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The relations between bi-periodic jacobsthal and bi-periodic jacobsthal lucas sequence

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Abstract

In this paper, one of the special integer sequences, Jacobsthal and Jacobsthal Lucas sequences which are encountered in computer science is generalized according to parity of the index of the entries of the sequences, called bi-periodic Jacobsthal and Jacobsthal Lucas sequences. The definitions of the bi-periodic Jacobsthal and Jacobsthal Lucas sequences are given by using classic Jacobsthal and Jacobsthal Lucas sequences. In literature, there were some relations for the bi-periodic Jacobsthal and Jacobsthal Lucas sequences. We find new identities for these sequences. If we substitute $a = b = 1$ in the results, we get identities for classic Jacobsthal and Jacobsthal Lucas sequences.

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1. Introduction

The classical Jacobsthal sequence is defined as $j_n = j_{n-1} + 2j_{n-2}$ with initial conditions $j_0 = 0$, $j_1 = 1$ and the Jacobsthal Lucas sequence is defined as with the initial conditions $c_0 = 2$, $c_1 = 1$ [1]. There are many generalizations on special integer sequences. With this in mind, we proceed with the introduction as follows. In the year 2009, a paper entitled, the generalized Fibonacci sequence has been published by Edson and Yayenie [2,3]. Jun, Choi gave some properties of the bi-periodic Fibonacci sequence by using a special matrix in [4]. Bilgici [5] introduced the bi-periodic Lucas sequence into literature in 2014. Uygun and Owusu demonstrated a new generalization for the Jacobsthal sequence called bi-periodic Jacobsthal sequence in [6]. The authors evaluated some relations for bi-periodic Jacobsthal sequence in [7]. In [8], Uygun and Owusu demonstrated the bi-periodic Jacobsthal Lucas sequence. Uygun, Karatas defined a new generalization of Pell-Lucas numbers which is called bi-periodic Pell-Lucas sequence in [9]. Choo studied some identities of generalized bi-periodic Fibonacci sequence in [10]. Gul studied bi-periodic Jacobsthal and Jacobsthal-Lucas quaternions in [11]. Komatsu, Ramírez studied on convolutions of the bi-periodic Fibonacci numbers in [12].

For every integer n, any nonzero real numbers a, b the generalized Jacobsthal sequence $\{J_m\}_{m=0}^{\infty}$ and the generalized Jacobsthal Lucas sequence $\{C_m\}_{m=0}^{\infty}$ satisfy the following equations.

$$J_n = \begin{cases} aJ_{n-1} + 2J_{n-2} & \text{if } n \text{ is even} \\ bJ_{n-1} + 2J_{n-2} & \text{if } n \text{ is odd} \end{cases} \quad (n \geq 2)$$

with initial values $J_0 = 0$ and $J_1 = 1$.

$$C_n = \begin{cases} bC_{n-1} + 2C_{n-2} & \text{if } n \text{ is even} \\ aC_{n-1} + 2C_{n-2} & \text{if } n \text{ is odd} \end{cases} \quad (n \geq 2)$$

with initial values, $C_0 = 2$ and $C_1 = a$ in [6,8].
If we take $a = b = 1$ then we have the classic Jacobsthal and classic Jacobsthal Lucas sequences respectively.

The sequence $\{J_n\}_{n=0}^{\infty}$ satisfies the following properties

$$J_{2n} = (ab + 4)J_{2n-2} - 4J_{2n-4}$$
$$J_{2n+1} = (ab + 4)J_{2n-1} - 4J_{2n-3}\,.$$

The sequence $\{C_n\}_{n=0}^{\infty}$ also satisfies the following properties

$$C_{2n} = (ab + 4)C_{2n-2} - 4C_{2n-4}$$
$$C_{2n+1} = (ab + 4)C_{2n-1} - 4C_{2n-3}\,.$$

For every n belonging to the set of natural numbers, the Binet formula for the bi-periodic Jacobsthal sequence is given by

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$$J_n = \frac{a^{1-\xi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}}\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right) \quad (1)$$

where

$\alpha = \frac{ab+\sqrt{a^2b^2+8ab}}{2}$ and $\beta = \frac{ab-\sqrt{a^2b^2+8ab}}{2}$

are the roots of the characteristic polynomial given by $x^2 - abx - 2ab = 0$. Similarly, Binet formula for the bi-periodic Jacobsthal Lucas sequence is given by

$$C_n = \frac{a^{\xi(n)}}{(ab)^{\lfloor \frac{n+1}{2} \rfloor}}(\alpha^n + \beta^n). \quad (2)$$

Lemma The roots of the characteristic polynomial are satisfied the following conditions:

1. $(\alpha + 2)(\beta + 2) = 4$
2. $\alpha + \beta = ab,\ \ \alpha\beta = -2ab$
3. $\alpha + 2 = \frac{\alpha^2}{ab},\ \beta + 2 = \frac{\beta^2}{ab}$
4. $-\beta(\alpha + 2) = 2\alpha,\ -\alpha(\beta + 2) = 2\beta$ [6].

The authors evaluated some relations of bi-periodic Jacobsthal sequence in [7] as:

a) $\alpha^n = a^{\lfloor \frac{n}{2} \rfloor - \xi(n-1)} b^{\lfloor \frac{n}{2} \rfloor} J_n \alpha + 2a^{\lfloor \frac{n}{2} \rfloor} b^{\lfloor \frac{n}{2} \rfloor + \xi(n)} J_{n-1}$

b) $\beta^n = a^{\lfloor \frac{n}{2} \rfloor - \xi(n-1)} b^{\lfloor \frac{n}{2} \rfloor} J_n \beta + 2a^{\lfloor \frac{n}{2} \rfloor} b^{\lfloor \frac{n}{2} \rfloor + \xi(n)} J_{n-1}$

c) $J_{n+6} = (ab + 6)a^{1-\xi(n)} b^{\xi(n)} J_{n+3} - 8J_n$

d) $aJ_{2n-1} = J_{n+1}J_n + J_{n-1}J_{n-2}$

e) $J_{m+n-1} = \left(\frac{b}{a}\right)^{1-\xi(mn+n-m)} J_m J_n + 2\left(\frac{b}{a}\right)^{\xi(mn)} J_{m-1} J_{n-1}$

f) $J_{2m-1} = \left(\frac{b}{a}\right)^{\xi(m+1)} (J_m)^2 + 2\left(\frac{b}{a}\right)^{\xi(m)} (J_{m-1})^2$

g) $J_n J_{n+2} = \left(\frac{a}{b}\right)^{\xi(n+1)} \left[\left(\frac{b}{a}\right)^{\xi(n)} J_{n+1}^2 - (-2)^{n+1}\right]$

h) $J_m = a^{\xi(m-1)} \sum_{k=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m-k-1}{k} (ab)^{\lfloor \frac{m-1}{2} \rfloor - k} 2^k$

i) $J_m = \frac{a^{\xi(m+1)}}{2^{m-1}} \sum_{k=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m}{2k+1} (ab)^{\lfloor \frac{m-1}{2} \rfloor - k} (ab + 8)^k$

