(b_1 - a_{11}x_t) + a_{12}x_t y_t) = f(x_t, y_t, s) \\ y_{t+1} = y_t + \delta(y_t(b_2 - a_{22}y_t) + s) = g(x_t, y_t, s) \end{cases} \quad (12)$$

where s is taken as the controlling parameter.On the other hand, it is assumed that controlparameter s satisfies $|s - s_0| < \mu$ where $\mu > 0$ and s_0 represents some nominal value, which is locatedin the chaotic region. Next, we assume that (x^*, y^*) is an interior unstable fixed point of system (4). Also,suppose that $(x^*, y^*)$is located in some chaotic region. Ourmain aim is to move the unstable fixed point towards a stable one. For this, system (12) is linearizedabout the unstable fixed point (x^*, y^*) as follows:

$$\begin{bmatrix} x_{t+1} - x^* \\ y_{t+1} - y^* \end{bmatrix} \approx A \begin{bmatrix} x_t - x^* \\ y_t - y^* \end{bmatrix} + B[s - s_0] \quad (13)$$

where

$$A = \begin{bmatrix} \dfrac{\partial f(x^*, y^*, s_0)}{\partial x_t} & \dfrac{\partial f(x^*, y^*, s_0)}{\partial y_t} \\ \dfrac{\partial g(x^*, y^*, s_0)}{\partial x_t} & \dfrac{\partial g(x^*, y^*, s_0)}{\partial y_t} \end{bmatrix}$$

$$= \begin{bmatrix} 1 + \delta\left[b_1 - \dfrac{2(b_1 a_{22} + a_{12}b_2 + a_{12}s_0)}{a_{22}}\right] + \dfrac{a_{12}(b_2 + s_0)}{a_{22}} & \dfrac{\delta a_{12}(b_1 a_{22} + a_{12}b_2 + a_{12}s_0)}{a_{11}a_{22}} \\ 0 & 1 - \delta b_2 - \delta s_0 \end{bmatrix}$$

and

$$B = \begin{bmatrix} \dfrac{\partial f(x^*, y^*, s_0)}{\partial s} \\ \dfrac{\partial g(x^*, y^*, s_0)}{\partial s} \end{bmatrix} = \begin{bmatrix} 0 \\ \dfrac{\delta(b_2 + s_0)}{a_{22}} \end{bmatrix}.$$

Now we define the following controllability matrix for the system (12):

$$C = [B : AB] = \begin{bmatrix} 0 & \dfrac{\delta^2 a_{12}(b_1 a_{22} + a_{12}b_2 + a_{12}s_0)(b_2 + s_0)}{a_{11}(a_{22})^2} \\ \dfrac{\delta(b_2 + s_0)}{a_{22}} & \dfrac{(1 - \delta b_2 - \delta s_0)\delta(b_2 + s_0)}{a_{22}} \end{bmatrix}. \quad (14)$$

Then it is easy to see that rank of C is 2. Now suppose that $[s - s_0] = -K \begin{bmatrix} x_t - x^* \\ y_t - y^* \end{bmatrix}$, where $R = [p_1 \quad p_2]$. Consequently, the system (13) takes the following form:

$$\begin{bmatrix} x_{t+1} - x^* \\ y_{t+1} - y^* \end{bmatrix} \approx [A - BR] \begin{bmatrix} x_t - x^* \\ y_t - y^* \end{bmatrix} \quad (15)$$

Moreover, the fixed point$(x^*, y^*)$is locally asymptotically stable if and onlyif both eigenvalues of the matrix$A - BR$lie in an open unit disk. The Jacobian matrix$A - BR$ of the controlled system (15) can be written as follows:

$$A - BR = \begin{bmatrix} -\dfrac{-a_{22} + \delta b_1 a_{22} + \delta a_{12}b_2 + \delta a_{12}s_0}{a_{22}} & \dfrac{\delta a_{12}(b_1 a_{22} + a_{12}b_2 + a_{12}s_0)}{a_{11}a_{22}} \\ -\dfrac{\delta(b_2 + s_0)p_1}{a_{22}} & -\dfrac{-a_{22} + \delta b_1 a_{22} + \delta s_0 a_{22} + \delta p_2 b_2 + \delta p_2 s_0}{a_{22}} \end{bmatrix}$$

