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Transmuted complementary exponential power distribution: properties and applications

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Abstract

In this study, we introduce a new lifetime distribution by using quadratic rank transmutation map. The some properties of this new distribution is provided. Furthermore, the parameters of this new distribution are estimated by the maximum likelihood method. The performances of the estimates are examined according to bias and mean squared errors (MSEs) criteria through a Monte Carlo simulation study. Finally, two applications with real data are presented to evaluate the fits of introduced distribution.

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1. Introduction

In reliability analysis, every statistical tools are presented based on assumption of the distribution of lifetimes. Therefore, lifetime distributions are hearts of survival and reliability theory. Nowadays, introducing the new lifetime distributions is gaining much more attention. There are a lot of distributions introduced in last two decades. One of the them is complementary exponential power (CEP) distribution suggested by [1] using exponential power (EP) distribution introduced in [2]. The probability density function (pdf) and cumulative distribution function (cdf) of CEP distribution are given, respectively, by

$$g(x;\boldsymbol{\gamma})=\frac{\beta\theta x^{\beta-1}}{\alpha^{\beta}}\exp\left(1+\left(\frac{x}{\alpha}\right)^{\beta}-\exp\left\{\left(\frac{x}{\alpha}\right)^{\beta}\right\}\right)\times\left\{1-\exp\left(1-\exp\left\{\left(\frac{x}{\alpha}\right)^{\beta}\right\}\right)\right\}^{\theta-1}I_{\mathbb{R}_+}(x) \tag{1}$$

and

$$G(x;\boldsymbol{\gamma})=\left[1-\exp\left(1-\exp\left\{\left(\frac{x}{\alpha}\right)^{\beta}\right\}\right)\right]^{\theta}, \tag{2}$$

where $I_A(x)$ is indicator function on set A, $\boldsymbol{\gamma}=(\alpha,\beta,\theta)$ is parameter vector, $\alpha>0$ is a scale parameter, $\beta>0$ and $\theta>0$ are shape parameters.

In this study, we aim to introduce a new distribution named transmuted complementary exponential power (TCEP) using Quadratic Rank Transmutation Map (QRTM) proposed by [3]. In the literature, there are many lifetime distributions generated by QRTM such as [4], [5], [6] and [7]. The pdf and cdf of QRTM family are given by

$$f(x;\boldsymbol{\delta})=g(x;\boldsymbol{\gamma})\left[1+\lambda-2\lambda G(x;\boldsymbol{\gamma})\right] \tag{3}$$

and

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$$F(x;\boldsymbol{\delta}) = (1+\lambda)G(x;\boldsymbol{\gamma}) - \lambda G(x;\boldsymbol{\gamma})^2, \tag{4}$$

where G and g are cdf and corresponding pdf of any lifetime, $\boldsymbol{\gamma}$ is parameter vector of distribution with cdf G and $\lambda \in [-1,1]$ is extra parameter beside $\boldsymbol{\delta} = (\boldsymbol{\gamma}, \lambda)$. Hence, new distribution includes a parameter to baseline distribution G. For more information on QRTM, see [3].

In this paper, a new lifetime distribution is introduced by QRTM family. In Section 2, the pdf and cdf of distribution are described. The raw moments are derived under a condition, quantile, survival and hazard functions are also given. In Section 3, the point and interval estimations are discussed by maximum likelihood (ML) methodology. A simulation study is conducted to observe the behaviours of ML estimates (MLEs) in Section 4. In Section 5 and Section 6, two numerical examples are also provided to close the paper.

2. TCEP Distribution

In this study, we introduce a new lifetime distribution obtained by using Eqs. (1-2) in Eqs. (3-4). Then pdf and cdf of introduced distribution are given, respectively, by

$$\begin{aligned} f(x;\boldsymbol{\delta}) = {} & \frac{\beta\theta x^{\beta-1}}{\alpha^{\beta}} \exp\left(1+\left(\frac{x}{\alpha}\right)^{\beta} - \exp\left\{\left(\frac{x}{\alpha}\right)^{\beta}\right\}\right) \\ & \times \left[1-\exp\left(1-\exp\left\{\left(\frac{x}{\alpha}\right)^{\beta}\right\}\right)\right]^{\theta-1} \left[1+\lambda-2\lambda\left[1-\exp\left(1-\exp\left\{\left(\frac{x}{\alpha}\right)^{\beta}\right\}\right)\right]^{\theta}\right] I_{\mathbb{R}_+}(x) \end{aligned} \tag{5}$$

and

$$F(x;\boldsymbol{\delta}) = (1+\lambda)\left[1-\exp\left(1-\exp\left\{\left(\frac{x}{\alpha}\right)^{\beta}\right\}\right)\right]^{\theta} - \lambda\left[1-\exp\left(1-\exp\left\{\left(\frac{x}{\alpha}\right)^{\beta}\right\}\right)\right]^{2\theta}, \tag{6}$$

where $\boldsymbol{\delta} = (\alpha, \beta, \theta, \lambda)$ is parameter vector, $\lambda \in [-1,1]$, $\alpha, \beta, \theta \in \mathbb{R}_+$ are parameters. The distribution with pdf (5) and cdf (6) is called Transmuted Complementary Exponential Power (TCEP) $(\boldsymbol{\delta})$ distribution. When $\lambda = 0$, TCEP $(\boldsymbol{\delta})$ reduces to CEP$(\boldsymbol{\gamma})$. In Fig. 1, the pdf of TCEP $(\boldsymbol{\delta})$ are plotted for some selected parameter values.