The authors evaluated some relations between the bi-periodic Jacobsthal sequence and the bi-periodic Jacobsthal Lucas sequences for all integers m and n in as:

a) $(ab + 8)J_m = 2C_{m-1} + C_{m+1}$

b) $J_{m+n} = \frac{1}{2}\left[\left(\frac{b}{a}\right)^{\xi(m+1)\xi(n)} J_m C_n + \left(\frac{b}{a}\right)^{\xi(m)\xi(n+1)} J_n C_m\right]$

c) $J_{m-n} = \frac{(-1)^n}{2^{n+1}}\left[\left(\frac{b}{a}\right)^{\xi(m+1)\xi(n)} J_m C_n - \left(\frac{b}{a}\right)^{\xi(m)\xi(n+1)} J_n C_m\right]$

d) $C_{m+n} = \frac{1}{2}\left[(a^2b^2 + 8ab)\left(\frac{1}{a^2}\right)^{\xi(m+1)\xi(n+1)} \left(\frac{1}{ab}\right)^{1-\xi(m+1)\xi(n+1)} J_m J_n + \left(\frac{b}{a}\right)^{\xi(m)\xi(n)} C_m C_n\right]$

e) $C_{m-n} = \frac{(-1)^n}{2^{n+1}} \begin{bmatrix} \left(\frac{b}{a}\right)^{\xi(m)\xi(n)} C_m C_n \\ -(a^2b^2+8ab)\left(\frac{1}{a^2}\right)^{\xi(m+1)\xi(n+1)} \left(\frac{1}{ab}\right)^{1-\xi(m+1)\xi(n+1)} J_m J_n \end{bmatrix}$

f) $J_{n+1} = \frac{1}{2}\left(C_n + a^{\xi(n)} b^{\xi(n+1)} J_n\right)$

g) $C_{n+1} = \frac{1}{2}\left[(ab+8)J_n + a^{\xi(n+1)} b^{\xi(n)} C_n\right]$

h) $\left(\frac{b}{a}\right)^{\xi(n)} {C_n}^2 - (a^2b^2+8ab)\left(\frac{1}{a^2}\right)^{\xi(n+1)} \left(\frac{1}{ab}\right)^{\xi(n)} {J_n}^2 = 4(-2)^n$

i) $C_{2n} = \frac{1}{2}\left[(a^2b^2+8ab)\left(\frac{1}{a^2}\right)^{\xi(n+1)} \left(\frac{1}{ab}\right)^{\xi(n)} {J_n}^2 + \left(\frac{b}{a}\right)^{\xi(n)} {C_n}^2\right]$

j) $C_{2m}C_{2n} = C_{2m+2n} + 4^m C_{2n-2m}$

k) $C_{2m}C_{2n} = \left(\frac{b}{a}\right)^{\xi(m+n)} [C^2{}_{m+n} + (2)^{2n} C^2{}_{m-n}] - 4(-2)^{m+n}$

l) $C_{2m}C_{2n} = a^{-2\xi(m+n+1)}(ab)^{-\xi(m+n)}(a^2b^2+8ab)J^2{}_{m+n} + 2^{2n}\left(\frac{b}{a}\right)^{\xi(m+n)} C^2{}_{m-n}$

m) $C_{2m}C_{2n} = a^{-2\xi(m+n+1)}(ab)^{-\xi(m+n)}(a^2b^2+8ab)J^2{}_{m+n} + 2^{2n}\left(\frac{b}{a}\right)^{\xi(m+n)} C^2{}_{m-n}$

n) $C_{2m}C_{2n} = 2^{2n} a^{-2\xi(m+n+1)}(ab)^{-\xi(m+n)}(a^2b^2+ +8ab)J^2{}_{m-n}\left(\frac{b}{a}\right)^{\xi(m+n)} C^2{}_{m+n}$

o) $C_nC_{n+1} = C_{2n+1} + (-2)^n a$

p) $2\left(\frac{b}{a}\right)^{\xi(n)} C^2{}_n + \left(\frac{b}{a}\right)^{\xi(n+1)} C^2{}_{n+1} = (ab+8)J_{2n+1}$

2. Main Results

2.1. New properties between the bi-periodic jacobsthal sequence and the bi-periodic jacobsthal lucas sequences

Jacobsthal numbers have applications in such areas as tiling, graph matching, alternating sign matrices, etc. [13-16]. So, in this part we want to develop this number sequence and find new properties of the sequence.

Theorem 2.1 For any integers m and n, we have

a) $J_{3n} = J_n[C_{2n} + (-2)^n]$,

b) $J_{3n} = \left(\frac{b}{a}\right)^{\xi(n)} [J_{2n}C_n - (-2)^n J_n]$.

Proof: For the proof of a) we use Binet formula for bi-periodic Jacobsthal sequence and bi-periodic Jacobsthal Lucas sequences

$$J_nC_{2n} = \left(\frac{a^{1-\xi(n)}}{(ab)^{\lfloor\frac{n}{2}\rfloor}}\right)\left(\frac{a^{\xi(2n)}}{(ab)^{\lfloor\frac{2n+1}{2}\rfloor}}\right)(\alpha^{2n}+\beta^{2n})\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)$$

$$= \frac{a^{\xi(n+1)}}{(ab)^{\lfloor\frac{n}{2}\rfloor+\lfloor\frac{2n+1}{2}\rfloor}}\frac{1}{\alpha-\beta}(\alpha^{3n}-\beta^{3n}+\alpha^n\beta^{2n}-\alpha^{2n}\beta^n$$

$$= \frac{a^{\xi(n+1)}}{(ab)^{\lfloor\frac{3n}{2}\rfloor}}\frac{1}{\alpha-\beta}\left(\alpha^{3n}-\beta^{3n}-\alpha^n\beta^n(\alpha^n-\beta^n)\right)$$

$$= \frac{a^{\xi(n+1)}}{(ab)^{\lfloor\frac{3n}{2}\rfloor}}\frac{\alpha^{3n}-\beta^{3n}}{\alpha-\beta} - \frac{a^{\xi(n+1)}}{(ab)^{\lfloor\frac{3n}{2}\rfloor}}(-2ab)^n\frac{\alpha^n-\beta^n}{\alpha-\beta}$$