The characteristic equation of the Jacobian matrix$A - BR$is given by

$$P(\lambda) = \lambda^2 - \left[2 + \delta\left(b_1 - \frac{2(b_1 a_{22} + a_{12}b_2 + a_{12}s_0)}{a_{22}} + \frac{a_{12}(b_2 + s_0)}{a_{22}}\right) - \delta b_2 - \delta s - \frac{\delta(b_2 + s_0)p_2}{a_{22}}\right]\lambda +$$

$$+1 - \delta b_2 - \delta s_0 - \frac{\delta p_2 b_2}{a_{22}} - \frac{\delta p_2 s_0}{a_{22}} - \delta b_1 + \delta^2 b_1 b_2 + \delta^2 b_1 s_0 +$$

$$+\frac{\delta^2 b_1 b_2 p_2 + \delta^2 b_1 p_2 s_0 - \delta a_{12} b_2 + \delta^2 a_{12}(b_2)^2 + 2\delta^2 a_{12} b_2 s_0 - \delta a_{12} s_0 + \delta^2 a_{12}(s_0)^2}{a_{22}} +$$

$$+\frac{\delta^2 a_{12}[(b_2)^2 p_2 + 2 b_2 p_2 s_0 + (s_0)^2 p_2]}{(a_{22})^2} + \frac{\delta^2 a_{12}[p_1 b_1 b_2 + p_1 b_1 s_0]}{a_{11} a_{22}} +$$

$$+\frac{\delta^2 (a_{12})^2 p_1}{a_{11}(a_{22})^2}[b_2 + s_0]^2 \qquad (16)$$

Let$\lambda_1$and$\lambda_2$be the eigenvalues of the characteristic equation (16),then

$$\lambda_1 + \lambda_2 = 2 + \delta\left(b_1 - \frac{2(b_1 a_{22} + a_{12} b_2 + a_{12} s_0)}{a_{22}} + \frac{a_{12}(b_2 + s_0)}{a_{22}}\right) - \delta b_2 - \delta s - \frac{\delta(b_2 + s_0) p_2}{a_{22}}$$

and

$$\lambda_1 \lambda_2 = 1 - \delta b_2 - \delta s_0 - \frac{\delta p_2 b_2}{a_{22}} - \frac{\delta p_2 s_0}{a_{22}} - \delta b_1 + \delta^2 b_1 b_2 + \delta^2 b_1 s_0 +$$

$$+\frac{\delta^2 b_1 b_2 p_2 + \delta^2 b_1 p_2 s_0 - \delta a_{12} b_2 + \delta^2 a_{12}(b_2)^2 + 2\delta^2 a_{12} b_2 s_0 - \delta a_{12} s_0 + \delta^2 a_{12}(s_0)^2}{a_{22}} +$$

$$+\frac{\delta^2 a_{12}[(b_2)^2 p_2 + 2 b_2 p_2 s_0 + (s_0)^2 p_2]}{(a_{22})^2} + \frac{\delta^2 a_{12}[p_1 b_1 b_2 + p_1 b_1 s_0]}{a_{11} a_{22}} +$$

$$+\frac{\delta^2 (a_{12})^2 p_1}{a_{11}(a_{22})^2}[b_2 + s_0]^2 \qquad (17)$$

are obtained.In order to get the lines of marginal stability we must solve the equations$\lambda_1 = \pm 1$and$\lambda_1 \lambda_2 = 1$.These restrictions make sure that$|\lambda_1| < 1$and$|\lambda_2| < 1$. Using $\lambda_1 \lambda_2 = 1$in equation (17) then,

$$L_1 = \left[\frac{(s_0 + b_2)^2 (a_{12})^2}{a_{11}(a_{22})^2} + \frac{a_{12} b_1 (1 + b_2)}{a_{11} a_{22}}\right]\delta^2 p_1$$