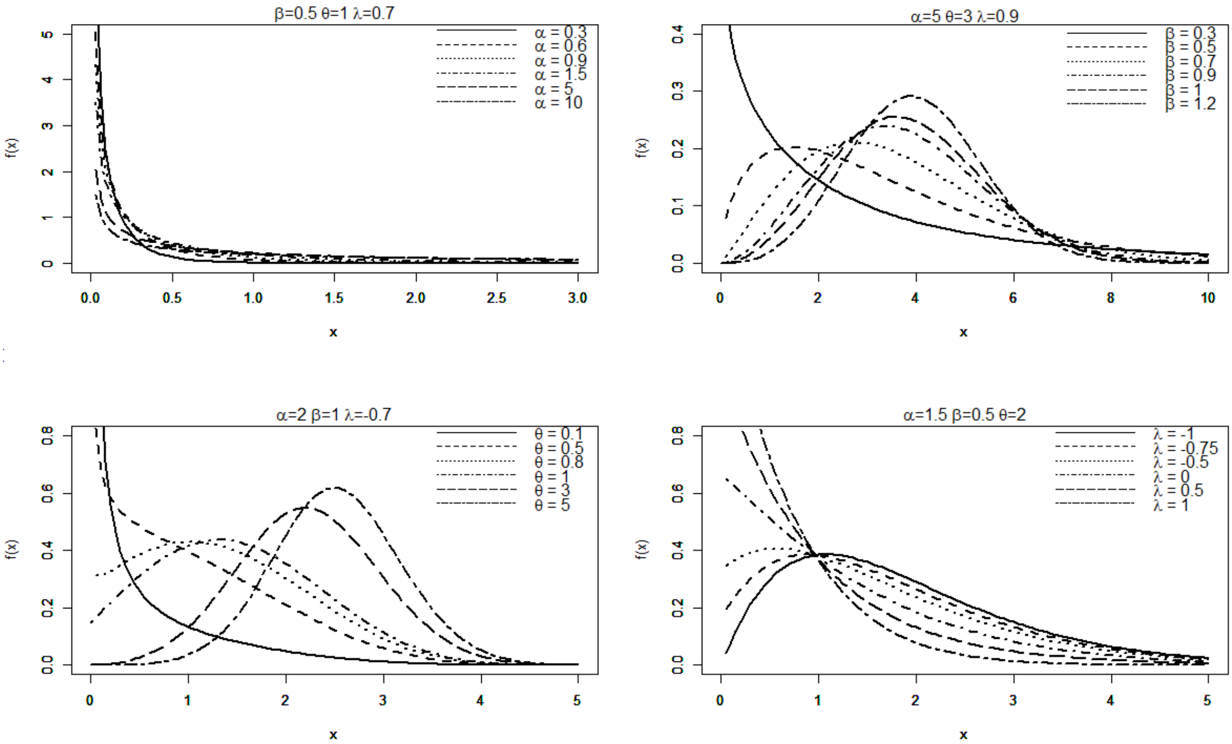

Figure 1. TCEP density functions

2.1. Moments

In this subsection, the raw moments of the TCEP$(\boldsymbol{\delta})$ distribution are derived, explicitly. We obtain the raw moments using following lemma under the condition that r/β is an integer.

Lemma 1 For $\upsilon,\mu>0$ and $m\in\mathbb{N}$

$$\int_1^{\infty} x^{\upsilon-1}(\log x)^m \exp(-\mu x)dx=\frac{\partial^m \mu^{-\nu}\Gamma(\nu,\mu)}{\partial \nu^m}, \quad m=0,1,2,... \tag{7}$$

where $\Gamma(a,x)=\int_x^{\infty} t^{a-1}\exp\{-t\}\,dt$ is the incomplete gamma function [8]. Using Lemma 1, following theorem gives the raw moments of TCEP$(\boldsymbol{\delta})$ distribution.

Theorem 1

If r/β is an positive integer, the r th moments of TCEP$(\boldsymbol{\delta})$ distribution are given by

$$\begin{aligned}E\left(X^r\right)=&(1+\lambda)\alpha^r\sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(\theta+1)\exp(j+1)}{\Gamma(\theta-j)\,j!}\left(\frac{\partial^m}{\partial\nu^m}(j+1)^{-\nu}\Gamma(\nu,j+1)\Big|_{\nu=1}\right)\\&-\lambda\alpha^r\sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(2\theta+1)\exp(j+1)}{\Gamma(2\theta-j)\,j!}\left(\frac{\partial^m}{\partial\nu^m}(j+1)^{-\nu}\Gamma(\nu,j+1)\Big|_{\nu=1}\right),\end{aligned} \tag{8}$$

where $r=1,2,\ldots$ and $\Gamma(\alpha)=\int_0^{\infty}t^{\alpha-1}\exp(-t)dt$ is the well-known gamma function.