$$= \frac{a^{\xi(n+1)}}{(ab)^{\lfloor\frac{3n}{2}\rfloor}}\frac{\alpha^{3n}-\beta^{3n}}{\alpha-\beta} - (-2)^n\frac{a^{1-\xi(n)}}{(ab)^{\lfloor\frac{n}{2}\rfloor}}\frac{\alpha^n-\beta^n}{\alpha-\beta}$$

$$= J_{3n} - (-2)^n J_n$$

For the proof of b)

$$J_{2n}C_n = \left(\frac{a^{1-\xi(2n)}}{(ab)^n}\right)\left(\frac{a^{\xi(n)}}{(ab)^{\lfloor\frac{n+1}{2}\rfloor}}\right)(\alpha^n+\beta^n)\left(\frac{\alpha^{2n}-\beta^{2n}}{\alpha-\beta}\right)$$

$$= \frac{a^{1+\xi(n)}}{(ab)^{n+\lfloor\frac{n+1}{2}\rfloor}}\frac{1}{\alpha-\beta}(\alpha^{3n}-\beta^{3n}+\alpha^{2n}\beta^n-\alpha^n\beta^{2n})$$

If n is odd, we have

$$J_{2n}C_n = \frac{a^2}{(ab)^{\lfloor\frac{3n}{2}\rfloor+1}}\frac{1}{\alpha-\beta}\left(\alpha^{3n}-\beta^{3n}+\alpha^n\beta^n(\alpha^n-\beta^n)\right)$$

$$= \frac{a}{b}\frac{ab}{(ab)^{\lfloor\frac{3n}{2}\rfloor+1}}\frac{1}{\alpha-\beta}\left(\alpha^{3n}-\beta^{3n}+(-2ab)^n(\alpha^n-\beta^n)\right)$$

$$= \frac{a}{b}\frac{1}{(ab)^{\lfloor\frac{3n}{2}\rfloor}}\frac{\alpha^{3n}-\beta^{3n}}{\alpha-\beta}+\frac{a}{b}\frac{1}{(ab)^{\lfloor\frac{3n}{2}\rfloor-n}}(-2)^n\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)$$

$$= \frac{a}{b}\frac{1}{(ab)^{\lfloor\frac{3n}{2}\rfloor}}\frac{\alpha^{3n}-\beta^{3n}}{\alpha-\beta}+\frac{a}{b}(-2)^n\frac{1}{(ab)^{\lfloor\frac{n}{2}\rfloor}}\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)$$

$$= \frac{a}{b}J_{3n}+\frac{a}{b}(-2)^nJ_n$$

Now if n is even, we have

$$J_{2n}C_n = \frac{a}{(ab)^{\lfloor\frac{3n}{2}\rfloor}}\frac{1}{\alpha-\beta}\left(\alpha^{3n}-\beta^{3n}-\alpha^n\beta^n(\alpha^n-\beta^n)\right)$$

$$= \frac{a}{(ab)^{\lfloor\frac{3n}{2}\rfloor}}\frac{1}{\alpha-\beta}\left(\alpha^{3n}-\beta^{3n}-(-2ab)^n(\alpha^n-\beta^n)\right)$$

$$= \frac{a}{(ab)^{\lfloor\frac{3n}{2}\rfloor}}\frac{\alpha^{3n}-\beta^{3n}}{\alpha-\beta}-\frac{a}{(ab)^{\lfloor\frac{3n}{2}\rfloor-n}}(-2)^n\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)$$

$$= \frac{a}{(ab)^{\lfloor\frac{3n}{2}\rfloor}}\frac{\alpha^{3n}-\beta^{3n}}{\alpha-\beta}-(-2)^n\frac{a}{(ab)^{\lfloor\frac{n}{2}\rfloor}}\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)$$

$$= J_{3n}-(-2)^nJ_n$$ ■

By condensing the result gives us the desired result.

Now taking $a=b=1$ into Theorem 2.1 gives

$$j_{3n} = j_n[c_{2n}+(-2)^n] = j_{2n}c_n+(-2)^nj_n.$$

Theorem 2.2 For any integers m and n, we have the following property for bi-periodic Jacobsthal sequence and bi-periodic Jacobsthal Lucas sequences

$$J_{2m+n} = \left(\frac{b}{a}\right)^{\xi(m+1)\xi(n)}J_mC_{m+n}+(-2)^mJ_n.$$

Proof:

$$J_mC_{m+n} = \left(\frac{a^{1-\xi(m)}}{(ab)^{\lfloor\frac{m}{2}\rfloor}}\right)\left(\frac{a^{\xi(m+n)}}{(ab)^{\lfloor\frac{m+n+1}{2}\rfloor}}\right)(\alpha^{m+n}+\beta^{m+n})\left(\frac{\alpha^m-\beta^m}{\alpha-\beta}\right)$$

$$= \frac{a^{1-\xi(m)+\xi(m+n)}}{(ab)^{\lfloor\frac{m}{2}\rfloor+\lfloor\frac{m+n+1}{2}\rfloor}}\frac{\alpha^{2m+n}-\beta^{2m+n}+\alpha^m\beta^{m+n}-\alpha^{m+n}\beta^m}{\alpha-\beta}$$

If both m and n are even, $m+n$ is even as well and hence we have

$$J_mC_{m+n} = \frac{a}{(ab)^{\lfloor\frac{2m+n}{2}\rfloor}}\frac{\alpha^{2m+n}-\beta^{2m+n}-\alpha^m\beta^m(\alpha^n-\beta^n)}{\alpha-\beta}$$

$$= \frac{a}{(ab)^{\lfloor\frac{2m+n}{2}\rfloor}}\frac{\alpha^{2m+n}-\beta^{2m+n}}{\alpha-\beta}-(-2t)^m\frac{a}{(ab)^{\lfloor\frac{2m+n}{2}\rfloor-m}}\frac{\alpha^n-\beta^n}{\alpha-\beta}$$

$$= J_{2m+n}-(-2)^mJ_n$$

If m and n are both odd, $m+n$ even and hence we have

$$J_mC_{m+n} = \frac{1}{(ab)^{\lfloor\frac{2m+n}{2}\rfloor}}\frac{\alpha^{2m+n}-\beta^{2m+n}+\alpha^m\beta^{m+n}-\alpha^{m+n}\beta^m}{\alpha-\beta}$$