$$+\left[\left[\frac{(s_0 + b_2)^2 a_{12}}{(a_{22})^2} + \frac{b_1(1 + b_2)}{a_{22}}\right]\delta^2 - \frac{(s_0 + b_2)\delta}{a_{22}}\right] p_2$$

$$+\left[\frac{(s_0 + b_2)^2 (a_{12})^2}{a_{22}} + b_1(1 + b_2)\right]\delta^2 - \left[\left(1 + \frac{a_{12}}{a_{22}}\right) s_0 + b_1 + \left(1 + \frac{a_{12}}{a_{22}}\right) b_2\right]\delta = 0.$$

Furthermore, suppose that$\lambda_1 = 1$,then

$$L_2 = \left[\frac{(s_0 + b_2)^2 (a_{12})^2}{a_{11}(a_{22})^2} + \frac{a_{12} b_1 (1 + b_2)}{a_{11} a_{22}}\right]\delta^2 p_1$$

$$+\left[\frac{(s_0 + b_2)^2 a_{12}}{(a_{22})^2} + \frac{b_1(1 + b_2)}{a_{22}}\right]\delta^2 p_2$$

$$+\left[\frac{(s_0+b_2)^2(a_{12})^2}{a_{22}}+b_1(1+b_2)\right]\delta^2=0.$$

Finally, suppose that$\lambda_1=-1$,then

$$L_3=\left[\frac{(s_0+b_2)^2(a_{12})^2}{a_{11}(a_{22})^2}+\frac{a_{12}b_1(1+b_2)}{a_{11}a_{22}}\right]\delta^2p_1$$

$$+\left[\left[\frac{(s_0+b_2)^2a_{12}}{(a_{22})^2}+\frac{b_1(1+b_2)}{a_{22}}\right]\delta^2-\frac{2(s_0+b_2)\delta}{a_{22}}\right]p_2+4$$

$$+\left[\frac{(s_0+b_2)^2a_{12}}{a_{22}}+b_1(1+b_2)\right]\delta^2-\left[\left(1+\frac{a_{12}}{a_{22}}\right)2s_0+2b_1+2\left(1+\frac{a_{12}}{a_{22}}\right)b_2\right]\delta=0.$$

Then, stable eigenvalues lie within the triangular region in$p_1p_2$plane bounded by thestraight linesL_1, L_2, $L_3$for particular parametric values.

5. Numerical Simulations

In this chapter to demonstrate the accuracy of theoretical studies, numerical examples are given using the software Maple12.

Example 5.1 In order to verify theoretical results we choose particular parametricvalues for the system (4) as follows [25]

$a_{11}=1.2,\ a_{12}=1.4,\ a_{22}=1,\ b_1=1,\ b_2=2,\ s=0.7,\ K=2.7,\ R=1.4$

The Jacobian matrix for these parameter values is

$$J=\begin{pmatrix}-0.912 & 2.230666\\ 0 & -0.08\end{pmatrix} \tag{18}$$

Characteristic values of the jakobian matrix(18) are

$\lambda_1=-0.912$

$\lambda_2=-0.08.$

Clearly $|\lambda_1|<1$ and $|\lambda_2|<1$. Moreover,

$\delta_1=0.4184100418$

$\delta_2=0.7407407408$

are obtained. For $\delta=0.4$ andinitial condition(x_0,y_0)=(3,2), the positive fixed point of the model (4) is obtained as $E_4=(3.98333,2.7)$.It isa local asymptotic stable which shows the correctness of our theoretical results.

Figure 3 shows that the fixed point of the model (4) is a local asymptotic stable where$X_t(prey)$and $Y_t(predator)$ population density, respectively.