Proof. Using pdf in Eq. (5), the raw moments can be written by

$$E\left(X^r\right)=\frac{(1+\lambda)\beta\theta}{\alpha^{\beta}}\int_0^{\infty}x^{r+\beta-1}\exp\left\{1+\left(\frac{x}{\alpha}\right)^{\beta}-\exp\left\{\left(\frac{x}{\alpha}\right)^{\beta}\right\}\right\}\left(1-\exp\left\{1-\exp\left\{\left(\frac{x}{\alpha}\right)^{\beta}\right\}\right\}\right)^{\theta-1}dx$$
$$-\frac{2\lambda\beta\theta}{\alpha^{\beta}}\int_0^{\infty}x^{r+\beta-1}\exp\left\{1+\left(\frac{x}{\alpha}\right)^{\beta}-\exp\left\{\left(\frac{x}{\alpha}\right)^{\beta}\right\}\right\}\left(1-\exp\left\{1-\exp\left\{\left(\frac{x}{\alpha}\right)^{\beta}\right\}\right\}\right)^{2\theta-1}dx \quad (9)$$

for $\theta>0$. Let us consider the identity

$$(1-z)^{\theta-1}=\sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(\theta)z^j}{\Gamma(\theta-j)\,j!}. \quad (10)$$

By using expansion (10) in (9), we can write

$$\mu_r'=\frac{(1+\lambda)\Gamma(\theta+1)\beta}{\alpha^{\beta}}\sum_{j=0}^{\infty}\frac{(-1)^j e^{j+1}}{\Gamma(\theta-j)\,j!}\int_0^{\infty}x^{r+\beta-1}\exp\left\{\left(\frac{x}{\alpha}\right)^{\beta}\right\}\exp\left\{-(j+1)\exp\left\{\left(\frac{x}{\alpha}\right)^{\beta}\right\}\right\}dx$$
$$-\frac{\lambda\Gamma(2\theta+1)\beta}{\alpha^{\beta}}\sum_{j=0}^{\infty}\frac{(-1)^j e^{j+1}}{\Gamma(2\theta-j)\,j!}\int_0^{\infty}x^{r+\beta-1}\exp\left\{\left(\frac{x}{\alpha}\right)^{\beta}\right\}\exp\left\{-(j+1)\exp\left\{\left(\frac{x}{\alpha}\right)^{\beta}\right\}\right\}dx. \quad (11)$$

Using transformation of $y=\exp\left\{\left(\frac{x}{\alpha}\right)^{\beta}\right\}$ in integrals in (11), we get

$$\mu_r'=(1+\lambda)\alpha^r\Gamma(\theta+1)\sum_{j=0}^{\infty}\frac{(-1)^j e^{j+1}}{\Gamma(\theta-j)\,j!}\int_0^{\infty}y^{1-1}(\log y)^{\frac{r}{\beta}}\exp\{-(j+1)y\}\,dy$$
$$-\lambda\alpha^r\Gamma(2\theta+1)\sum_{j=0}^{\infty}\frac{(-1)^j e^{j+1}}{\Gamma(2\theta-j)\,j!}\int_0^{\infty}y^{1-1}(\log y)^{\frac{r}{\beta}}\exp\{-(j+1)y\}\,dy \quad (12)$$

By using Lemma 1 in (12), the proof is completed.

2.2. Quantile function and random number generation

The quantile function of TCEP$(\boldsymbol{\delta})$ distribution is obtained by solving $F(x;\boldsymbol{\delta})=p$ for $p\in(0,1)$ and it is given by

$$x_p=\alpha\left[\log\left(1-\log\left(1-\left(\frac{1+\lambda-\sqrt{(\lambda+1)^2-4\lambda p}}{2\lambda}\right)^{1/\theta}\right)\right)\right]^{1/\beta}, \quad (13)$$

where $F(x;\boldsymbol{\delta})$ is given in (6).

2.3. Reliability and hazard functions

The reliability function and hazard function of TCEP$(\boldsymbol{\delta})$ distribution are given, respectively, by

$$R(t)=1-\left[(1+\lambda)\left[1-\exp\left(1-\exp\left\{\left(\frac{t}{\alpha}\right)^{\beta}\right\}\right)\right]^{\theta}-\lambda\left(\left[1-\exp\left(1-\exp\left\{\left(\frac{t}{\alpha}\right)^{\beta}\right\}\right)\right]^{\theta}\right)^2\right] \quad (14)$$

and

$$h(t)=\frac{g(t;\alpha,\beta)\left[1-\exp\left(1-\exp\left\{\left(\frac{t}{\alpha}\right)^{\beta}\right\}\right)\right]^{\theta-1}\left[1+\lambda-2\lambda\left[1-\exp\left(1-\exp\left\{\left(\frac{t}{\alpha}\right)^{\beta}\right\}\right)\right]^{\theta}\right]}{1-\left[(1+\lambda)\left[1-\exp\left(1-\exp\left\{\left(\frac{t}{\alpha}\right)^{\beta}\right\}\right)\right]^{\theta}-\lambda\left(\left[1-\exp\left(1-\exp\left\{\left(\frac{t}{\alpha}\right)^{\beta}\right\}\right)\right]^{\theta}\right)^{2}\right]}, \qquad (15)$$

where

$$g(t;\alpha,\beta)=\frac{\beta\theta t^{\beta-1}}{\alpha^{\beta}}\exp\left(1+\left(\frac{t}{\alpha}\right)^{\beta}-\exp\left\{\left(\frac{t}{\alpha}\right)^{\beta}\right\}\right).$$

Fig. 2 presents the shapes of the hazard function of TCEP$(\boldsymbol{\delta})$ distribution for selected parameter values.