$$= \frac{1}{(ab)^{\lfloor \frac{2m+n}{2} \rfloor}} \frac{\alpha^{2m+n}-\beta^{2m+n}}{\alpha-\beta} - (-2)^m \frac{1}{(ab)^{\lfloor \frac{2m+n}{2} \rfloor - m}} \left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)$$

$$= J_{2m+n} - (-2)^m J_n$$

If m is odd and n is even, then $m+n$ is odd and hence we have

$$J_m C_{m+n} = \frac{a}{(ab)^{\lfloor \frac{2m+n}{2} \rfloor}} \frac{1}{\alpha-\beta} (\alpha^{2m+n} - \beta^{2m+n} + \alpha^m \beta^{m+n} - \alpha^{m+n}\beta^m)$$

$$= \frac{a}{(ab)^{\lfloor \frac{2m+n}{2} \rfloor}} \frac{1}{\alpha-\beta} \left(\alpha^{2m+n} - \beta^{2m+n} - \alpha^m\beta^m(\alpha^n-\beta^n)\right)$$

$$= \frac{a}{(ab)^{\lfloor \frac{2m+n}{2} \rfloor}} \frac{\alpha^{2m+n}-\beta^{2m+n}}{\alpha-\beta} - \frac{(-2)^m a}{(ab)^{\lfloor \frac{2m+n}{2} \rfloor - m}} \left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)$$

$$= J_{2m+n} - (-2)^m J_n$$

Finally if m is even and n is odd, $m+n$ is odd

$$J_m C_{m+n} = \frac{a^2}{(ab)^{\lfloor \frac{2m+n}{2} \rfloor + 1}} \frac{\alpha^{2m+n}-\beta^{2m+n}+\alpha^m\beta^{m+n}-\alpha^{m+n}\beta^m}{\alpha-\beta}$$

$$= \frac{a}{b} \frac{ab}{(ab)^{\lfloor \frac{2m+n}{2} \rfloor + 1}} \frac{\alpha^{2m+n}-\beta^{2m+n}-\alpha^m\beta^m(\alpha^n-\beta^n)}{\alpha-\beta}$$

$$= \frac{a}{b} \frac{1}{(ab)^{\lfloor \frac{2m+n}{2} \rfloor}} \frac{\alpha^{2m+n}-\beta^{2m+n}}{\alpha-\beta} - \frac{a}{b}(-2)^m \frac{1}{(ab)^{\lfloor \frac{2m+n}{2} \rfloor - m}} \left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)$$

$$= \frac{a}{b} J_{2m+n} - \frac{a}{b}(-2)^m J_n.$$

Therefore, condensing the above with the help of the parity function gives

$$J_{2m+n} = \left(\frac{b}{a}\right)^{\xi(m+1)\xi(n)} J_m C_{m+n} + (-2)^m J_n$$

which completes the proof.

Now taking $a = b = 1$ into Theorem 2.2 gives

$$j_{2m+n} = j_m c_{m+n} + (-2)^m j_n.$$

Theorem 2.3 The following identity is satisfied by bi-periodic Jacobsthal sequence and bi-periodic Jacobsthal Lucas sequence for any integer n

$$(\alpha-\beta)^2 J_{2n+3} J_{2n-3} = C_{4n} - (-2)^{2n-3} C_6.$$

Proof: By (1) and (2), we establish

$$J_{2n+3} J_{2n-3} = \left(\frac{a^{1-\xi(2n+3)}}{(ab)^{\lfloor \frac{2n+3}{2} \rfloor}} \frac{\alpha^{2n+3}-\beta^{2n+3}}{\alpha-\beta}\right)\left(\frac{a^{1-\xi(2n-3)}}{(ab)^{\lfloor \frac{2n-3}{2} \rfloor}} \frac{\alpha^{2n-3}-\beta^{2n-3}}{\alpha-\beta}\right)$$

$$= \frac{1}{(ab)^{\lfloor \frac{2n+3}{2} \rfloor + \lfloor \frac{2n-3}{2} \rfloor}} \frac{\alpha^{4n}+\beta^{4n}-\alpha^{2n-3}\beta^{2n-3}(\alpha^6+\beta^6)}{(\alpha-\beta)^2}.$$

Multiplying through by $(\alpha-\beta)^2$gives the following

$$(\alpha-\beta)^2 J_{2n+3}J_{2n-3}=\frac{\alpha^{4n}+\beta^{4n}-(-2ab)^{2n-3}(\alpha^6+\beta^6)}{(ab)^{2n}}$$

$$=\frac{a^{\xi(4n)}}{(ab)^{\left\lfloor\frac{4n+1}{2}\right\rfloor}}(\alpha^{4n}+\beta^{4n})-(-2)^{2n-3}\frac{a^{\xi(6)}}{(ab)^{\left\lfloor\frac{6+1}{2}\right\rfloor}}(\alpha^6+\beta^6)$$

$$=C_{4n}-(-2)^{2n-3}C_6$$

which completes the proof.

Now taking $a=b=1$ into Theorem 2.3 gives

$$9j_{2n+3}j_{2n-3}=c_{4n}-(-2)^{2n-3}c_6.$$

Theorem 2.4 For any integer n, we have for bi-periodic Jacobsthal sequence and bi-periodic Jacobsthal Lucas sequences

$$2J_{2n+1}=\left(\frac{b}{a}\right)^{\xi(n)}J_{n+1}C_{n+2}-bJ_{n+2}C_n+(-2)^n(ab-2).$$

Proof:

$$\left(\frac{b}{a}\right)^{\xi(n)}J_{n+1}C_{n+2}=\left(\frac{b}{a}\right)^{\xi(n)}\left(\frac{a^{\xi(n)}}{(ab)^{\left\lfloor\frac{n+1}{2}\right\rfloor}}\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}\right)\left(\frac{a^{\xi(n+2)}(\alpha^{n+2}+\beta^{n+2})}{(ab)^{\left\lfloor\frac{n+3}{2}\right\rfloor}}\right)$$

$$=\left(\frac{b}{a}\right)^{\xi(n)}\left(\frac{a^{2\xi(n)}}{(ab)^{2\left\lfloor\frac{n+1}{2}\right\rfloor+1}}\frac{\alpha^{2n+3}-\beta^{2n+3}-(\alpha\beta)^{n+1}(\alpha-\beta)}{\alpha-\beta}\right)$$

$$=\left(\frac{(ab)^{\xi(n)}}{(ab)^{n+1+\xi(n)}}\frac{\alpha^{2n+3}-\beta^{2n+3}}{\alpha-\beta}\right)-\frac{(2ab)^{n+1}(\alpha-\beta)}{(ab)^{n+1}}$$