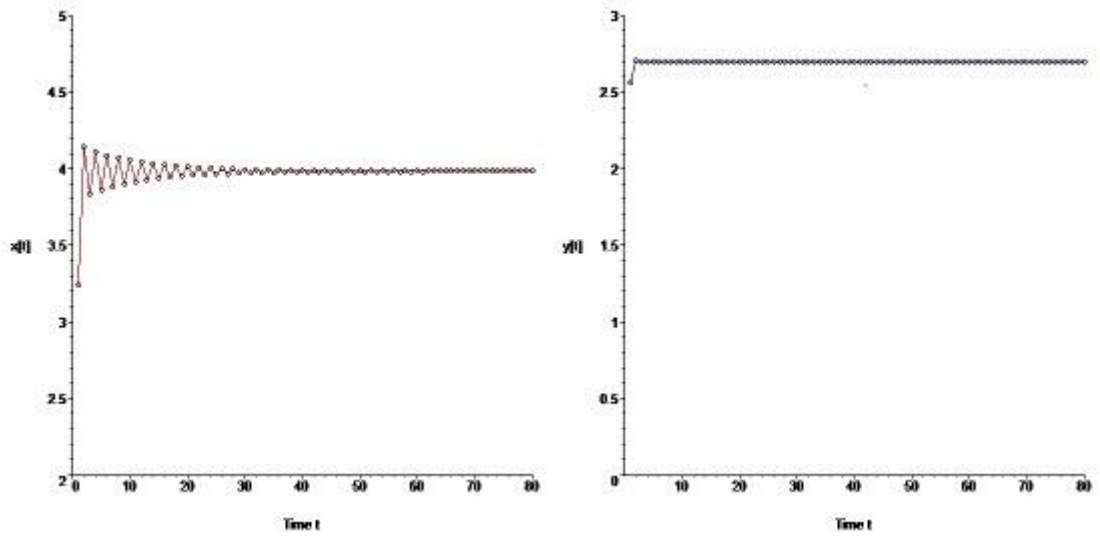


Figure 3. A stable fix point of the system (4) for $a_{11} = 1.2, a_{12} = 1.4, a_{22} = 1, b_1 = 1, b_2 = 2, \delta = 0.4, s = 0.7, K = 2.7, R = 1.4$ and initial value $(x_0, y_0) = (3,2)$.

Example 5.2 For the parameter values

$a_{11} = 1.2,\ a_{12} = 1.4, a_{22} = 1, b_1 = 1,\ b_2 = 2, s = 0.2, K = 2.2,\ R = 1.4$

the positive fixed point of system (4) is $E_4 = (3.4, 2.2)$. The Jacobian matrix for these parameter values

$$J = \begin{pmatrix} -0.428 & 1.666 \\ 0 & 0.23 \end{pmatrix} \tag{19}$$

and thecharacteristic values of the (13)

$\lambda_1 = -0.418$

$\lambda_2 = 0.23.$

Here $|\lambda_1| < 1$ and $|\lambda_2| < 1$. Also,

$\delta_1 = 0.4901960784$

$\delta_2 = 0.9090909090$

is obtained. For $\delta = 0.35$ and(x_0, y_0)=(3, 1.9), the fixed point is$E_4 = (3.4,\ 2.2)$. It is local asymptotic stable for all above parameter values

Figure 4 shows graphs showing $X_t(prey)$and $Y_t(predator)$ population density, respectively.