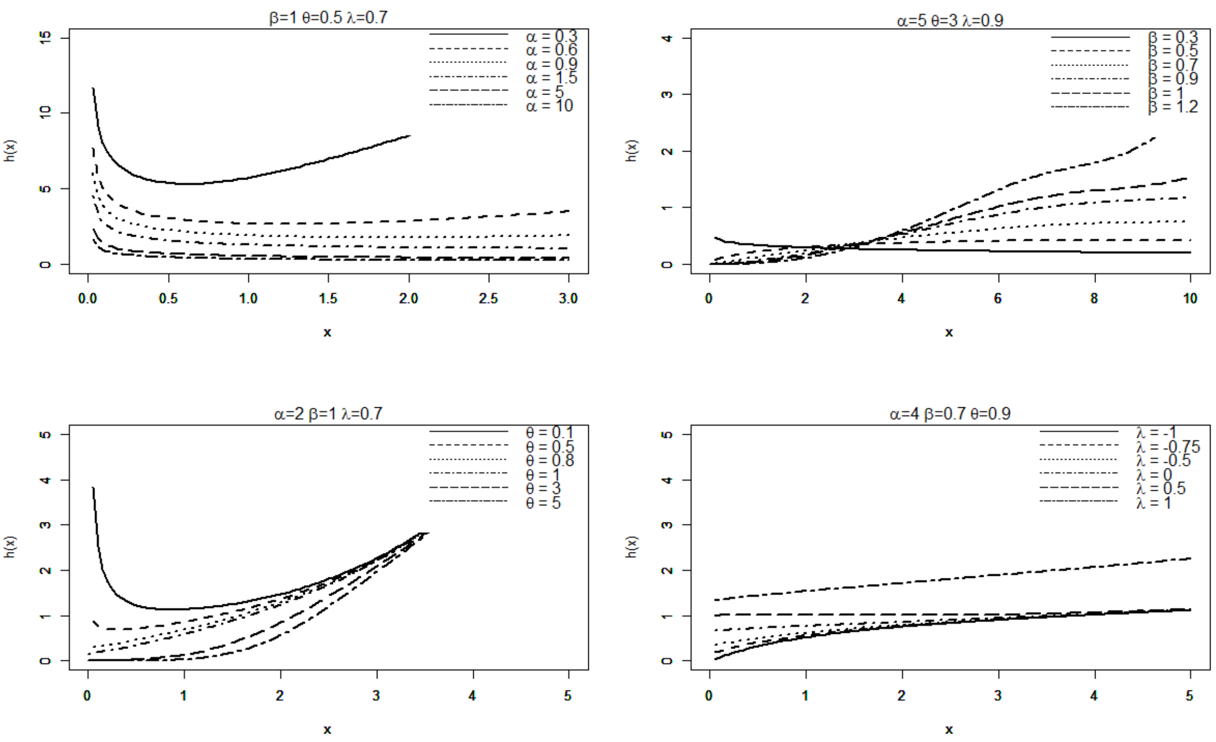

Figure 2. TCEP hazard functions

3. Maximum Likelihood Estimation and Asymtotic Confidence Intervals

Let $X_1, X_2, ..., X_n$ be the independent random variables having TCEP$(\boldsymbol{\delta})$ distribution. The log-likelihood function based on this sample is given by

$$\ell(\boldsymbol{\delta}\,|\,\mathbf{x}) = n\left(1+\log(\beta)+\log(\theta)-\beta\log(\alpha)\right)$$
$$+(\beta-1)\sum_{i=1}^{n}\log(x_i)+\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\beta}-\sum_{i=1}^{n}\exp\left(\left(\frac{x_i}{\alpha}\right)^{\beta}\right) \tag{16}$$
$$+\sum_{i=1}^{n}\log\left[\left(1-\exp\left(1-\exp\left\{\left(\frac{x_i}{\alpha}\right)^{\beta}\right\}\right)\right)^{\theta-1}\left(1+\lambda-2\lambda\left(1-\exp\left(1-\exp\left\{\left(\frac{x_i}{\alpha}\right)^{\beta}\right\}\right)\right)^{\theta}\right)\right]$$

and associated gradients found to be

$$\frac{\partial\ell(\boldsymbol{\delta}\,|\,\mathbf{x})}{\partial\alpha}=\frac{2\lambda\theta(\theta-1)\left[\sum_{i=1}^{n}\log\left(k(x,\alpha,\beta)\right)\left(k(x,\alpha,\beta)\right)^{\theta}\right]}{k(x,\alpha,\beta)}$$
$$+\frac{(\theta-1)\left(\frac{1}{2}+\frac{\lambda}{2}-\lambda\left(k(x,\alpha,\beta)\right)^{\theta}\right)\beta\left(\frac{x_i}{\alpha}\right)^{\beta}\exp\left(\left(\frac{x_i}{\alpha}\right)^{\beta}\right)\left(1-k(x,\alpha,\beta)\right)}{k(x,\alpha,\beta)} \tag{17}$$
$$+\frac{\beta}{\alpha}\left[\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\beta}\exp\left(\left(\frac{x_i}{\alpha}\right)^{\beta}\right)+\left(n+\frac{\sum_{i=1}^{n}x_i}{\alpha}\right)^{\beta}\right]$$