$$=J_{2n+3}-(-2)^{n+1}$$

$$bJ_{n+2}C_n=b\left(\frac{a^{1-\xi(n+2)}}{(ab)^{\left\lfloor\frac{n+2}{2}\right\rfloor}}\frac{\alpha^{n+2}-\beta^{n+2}}{\alpha-\beta}\right)\left(a^{\xi(n)}\frac{\alpha^n+\beta^n}{(ab)^{\left\lfloor\frac{n+1}{2}\right\rfloor}}\right)$$

$$=\left(\frac{ab}{(ab)^{n+1}}\frac{\alpha^{2n+2}-\beta^{2n+2}+(\alpha\beta)^n(\alpha^2-\beta^2)}{\alpha-\beta}\right)$$

$$=b\left(\frac{a^{1-\xi(2n+2)}}{(ab)^{n+1}}\frac{\alpha^{2n+2}-\beta^{2n+2}}{\alpha-\beta}\right)+\frac{(-2ab)^n ab(\alpha+\beta)}{(ab)^{n+1}}$$

$$=bJ_{2n+2}+(-2)^n\mathrm{ab}$$

Therefore we have

$$\left(\frac{b}{a}\right)^{\xi(n)}J_{n+1}C_{n+2}-bJ_{n+2}C_n+(-2)^n(ab-2)$$

$$=J_{2n+3}-(-2)^{n+1}-bJ_{2n+2}-(-2)^n\mathrm{ab}+(-2)^n(ab-2)$$

$$=J_{2n+3}-bJ_{2n+2}=2J_{2n+1}.$$

Now taking $a=b=1$ into Theorem 2.4 gives

$2j_{2n+1} = j_{n+1}c_{n+2} - j_{n+2}c_n - (-2)^n.$

Theorem 2.5 For any n, nth element of bi-periodic Jacobsthal Lucas sequence is demonstrated by the following equality:

$$C_n = \left(\frac{b}{a}\right)^{\xi(n+1)\xi(m)} C_m J_{n-m+1} + 2\left(\frac{b}{a}\right)^{\xi(n+1)\xi(m+1)} C_{m-1} J_{n-m}.$$

Proof: For proof we use the following properties

$$\frac{1}{2}\left(\xi(n) + \xi(m) - \xi(m+n)\right) = \xi(n)\xi(m)$$

$$n - \xi(n) = 2\left\lfloor\frac{n}{2}\right\rfloor.$$

By Binet formulas (1, 2), we have

$$\left(\frac{b}{a}\right)^{\xi(n+1)\xi(m)} C_m J_{n-m+1} + 2\left(\frac{b}{a}\right)^{\xi(n+1)\xi(m+1)} C_{m-1} J_{n-m}$$

$$=\left(\frac{b}{a}\right)^{\frac{1}{2}(\xi(n+1)+\xi(m)-\xi(m+n+1))} \left(\frac{a^{\xi(m)}}{(ab)^{\left\lfloor\frac{m+1}{2}\right\rfloor}}(\alpha^m + \beta^m)\right)\left(\frac{a^{1-\xi(n-m+1)}}{(ab)^{\left\lfloor\frac{n-m+1}{2}\right\rfloor}}\frac{\alpha^{n-m+1} - \beta^{n-m+1}}{\alpha-\beta}\right) +$$
$$2\left(\frac{b}{a}\right)^{\frac{1}{2}(\xi(n+1)+\xi(m+1)-\xi(m+n+2))} \left(\frac{a^{\xi(m-1)}}{(ab)^{\left\lfloor\frac{m}{2}\right\rfloor}}(\alpha^{m-1} + \beta^{m-1})\right)\left(\frac{a^{1-\xi(n-m)}}{(ab)^{\left\lfloor\frac{n-m}{2}\right\rfloor}}\frac{\alpha^{n-m} - \beta^{n-m}}{\alpha-\beta}\right)$$

$$=\frac{1}{a^{\left\lfloor\frac{n}{2}\right\rfloor}b^{\left\lfloor\frac{n+1}{2}\right\rfloor}}\frac{1}{\alpha-\beta}\begin{pmatrix}\alpha^{n+1} - \beta^{n+1} - \alpha^m\beta^{n-m+1} + \alpha^{n-m+1}\beta^m \\ +2ab(\alpha^{n-1} - \beta^{n-1} - \alpha^{m-1}\beta^{n-m} + \alpha^{n-m}\beta^{m-1}\end{pmatrix}$$

$$=\frac{1}{a^{\left\lfloor\frac{n}{2}\right\rfloor}b^{\left\lfloor\frac{n+1}{2}\right\rfloor}}\frac{1}{\alpha-\beta}\begin{pmatrix}\alpha^{n+1} - \beta^{n+1} - \alpha^m\beta^{n-m+1} + \alpha^{n-m+1}\beta^m \\ -\alpha^n\beta + \alpha\beta^n + \alpha^m\beta^{n-m+1} - \alpha^{n-m+1}\beta^m\end{pmatrix}$$

After simplifications we have

$$=\frac{1}{a^{\left\lfloor\frac{n}{2}\right\rfloor}b^{\left\lfloor\frac{n+1}{2}\right\rfloor}}\frac{1}{\alpha-\beta}(\alpha^{n+1} - \beta^{n+1} - \alpha^n\beta + \alpha\beta^n) = \frac{1}{a^{\left\lfloor\frac{n}{2}\right\rfloor}b^{\left\lfloor\frac{n+1}{2}\right\rfloor}}\frac{1}{\alpha-\beta}(\alpha^n + \beta^n)(\alpha - \beta) = C_n.$$

Now taking $a = b = 1$ into Theorem 2.5 gives

$$c_n = c_m j_{n-m+1} + 2c_{m-1} j_{n-m}.$$

Theorem 2.6 For any m, n the mnth element of bi-periodic Jacobsthal sequence is given by

$$J_{mn} = \left(\frac{b}{a}\right)^{\xi(n)\xi(m)} C_m J_{m(n-1)} - (-2)^m J_{m(n-2)}.$$