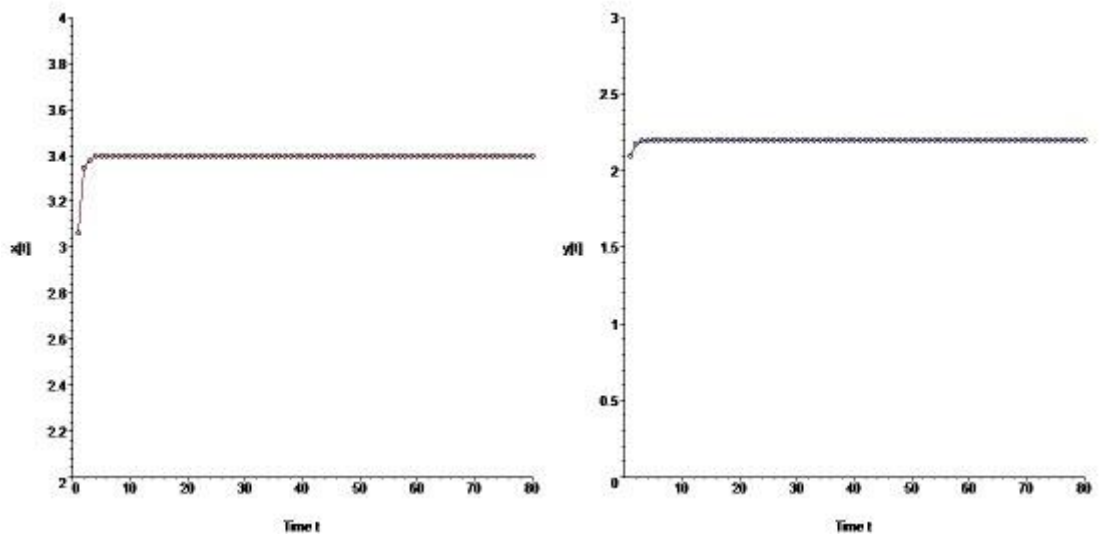


Figure 4. A stable fix point of the system (4) for $a_{11} = 1.2, a_{12} = 1.4, a_{22} = 1, b_1 = 1, b_2 = 2, \delta = 0.35, s = 0.2$, $K = 2.2, R = 1.4$ and initial value $(x_0, y_0) = (3,1.9)$.

Example 5.3 For the following parameter values

$a_{11} = 1.2, a_{12} = 1.4,\ a_{22} = 1,\ b_1 = 1,\ b_2 = 2,\ s = 0.3,\ K = 2.9,\ R = 1.4$

the positive fixed point of system (4) is $E_4 = (4.21666, 2.9)$. Jacobian matrix

$$J = \begin{pmatrix} -0.518 & 1.771 \\ 0 & 0.13 \end{pmatrix} \tag{20}$$

andcharacteristic values

$\lambda_1 = -0.518$

$\lambda_2 = 0.13.$

Here $|\lambda_1| < 1$ and $|\lambda_2| < 1$. So

$\delta_1 = 0.3952569170$

$\delta_2 = 0.6896551724$

it is obtained. If we choose $\delta = 0.3$ and$(x_0, y_0) = (4, 2.7)$we get fixed point$E_4 = (4.21666, 2.9)$and it is local asymptotic stablefor above parameter values.