$$\frac{\partial\ell(\boldsymbol{\delta}\,|\,\mathbf{x})}{\partial\beta}=\frac{n}{\beta}-n\log(\alpha)+\sum_{i=1}^{n}\log(x_i)+\left(\frac{\sum_{i=1}^{n}x_i}{\alpha}\right)^{\beta}$$
$$+\log\left(\frac{\sum_{i=1}^{n}x_i}{\alpha}\right)-\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\beta}\log\left(\frac{x_i}{\alpha}\right)\exp\left(\left(\frac{x_i}{\alpha}\right)^{\beta}\right) \tag{18}$$
$$-2\lambda\theta\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\beta}\log\left(k(x,\alpha,\beta)\right)\left(k(x,\alpha,\beta)\right)^{\theta}$$
$$-\frac{(\theta-1)\left(\frac{1}{2}+\frac{\lambda}{2}-\lambda\left(k(x,\alpha,\beta)\right)^{\theta}\right)\exp\left(\left(\frac{x_i}{\alpha}\right)^{\beta}\right)\log\left(\frac{x_i}{\alpha}\right)}{k(x,\alpha,\beta)}$$

$$\frac{\partial\ell(\boldsymbol{\delta}\,|\,\mathbf{x})}{\partial\theta}=n\left(1+\frac{1}{\theta}+\lambda\right)-2\lambda\sum_{i=1}^{n}\left(k(x,\alpha,\beta)\right)^{\theta}+2\lambda(\theta+1)\log\left(k(x,\alpha,\beta)\right)\left(k(x,\alpha,\beta)\right)^{\theta} \tag{19}$$

$$\frac{\ell(\boldsymbol{\delta}\,|\,\mathbf{x})}{\partial\lambda}=\sum_{i=1}^{n}\log\left(\left(k(x,\alpha,\beta)\right)^{(\theta-1)}\left(1-2\left(k(x,\alpha,\beta)\right)^{\theta}\right)\right), \tag{20}$$

where

$$k(x,\alpha,\beta)=1-\exp\left(1-\exp\left(\left(\frac{x_i}{\alpha}\right)^{\beta}\right)\right).$$

The log-likelihood function $\ell(\boldsymbol{\delta}\,|\,\mathbf{x})$ can be maximized by using numerical methods such as Nelder-Mead. Let $\hat{\boldsymbol{\delta}}$ denote the MLEs of $\boldsymbol{\delta}$. Under some mild regularity conditions, one can write

$$\sqrt{n}\left(\hat{\boldsymbol{\delta}}-\boldsymbol{\delta}\right)\xrightarrow{d}N\left(\mathbf{0},I^{-1}(\boldsymbol{\delta})\right),$$

where

$$I(\boldsymbol{\delta})=\begin{pmatrix} -E\left[\frac{\ell^2(\boldsymbol{\delta}|\mathbf{x})}{\partial\alpha^2}\right] & -E\left[\frac{\ell^2(\boldsymbol{\delta}|\mathbf{x})}{\partial\alpha\partial\beta}\right] & -E\left[\frac{\ell^2(\boldsymbol{\delta}|\mathbf{x})}{\partial\alpha\partial\theta}\right] & -E\left[\frac{\ell^2(\boldsymbol{\delta}|\mathbf{x})}{\partial\alpha\partial\lambda}\right] \\ -E\left[\frac{\ell^2(\boldsymbol{\delta}|\mathbf{x})}{\partial\beta\partial\alpha}\right] & -E\left[\frac{\ell^2(\boldsymbol{\delta}|\mathbf{x})}{\partial\beta^2}\right] & -E\left[\frac{\ell^2(\boldsymbol{\delta}|\mathbf{x})}{\partial\beta\partial\theta}\right] & -E\left[\frac{\ell^2(\boldsymbol{\delta}|\mathbf{x})}{\partial\beta\partial\lambda}\right] \\ -E\left[\frac{\ell^2(\boldsymbol{\delta}|\mathbf{x})}{\partial\theta\partial\alpha}\right] & -E\left[\frac{\ell^2(\boldsymbol{\delta}|\mathbf{x})}{\partial\theta\partial\beta}\right] & -E\left[\frac{\ell^2(\boldsymbol{\delta}|\mathbf{x})}{\partial\theta^2}\right] & -E\left[\frac{\ell^2(\boldsymbol{\delta}|\mathbf{x})}{\partial\theta\partial\lambda}\right] \\ -E\left[\frac{\ell^2(\boldsymbol{\delta}|\mathbf{x})}{\partial\lambda\partial\alpha}\right] & -E\left[\frac{\ell^2(\boldsymbol{\delta}|\mathbf{x})}{\partial\lambda\partial\beta}\right] & -E\left[\frac{\ell^2(\boldsymbol{\delta}|\mathbf{x})}{\partial\lambda\partial\theta}\right] & -E\left[\frac{\ell^2(\boldsymbol{\delta}|\mathbf{x})}{\partial\lambda^2}\right] \end{pmatrix}$$

is expected Fisher information matrix. $I(\boldsymbol{\delta})$ can be approximated by $I(\hat{\boldsymbol{\delta}})$ which is observed Fisher Information Matrix. Using asymptotic normality of MLEs, we can write the approximate confidence intervals (CIs)

$$P\left(\hat{\delta}_i - z_{\frac{\eta}{2}}\sqrt{Var\left(\hat{\delta}_i\right)} < \delta_i < \hat{\delta}_i + z_{\frac{\eta}{2}}\sqrt{Var\left(\hat{\delta}_i\right)}\right) = 1-\eta,\ i=1,2,3,4, \tag{21}$$

where $Var\left(\hat{\delta}_i\right)$ is (i,i) (diagonal) elements of $I^{-1}(\hat{\boldsymbol{\delta}})$, $\boldsymbol{\delta}=(\delta_1,\delta_2,\delta_3,\delta_4)=(\alpha,\beta,\theta,\lambda)$ and $\hat{\boldsymbol{\delta}}=(\hat{\delta}_1,\hat{\delta}_2,\hat{\delta}_3,\hat{\delta}_4)=(\hat{\alpha},\hat{\beta},\hat{\theta},\hat{\lambda})$.