Proof: By using the following property,

$$\xi(n)\xi(m) = \left\lfloor\frac{m+n}{2}\right\rfloor - \left\lfloor\frac{m}{2}\right\rfloor - \left\lfloor\frac{n}{2}\right\rfloor$$

we have

$$\left(\frac{b}{a}\right)^{\xi(n)\xi(m)} C_m J_{m(n-1)} + J_{m(n-2)}$$

$$= \left(\frac{a^{-\left\lfloor\frac{m+n}{2}\right\rfloor+\left\lfloor\frac{n}{2}\right\rfloor-\left\lfloor\frac{m(n-1)-1}{2}\right\rfloor} b^{\left\lfloor\frac{m+n}{2}\right\rfloor-\left\lfloor\frac{n}{2}\right\rfloor-\left\lfloor\frac{m}{2}\right\rfloor-\left\lfloor\frac{m+1}{2}\right\rfloor-\left\lfloor\frac{m(n-1)}{2}\right\rfloor}}{\alpha-\beta} (\alpha^m+\beta^m)(\alpha^{m(n-1)}-\beta^{m(n-1)}) \right) +$$

$$(-2)^{m+1} \left(\frac{a^{-\left\lfloor\frac{m(n-2)-1}{2}\right\rfloor} b^{-\left\lfloor\frac{m(n-2)}{2}\right\rfloor}}{\alpha-\beta} (\alpha^{m(n-2)}-\beta^{m(n-2)}) \right)$$

$$= \frac{1}{a^{\left\lfloor\frac{mn-1}{2}\right\rfloor} b^{\left\lfloor\frac{mn}{2}\right\rfloor}} \frac{\alpha^{mn}-\beta^{mn}}{\alpha-\beta} + \frac{1}{a^{\left\lfloor\frac{mn-1}{2}\right\rfloor} b^{\left\lfloor\frac{mn}{2}\right\rfloor}} \frac{(-\alpha^m\beta^{m(n-1)}+\alpha^{m(n-1)}\beta^m-(-2)^m(ab)^m(\alpha^{m(n-2)}-\beta^{m(n-2)})}{\alpha-\beta}$$

$$= \frac{1}{a^{\left\lfloor\frac{mn-1}{2}\right\rfloor} b^{\left\lfloor\frac{mn}{2}\right\rfloor}} \frac{\alpha^{mn}-\beta^{mn}}{\alpha-\beta} = J_{mn}.$$

Now taking $a = b = 1$ into Theorem 2.6 gives

$$j_{mn} = j_{m(n-1)} - (-2)^m j_{m(n-2)}.$$

Theorem 2.7 For any n, we have the following identities for bi-periodic Jacobsthal sequence and bi-periodic Jacobsthal Lucas sequence:

$$J_{4n+1} - 2^{2n} = \left(\frac{b}{a}\right) C_{2n+1} J_{2n},$$

and

$$J_{4n+3} - 2^{2n+1} = \left(\frac{b}{a}\right) C_{2n+1} J_{2n+2}.$$

Proof: (1)

$$C_{2n+1} J_{2n} = \left(\frac{a^{\xi(2n+1)}}{(ab)^{\left\lfloor\frac{2n+2}{2}\right\rfloor}} (\alpha^{2n+1}+\beta^{2n+1}) \right) \left(\frac{a^{1-\xi(2n)}}{(ab)^{\left\lfloor\frac{2n}{2}\right\rfloor}} \frac{\alpha^{2n}-\beta^{2n}}{\alpha-\beta} \right)$$

$$= \frac{a^2}{(ab)^{2n+1}} \left(\frac{\alpha^{4n+1}-\beta^{4n+1}-\alpha^{2n+1}\beta^{2n}+\beta^{2n+1}\alpha^{2n}}{\alpha-\beta} \right)$$

$$= \frac{a^2}{(ab)^{2n+1}} \left(\frac{\alpha^{4n+1}-\beta^{4n+1}-\alpha^{2n}\beta^{2n}(\alpha-\beta)}{\alpha-\beta} \right)$$

$$= \frac{a}{b} \left[\frac{1}{(ab)^{2n}} \frac{\alpha^{4n+1}-\beta^{4n+1}}{\alpha-\beta} \right] - \frac{a}{b} \frac{1}{(ab)^{2n}} (ab)^{2n} (-2)^{2n}$$

$$= \frac{a}{b} \left[\frac{1}{(ab)^{2n}} \frac{\alpha^{4n+1}-\beta^{4n+1}}{\alpha-\beta} \right] - \frac{a}{b} \frac{1}{(ab)^{2n}} (ab)^{2n} (-2)^{2n}$$

$$= \frac{a}{b} \left[\frac{a^{1-\xi(4n+1)}}{(ab)^{\left\lfloor\frac{4n+1}{2}\right\rfloor}} \frac{\alpha^{4n+1}-\beta^{4n+1}}{\alpha-\beta} \right] - \frac{a}{b} 2^{2n}$$

$$= \frac{a}{b} J_{4n+1} - \frac{a}{b} 2^{2n}$$

(2)

$$C_{2n+1} J_{2n+2} = \left(\frac{a^{\xi(2n+1)}}{(ab)^{\left\lfloor\frac{2n+2}{2}\right\rfloor}} (\alpha^{2n+1}+\beta^{2n+1}) \right) \left(\frac{a^{1-\xi(2n+2)}}{(ab)^{\left\lfloor\frac{2n+2}{2}\right\rfloor}} \frac{\alpha^{2n+2}-\beta^{2n+2}}{\alpha-\beta} \right)$$

$$= \frac{a^2}{(ab)^{2n+2}} \left(\frac{\alpha^{4n+3}-\beta^{4n+3}+\alpha^{2n+1}\beta^{2n+1}(\alpha-\beta)}{\alpha-\beta} \right)$$

$$= \frac{a}{b}\frac{ab}{(ab)^{2n+2}}\frac{\alpha^{4n+3}-\beta^{4n+3}}{\alpha-\beta} + \frac{a}{b}\frac{ab}{(ab)^{2n+2}}\frac{(\alpha\beta)^{2n+1}(\alpha-\beta)}{\alpha-\beta}$$

$$= \frac{a}{b}\frac{1}{(ab)^{2n+1}}\frac{\alpha^{4n+3}-\beta^{4n+3}}{\alpha-\beta} + \frac{a}{b}\frac{1}{(ab)^{2n+1}}(-2ab)^{2n+1}$$

$$= \frac{a}{b}\frac{a^{1-\xi(4n+3)}}{(ab)^{\lfloor\frac{4n+3}{2}\rfloor}}\frac{\alpha^{4n+3}-\beta^{4n+3}}{\alpha-\beta} + \frac{a}{b}(-2)^{2n+1}$$

$$= \frac{a}{b}J_{4n+3} - \frac{a}{b}2^{2n+1}.$$

Now if we take $a = b = 1$, we obtain the following

$$c_{4n+1} - (-2)^{2n} = c_{2n+1}j_{2n},$$

and

$$c_{4n+3} - (-2)^{2n+1} = c_{2n+1}j_{2n+2}.$$

Theorem 2.8 For any integer n, the identity for bi-periodic Jacobsthal sequence and bi-periodic Jacobsthal Lucas sequence is denoted by

$$C_nC_{n+2} - (ab+8)\,J_{n-1}\,J_{n+3} = (-2)^{n-1}\,a^{1+\xi(n)}b^{\xi(n+1)}(ab+6).$$