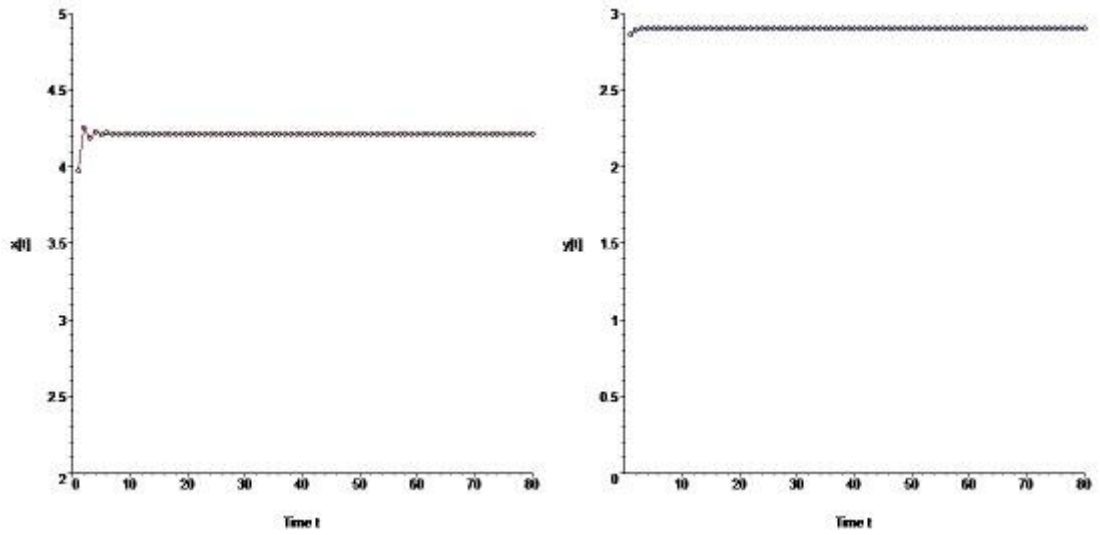


Figure 5. A stable fix point of the system (4) for $a_{11} = 1.2, a_{12} = 1.4, a_{22} = 1, b_1 = 1, b_2 = 2, \delta = 0.3, s = 0.3, K = 2.9, R = 1.4$ and initial value $(x_0, y_0) = (4,2.7)$.

Example 5.4 For the following parameter values

$a_{11} = 1.2, a_{12} = 1.4,\ a_{22} = 1, b_1 = 1, b_2 = 2,\ s = 0.5,\ K = 2.5,\ R = 1.4$

the positive fixed point of system (4) is $E_4 = (3.41666,\ 2.5)$. Jacobian matrix for these parameter values

$$J = \begin{pmatrix} -0.8 & 2.1 \\ 0 & 0 \end{pmatrix} \tag{21}$$

and characteristic values

$\lambda_1 = -0.8$

$\lambda_2 = 0.$

Clearly$|\lambda_1| < 1$ and $|\lambda_2| < 1$. Therefore

$\delta_1 = 0.4444444$

$\delta_2 = 0.8000000$

are obtained. Here, choosing $\delta = 0.4$and (x_0, y_0)=$(3.5, 2)$ the fixed point $E_4 = (3.41666, 2.5)$is obtained. For the above parameter values it is local asymptotic stable.

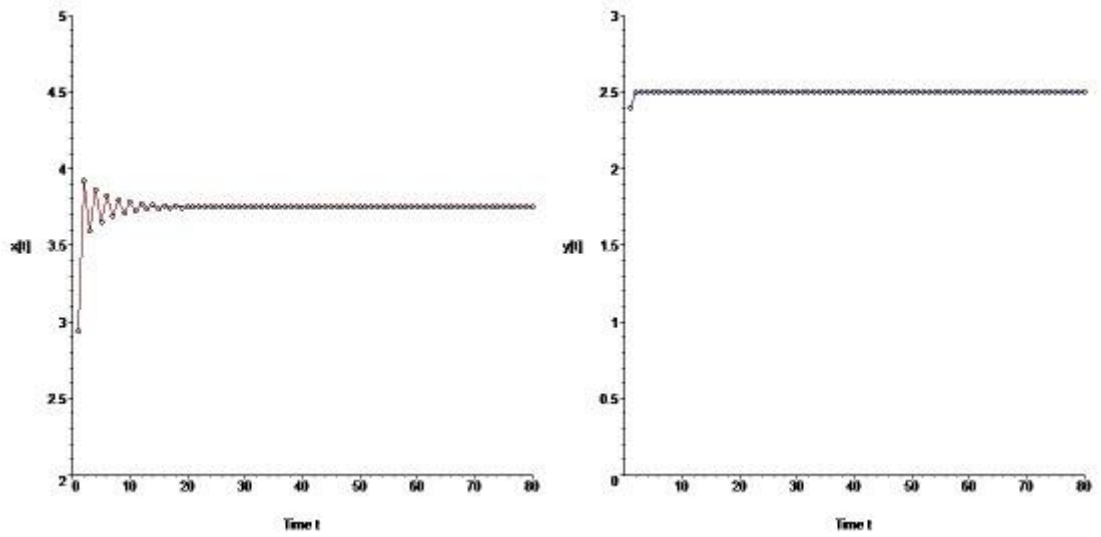


Figure 6. A stable fix point of the system (4) for $a_{11} = 1.2, a_{12} = 1.4, a_{22} = 1, b_1 = 1, b_2 = 2, \delta = 0.4, s = 0.5, K = 2.5, R = 1.4$ and initial value $(x_0, y_0) = (3.5,2)$.