4. Simulation Study

In this section, Monte Carlo simulation study is performed in order to compare the performances of the MLEs of $\boldsymbol{\delta}$ according to MSE and bias. In the simulation study, the biases and MSEs of the MLEs are emprically estimated by 1000 trials. The sample sizes are fixed as 50,100,250,500,750, 1000, 5000 and four different parameter settings are considered. The bias and MSEs of MLEs are given in Table 1 while the avarage lengths (AL) and covarage probabilities (CPs) of MLEs for TCEP$(\boldsymbol{\delta})$ are presented in Table 2.

According to Table 1, when the sample size increases, the MSEs and bias of MLEs decrease for all selected parameters settings. On the other hand, it is observed that the CPs of confidence intervals approache to nominal level 0.95 and AL of intervals decrease when the sample size increases for all the parameters.

Table 1. Biases and MSEs of MLEs for TCEP $(\mathbf{\delta})$ parameters

					$\hat{\alpha}$		$\hat{\beta}$		$\hat{\theta}$		$\hat{\lambda}$	
α	β	θ	λ	n	bias	MSE	bias	MSE	bias	MSE	bias	MSE
1	0.4	0.5	0.2	50	0.3767	1.0402	0.3563	0.3860	-0.0253	0.2618	-0.0370	0.2322
				100	0.3976	0.8813	0.2401	0.1978	-0.0832	0.0744	-0.0249	0.2307
				250	0.3694	0.8274	0.1470	0.0937	-0.0739	0.0341	-0.0100	0.1889
				500	0.2476	0.5415	0.0859	0.0469	-0.0439	0.0228	-0.0069	0.1152
				750	0.2005	0.3822	0.0658	0.0300	-0.0370	0.0162	0.0079	0.0865
				1000	0.1567	0.2944	0.0511	0.0230	-0.0282	0.0134	0.0051	0.0642
				5000	0.0505	0.0511	0.0134	0.0029	-0.01	0.0028	0.0103	0.0135
0.5	0.7	0.8	-0.5	50	0.1391	0.0696	0.8293	2.4541	-0.1237	0.2535	0.3079	0.3827
				100	0.0896	0.0455	0.4032	0.9083	-0.0723	0.1339	0.1925	0.2941
				250	0.0685	0.0234	0.1779	0.1976	-0.0784	0.0568	0.1264	0.2003
				500	0.0470	0.0133	0.1037	0.0778	-0.0561	0.0380	0.0778	0.1330
				750	0.0309	0.0067	0.0613	0.0322	-0.0463	0.0267	0.0360	0.0912
				1000	0.0265	0.0041	0.0485	0.0164	-0.0329	0.0211	0.0472	0.0707
				5000	0.0066	0.0005	0.0104	0.0013	-0.0187	0.0076	-0.0106	0.0193
0.2	0.3	0.4	0.3	50	0.1110	0.0909	0.2334	0.1456	-0.0536	0.1047	-0.0963	0.2066
				100	0.1124	0.0790	0.1472	0.0741	-0.0629	0.0434	-0.0715	0.2099
				250	0.1062	0.0651	0.0952	0.0368	-0.0554	0.0214	-0.0324	0.1345
				500	0.0781	0.0384	0.0620	0.0198	-0.0361	0.0137	0.0100	0.0784
				750	0.0547	0.0276	0.0422	0.0133	-0.0228	0.0106	-0.0018	0.0619
				1000	0.0429	0.0213	0.0322	0.0100	-0.0175	0.0088	-0.0072	0.0470
				5000	0.0122	0.0035	0.0089	0.0016	-0.0066	0.0019	0.0049	0.0092
0.3	0.5	0.6	-0.2	50	0.1157	0.0596	0.5440	0.8602	-0.0728	0.3321	0.1192	0.2737
				100	0.1124	0.0469	0.3407	0.3864	-0.1340	0.0768	0.0939	0.2586
				250	0.0714	0.0275	0.1523	0.1123	-0.0872	0.0386	0.0447	0.1833
				500	0.0391	0.0123	0.0766	0.0398	-0.0586	0.0207	0.0116	0.1166
				750	0.0265	0.0076	0.0480	0.0223	-0.0467	0.0145	-0.0163	0.0905
				1000	0.0241	0.0052	0.0400	0.0136	-0.0370	0.0108	0.0110	0.0700
				5000	0.0034	0.0007	0.0055	0.0013	-0.0048	0.0015	0.0024	0.0113

Table 2. ALs and CPs of approximate CIs for TCEP$(\boldsymbol{\delta})$ parameters