Proof:

$$C_nC_{n+2} - (ab+8)\,J_{n-1}\,J_{n+3} = \frac{(\alpha^n+\beta^n)}{a^{\lfloor\frac{n}{2}\rfloor}b^{\lfloor\frac{n+1}{2}\rfloor}}\frac{(\alpha^{n+2}+\beta^{n+2})}{a^{\lfloor\frac{n+2}{2}\rfloor}b^{\lfloor\frac{n+3}{2}\rfloor}} - \frac{(ab+8)}{(\alpha-\beta)^2}\frac{(\alpha^{n-1}-\beta^{n-1})}{a^{\lfloor\frac{n-2}{2}\rfloor}b^{\lfloor\frac{n-1}{2}\rfloor}}\frac{(\alpha^{n+3}-\beta^{n+3})}{a^{\lfloor\frac{n+2}{2}\rfloor}b^{\lfloor\frac{n+3}{2}\rfloor}}$$

$$= \frac{(\alpha^{2n+2}+\beta^{2n+2}+\alpha^n\beta^{n+2}+\alpha^{n+2}\beta^n)}{a^{2\lfloor\frac{n}{2}\rfloor}b^{2\lfloor\frac{n+1}{2}\rfloor}(ab)} - \frac{(\alpha^{2n+2}+\beta^{2n+2}-\alpha^{n-1}\beta^{n+3}-\alpha^{n+3}\beta^{n-1})}{a^{2\lfloor\frac{n}{2}\rfloor}b^{2\lfloor\frac{n+1}{2}\rfloor}(ab)}$$

$$= \frac{\alpha^{2n+2}+\beta^{2n+2}+\alpha^n\beta^n(\beta^2+\alpha^2)-\left[\alpha^{2n+2}+\beta^{2n+2}-\alpha^n\beta^n\left(\frac{\beta^3}{\alpha}+\frac{\alpha^3}{\beta}\right)\right]}{a^{2\lfloor\frac{n}{2}\rfloor}b^{2\lfloor\frac{n+1}{2}\rfloor}(ab)}$$

$$= \frac{\alpha^n\beta^n(\beta^2+\alpha^2)+\alpha^n\beta^n\left(\frac{\beta^3}{\alpha}+\frac{\alpha^3}{\beta}\right)}{a^{2\lfloor\frac{n}{2}\rfloor}b^{2\lfloor\frac{n+1}{2}\rfloor}(ab)}$$

$$= \frac{\alpha^n\beta^n\left(\beta^2+\alpha^2+\frac{\beta^3}{\alpha}+\frac{\alpha^3}{\beta}\right)}{a^{2\lfloor\frac{n}{2}\rfloor}b^{2\lfloor\frac{n+1}{2}\rfloor}(ab)}$$

$$= \frac{\alpha^n\beta^n\left[\beta^2\left(1+\frac{\beta}{\alpha}\right)+\alpha^2\left(1+\frac{\alpha}{\beta}\right)\right]}{a^{2\lfloor\frac{n}{2}\rfloor}b^{2\lfloor\frac{n+1}{2}\rfloor}(ab)}$$

$$= \frac{\alpha^n\beta^n\left[(\alpha+\beta)\left(\frac{\beta^2}{\alpha}+\frac{\alpha^2}{\beta}\right)\right]}{a^{2\lfloor\frac{n}{2}\rfloor}b^{2\lfloor\frac{n+1}{2}\rfloor}(ab)}$$

$$= \frac{\alpha^n\beta^n\left[(\alpha+\beta)\left(\frac{\beta^3+\alpha^3}{\alpha\beta}\right)\right]}{a^{2\lfloor\frac{n}{2}\rfloor}b^{2\lfloor\frac{n+1}{2}\rfloor}(ab)}$$

$$= \frac{(\alpha\beta)^n\left[\frac{ab}{-2}[(ab)^2+6ab]\right]}{a^{2\left\lfloor\frac{n}{2}\right\rfloor}b^{2\left\lfloor\frac{n+1}{2}\right\rfloor}(ab)}$$

$$= \frac{a^2b^2}{-2}\frac{(-2ab)^n(ab+6)}{a^{2\left\lfloor\frac{n}{2}\right\rfloor}b^{2\left\lfloor\frac{n+1}{2}\right\rfloor}(ab)}$$

$$= \frac{(-2)^n}{-2}\frac{(ab)^{n+1}(ab+6)}{a^{2\left\lfloor\frac{n}{2}\right\rfloor}b^{2\left\lfloor\frac{n+1}{2}\right\rfloor}}$$

$$= (-2)^{n-1}a^{1+n-2\left\lfloor\frac{n}{2}\right\rfloor}b^{n+1-2\left\lfloor\frac{n+1}{2}\right\rfloor}(ab+6)$$

$$= (-2)^{n-1}\, a^{1+\xi(n)}b^{\xi(n+1)}(ab+6).$$

Now if we take $a = b = 1$, we obtain the following

$$c_n c_{n+2} - 9\, j_{n-1}\, j_{n+3} = 7(-2)^{n-1}.$$

Theorem 2.9 For any integer $m, n \geq 2$, the relation between bi-periodic Jacobsthal sequence and bi-periodic Jacobsthal Lucas sequence is given as

$$J_m J_n - 4J_{m-2}J_{n-2} = a^{1-\xi(mn)}b^{\xi(mn)}J_{m+n-2.}$$

Proof: By (1, 2), we get

$$J_m J_n = \frac{a^{2-\xi(m)-\xi(n)}[\alpha^{m+n}+\beta^{m+n}-(\alpha^n\beta^m+\alpha^m\beta^n)]}{(ab)^{\left\lfloor\frac{m}{2}\right\rfloor+\left\lfloor\frac{n}{2}\right\rfloor}(\alpha-\beta)^2}$$

$$J_{m-2}J_{n-2} = \frac{a^{2-\xi(m)-\xi(n)}(ab)^2[\alpha^{m+n-4}+\beta^{m+n-4}-(\alpha\beta)^{-2}(\alpha^n\beta^m+\alpha^m\beta^n)]}{(ab)^{\left\lfloor\frac{m}{2}\right\rfloor+\left\lfloor\frac{n}{2}\right\rfloor}(\alpha-\beta)^2}$$