Example 5.5. For the following parameter values

$a_{11} = 1, a_{12} = 1, a_{22} = 1, b_1 = 1, b_2 = 2,\ s_0 = 0.668,\ \delta = 0.75$, thesystem (4) has a unique positive fixed point(x^*, y^*)=$(3.668,\ 2.668)$. Then we get

$$A = \begin{bmatrix} -1.751 & 2.751 \\ 0 & -1.001 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 2.001 \end{bmatrix}, C = [B: AB] = \begin{bmatrix} 0 & 2.0632500 \\ 2.001 & -2.003001 \end{bmatrix}$$

It is easy to check that the rank of$C$matrix is 2. Therefore the system iscontrollable. Then,for $K = [p_1 \quad p_2]$the Jacobian matrix

$$A - BR = \begin{bmatrix} -1.751 & 2.751 \\ -2.001p_1 & -1.001 - 2.001p_2 \end{bmatrix}$$

and the characteristic polynomial

$$P(\lambda) = \lambda^2 - (-2.752 - 2.001p_2) + 1.752751 + 3.503751p_2 + 5.504751p_1$$

Also, the linesL_1,L_2 and$L_3$for marginal stability are given by

$$L_1 = 0.752751 + 3.503751p_2 + 5.504751p_1 = 0,$$

$$L_2 = 5.504751 + 5.504751p_2 + 5.504751p_1 = 0,$$

$$L_3 = 0.000751 + 1.502751p_2 + 5.504751p_1 = 0.$$

Then, the stable triangular region bounded by marginal linesL_1,L_2 and$L_3$is shown in Figure 7.

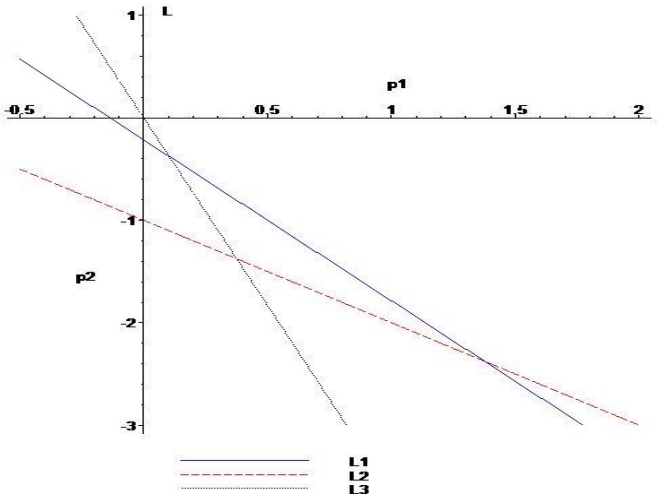


Figure 7. Triangular stability region bounded by L_1, L_2 and L_3 for the controlled system.

6. Conclusion

In this study, we deal with a discrete-time prey-predator system with constant rate of immigration on predatorwhich is obtained by implementing forward Euler's scheme and analyze existence fixed points. We showed that system(4) has fourpositive fixed points. Then we analyzed topological classifications and stability of these fixed points.Moreover, OGY feedback control method is to implement to control chaos caused by the Flip bifurcation. Finally, Flip bifurcation, maximum Lyapunov exponent, chaos control strategy, and asymptotic stability of the only positive fixed point are verified with the help of numerical simulations.

We can observe interesting dynamical behavior as immigration parameter s varies. When the immigration parameter rate $s$is less than critical value; a stable-steady state exists. With the increase of the immigration parameter rate s, the steady state losses the stability and it is interesting to observe the occurrance of Flip bifurcation whichleads to chaos. We show that the system may have rich dynamics with the change of immigration rate. Seems the immigration rate for predator can stabilize the ecosystem.

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Conflicts of interest

The authors state that did not have conflict of interests.

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