					α		β		θ		λ	
α	β	θ	λ	n	AL	CP	AL	CP	AL	CP	AL	CP
1	0.4	0.5	0.2	50	3.6072	0.6150	1.1817	0.5910	1.6266	0.6170	2.2592	0.8660
				100	3.5897	0.6340	1.0703	0.6690	1.0541	0.6820	1.9235	0.7900
				250	2.9860	0.6940	0.8319	0.7300	0.7323	0.7580	1.5404	0.7540
				500	2.3293	0.7470	0.6178	0.7910	0.5688	0.8090	1.2633	0.8190
				750	1.9973	0.7890	0.5281	0.8120	0.4747	0.8480	1.0483	0.8450
				1000	1.7505	0.8190	0.4572	0.8340	0.4230	0.8490	0.9403	0.8710
				5000	0.8132	0.9270	0.1954	0.9320	0.1990	0.9320	0.4342	0.9430
0.5	0.7	0.8	-0.5	50	0.9020	0.8130	2.7776	0.8280	2.1747	0.7710	2.3373	0.8520
				100	0.6988	0.8890	1.7721	0.8960	1.6363	0.8540	1.9090	0.8570
				250	0.5046	0.9410	1.0998	0.9630	1.0700	0.8890	1.5006	0.8550
				500	0.3435	0.9440	0.6911	0.9680	0.7879	0.9200	1.1748	0.8720
				750	0.2611	0.9560	0.4650	0.9730	0.6515	0.9210	0.9884	0.8540
				1000	0.2316	0.9620	0.4105	0.9790	0.5813	0.9300	0.9291	0.8820
				5000	0.0835	0.9510	0.1363	0.9610	0.2905	0.9130	0.4720	0.8800
0.2	0.3	0.4	0.3	50	1.0002	0.5600	0.6802	0.4830	1.0916	0.5180	2.1066	0.9210
				100	0.9593	0.5970	0.6419	0.5910	0.8196	0.6170	1.7801	0.8400
				250	0.7696	0.6350	0.5399	0.6640	0.5735	0.6890	1.3525	0.8000
				500	0.7097	0.6850	0.4583	0.7100	0.4927	0.7440	1.1476	0.8340
				750	0.5561	0.7170	0.3819	0.7590	0.4127	0.7880	0.9511	0.8190
				1000	0.4816	0.7400	0.3214	0.7780	0.3558	0.8000	0.8254	0.8440
				5000	0.2211	0.8890	0.1482	0.8910	0.1710	0.8920	0.3818	0.9150
0.3	0.5	0.6	-0.2	50	0.8327	0.6900	1.6557	0.6850	1.6997	0.6600	2.3687	0.8480
				100	0.7394	0.8030	1.3941	0.8130	1.0630	0.7440	1.9961	0.7930
				250	0.5439	0.9000	0.9379	0.9100	0.7644	0.8480	1.5755	0.8260
				500	0.3764	0.9370	0.5872	0.9440	0.5535	0.8870	1.2496	0.8800
				750	0.2855	0.9450	0.4248	0.9550	0.4371	0.8940	1.0186	0.8860
				1000	0.2587	0.9540	0.3786	0.9610	0.3810	0.9080	0.9058	0.9170
				5000	0.0984	0.9400	0.1330	0.9390	0.1500	0.9490	0.3973	0.9550

5. Real Data Analysis I

In this section, an application with real data is provided to compare the fitting ability of TCEP$(\boldsymbol{\delta})$ distribution with some lifetime distributions such as Complementary Exponential Power (CEP) [1], Log-Kumaraswamy (LKw) [9], Weibull and Exponentiated Exponential (EE) [10]. The pdfs of these distributions are given by

$$LKw: \quad f(x)=\alpha\beta e^{-x}\left(1-e^{-x}\right)^{\alpha-1}\left[1-\left(1-e^{-x}\right)^{\alpha}\right]^{\beta-1} I_{\mathbb{R}_+}(x)$$

$$Weibull: f(x)=\frac{\alpha}{\beta^{\alpha}}x^{\alpha-1}e^{-\left(\frac{x}{\beta}\right)^{\alpha}} I_{\mathbb{R}_+}(x)$$

$$EE: \quad f(x)=\alpha\beta\left(1-e^{-\beta x}\right)^{\alpha-1}e^{-\beta x}I_{\mathbb{R}_+}(x)$$

where $\alpha,\beta>0$ are parameters. We have considered the comparison criteria as the -2×log-likelihood value, Akaike's Information Criterion (AIC), Kolmogorov-Smirnov test statistics (KS) and its (p-value) as comparison criteria. The data related to the failure stresses of single carbon fibers (length 1mm) is considered in the analysis. Note that this data firstly analyzed by [11]. The MLEs and the selection criteria statistics are given in Table 3. Furthermore, Fig. 3 presents the fitted cdfs to real data.