Then we subtract the equalities

$$J_m J_n - 4J_{m-2}J_{n-2} = \frac{a^{2-\xi(m)-\xi(n)}}{(ab)^{\left\lfloor\frac{m}{2}\right\rfloor+\left\lfloor\frac{n}{2}\right\rfloor}(\alpha-\beta)^2}\begin{bmatrix}\alpha^{m+n}\left(1-\frac{4(ab)^2}{\alpha^4}\right)+\beta^{m+n}\left(1-\frac{4(ab)^2}{\beta^4}\right)- \\ (\alpha^n\beta^m+\alpha^m\beta^n)\left(1-\frac{4(ab)^2}{(\alpha\beta)^2}\right)\end{bmatrix}$$

By $\alpha\beta = -2ab$ and $\xi(m)+\xi(n)-\xi(mn) = \xi(m+n)$, we get

$$J_m J_n - 4J_{m-2}J_{n-2} = \frac{a^{2-\xi(m)-\xi(n)}(\alpha+\beta)}{(ab)^{\left\lfloor\frac{m}{2}\right\rfloor+\left\lfloor\frac{n}{2}\right\rfloor}(\alpha-\beta)}(\alpha^{m+n-2}-\beta^{m+n-2})$$

$$= \frac{a^{2-\xi(mn)-\xi(m+n)}}{(ab)^{\left\lfloor\frac{m}{2}\right\rfloor+\left\lfloor\frac{n}{2}\right\rfloor-1}(\alpha-\beta)}(\alpha^{m+n-2}-\beta^{m+n-2})$$

$$= a^{1-\xi(mn)}\frac{a^{1-\xi(m+n-2)}}{(ab)^{\left\lfloor\frac{m}{2}\right\rfloor+\left\lfloor\frac{n-2}{2}\right\rfloor}(\alpha-\beta)}(\alpha^{m+n-2}-\beta^{m+n-2})$$

$$= a^{1-\xi(mn)}b^{\xi(mn)}\frac{a^{1-\xi(m+n-2)}}{(ab)^{\left\lfloor\frac{m+n-2}{2}\right\rfloor}(\alpha-\beta)}(\alpha^{m+n-2}-\beta^{m+n-2})$$

$$= a^{1-\xi(mn)}b^{\xi(mn)}J_{m+n-2}$$

Therefore, the proof is completed.

Now if we take $a = b = 1$, we obtain the following

$j_m j_n - 4 j_{m-2} j_{n-2} = j_{m+n-2}$.

Theorem 2.10 A summation formula of bi-periodic Jacobsthal sequence with even index term is given by

$J_{2m} = -(ab+4)^{m-1}\left[-a + 4\sum_{k=0}^{m-2}(ab+4)^{-k-1}J_{2k}\right]$.

Proof:

$$\sum_{k=0}^{m-2}(ab+4)^{-k-1}J_{2k} = \sum_{k=0}^{m-2}(ab+4)^{-k-1}\frac{a^{1-\xi(2k)}}{(ab)^{\left\lfloor\frac{2k}{2}\right\rfloor}}\left(\frac{\alpha^{2k}-\beta^{2k}}{\alpha-\beta}\right)$$

$$= \frac{a}{(ab+4)(\alpha-\beta)}\sum_{k=0}^{m-2}\frac{\alpha^{2k}-\beta^{2k}}{(ab+4)^k(ab)^k}$$

$$= \frac{a}{(ab+4)(\alpha-\beta)}\sum_{k=0}^{m-2}\left[\left(\frac{\alpha^2}{ab(ab+4)}\right)^k - \left(\frac{\beta^2}{ab(ab+4)}\right)^k\right]$$

$$= \frac{a}{(ab+4)(\alpha-\beta)}\left(\frac{\left(\frac{\alpha^2}{ab(ab+4)}\right)^{m-1}-1}{\frac{\alpha^2}{ab(ab+4)}-1} - \frac{\left(\frac{\beta^2}{ab(ab+4)}\right)^{m-1}-1}{\frac{\beta^2}{ab(ab+4)}-1}\right)$$

$$= \frac{a}{(ab+4)(\alpha-\beta)}\left(\frac{\alpha^{2m-2}-(ab)^{m-1}(ab+4)^{m-1}}{(\alpha^2-ab(ab+4))(ab)^{m-2}(ab+4)^{m-2}} - \frac{\beta^{2m-2}-(ab)^{m-1}(ab+4)^{m-1}}{(\beta^2-ab(ab+4))(ab)^{m-2}(ab+4)^{m-2}}\right)$$

$$= \frac{a}{(ab+4)^{m-1}(ab)^{m-2}(\alpha-\beta)}\left(\frac{\alpha^{2m-2}-(ab)^{m-1}(ab+4)^{m-1}}{\alpha^2-ab(ab+4)} - \frac{\beta^{2m-2}-(ab)^{m-1}(ab+4)^{m-1}}{\beta^2-ab(ab+4)}\right)$$

It is noticed that,

$$\sum_{k=0}^{m-2}(ab+4)^{-k-1}J_{2k} = \frac{a}{(ab+4)^{m-1}(ab)^{m-2}(\alpha-\beta)}\left(\frac{4(ab)^2(\alpha^{2m-4}-\beta^{2m-4})-ab(ab+4)(\alpha^{2m-2}-\beta^{2m-2}) + (ab)^{m-1}(ab+4)^{m-1}(\alpha^2-\beta^2)}{4(ab)^2}\right)$$

$$= \frac{a(\alpha^{2m-4}-\beta^{2m-4})}{(ab+4)^{m-1}(ab)^{m-2}(\alpha-\beta)} - \frac{a(\alpha^{2m-2}-\beta^{2m-2})}{4(ab+4)^{m-2}(ab)^{m-1}(\alpha-\beta)} + \frac{a}{4}$$

$$= \frac{J_{2m-4}}{(ab+4)^{m-1}} - \frac{J_{2m-2}}{4(ab+4)^{m-2}} + \frac{a}{4}$$

$$= \frac{4J_{2m-4}-(ab+4)J_{2m-2}}{4(ab+4)^{m-1}} + \frac{a}{4}$$

Consequently,

$$\sum_{k=0}^{m-2}(ab+4)^{-k-1}J_{2k} = -\frac{1}{4(ab+4)^{m-1}}J_{2m} + \frac{a}{4}$$

The proof is completed.

Now if we take $a = b = 1$, we obtain the following

$j_{2m} = -5^{m-1}\left[-a + 4\sum_{k=0}^{m-2}(5)^{-k-1}j_{2k}\right]$.

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Conflicts of interest

The authors state that did not have a conflict of interests.

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