Table 3. Selection criteria statistics and MLEs for carbon fibres data

	Weibull	**TCEP**	**LKw**	**EE**	**CEP**
LogL	-71.0240	**-69.9704**	-72.0352	-73.7699	-70.0187
-2LogL	142.0479	**139.9408**	144.0705	147.5399	140.0375
AIC	146.0479	147.9408	148.0705	151.5399	**146.0375**
BIC	**150.1340**	156.1130	152.1566	155.6260	152.1666
CAIC	**146.2701**	148.7100	148.2927	151.7621	146.4903
HQIC	**147.6359**	151.1168	149.6585	153.1279	148.4195
K-S	0.0859	**0.0576**	0.0803	0.0967	0.0603
A*	0.3882	**0.1521**	0.4354	0.6913	0.1595
W*	0.0591	**0.0212**	0.0729	0.1067	0.0227
p-value(K-S)	0.7618	**0.9859**	0.8273	0.6261	0.9777
p-value (A*)	0.8598	**0.9985**	0.8124	0.5655	0.9978
p-value (W*)	0.8219	**0.9961**	0.7359	0.5545	0.9941
$\hat{\alpha}$	4.5752	4.0176	68.7284	114.5288	3.5979
$\hat{\beta}$	5.5930	1.5294	1.7687	1.2421	1.3105
$\hat{\theta}$	-	5.7577	-	-	7.5679
$\hat{\lambda}$	-	0.3693	-	-	
LB for α	4.3507	1.9357	47.7744	15.5453	2.0178
LB for β	4.4972	0.0795	1.0788	1.0039	0.2921
LB for θ	-	-4.7440	-	-	-5.2705
LB for λ	-	-1.1059	-	-	-
UB for α	4.7998	6.0995	89.6825	213.5122	5.1779
UB for β	6.6887	2.9793	2.4587	1.4804	2.3290
UB for θ	-	16.2593	-	-	20.4064
UB for λ	-	1.8444	-	-	-
SE of $\hat{\alpha}$	0.1146	1.0622	10.6910	50.5027	0.8062
SE of $\hat{\beta}$	0.5591	0.7397	0.3520	0.1215	0.5196
SE of $\hat{\theta}$	-	5.3581	-	-	6.5503
SE of $\hat{\lambda}$	-	0.7526	-	-	-

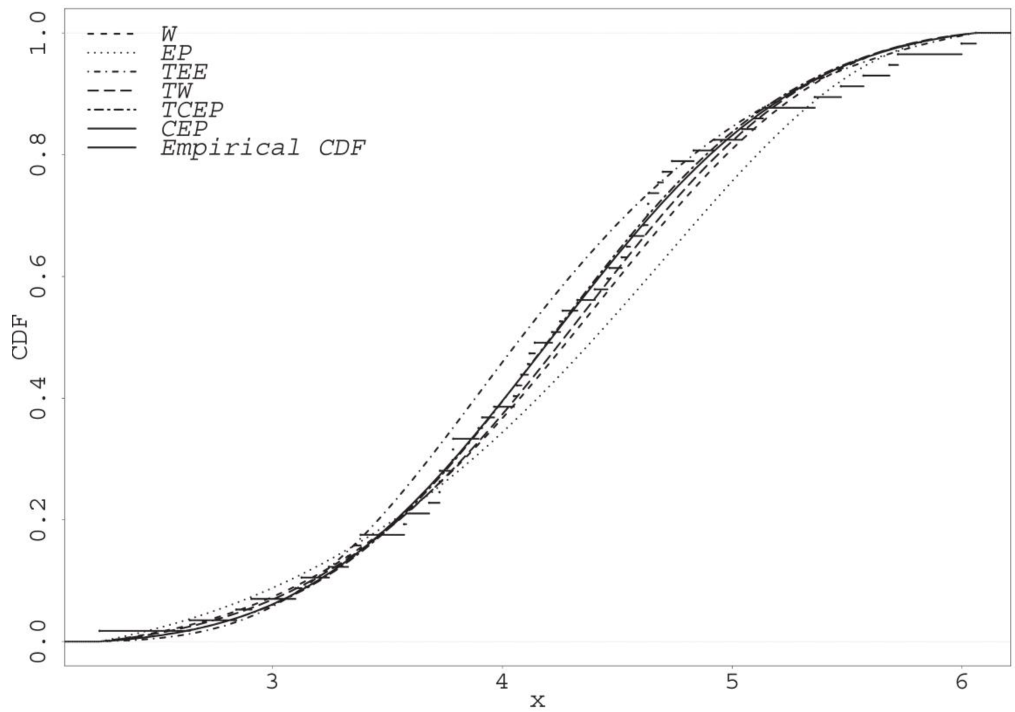

Figure 3. Fitted cdfs and empirical cdf for carbon fibres data

From Table 3 and Fig. 3, it can be said that the $\text{TCEP}(\boldsymbol{\delta})$ distribution is candidate to fitting the real data and it is competitor to the other existing models according to all criteria discussed here.

6. Real Data Analysis II

Let us consider lifetime regression analysis and let $Y=\log(X)$. Then cdf and pdf of Y is given by

$$F_1(y;\boldsymbol{\kappa})=(1+\lambda)\left[1-\exp\left(1-\exp\left\{\exp\left(\frac{y-\mu}{\sigma}\right)\right\}\right)\right]^{\theta} -\lambda\left[1-\exp\left(1-\exp\left\{\exp\left(\frac{y-\mu}{\sigma}\right)\right\}\right)\right]^{2\theta}, \tag{22}$$

and

$$f_1(y;\boldsymbol{\tau})=\frac{\theta}{\sigma}\exp\left(\frac{y-\mu}{\sigma}\right)\exp\left(1+\exp\left(\frac{y-\mu}{\sigma}\right)-\exp\left\{\exp\left(\frac{y-\mu}{\sigma}\right)\right\}\right) \times\left[1-\exp\left(1-\exp\left\{\exp\left(\frac{y-\mu}{\sigma}\right)\right\}\right)\right]^{\theta-1} \times\left[1+\lambda-2\lambda\left[1-\exp\left(1-\exp\left\{\exp\left(\frac{y-\mu}{\sigma}\right)\right\}\right)\right]^{\theta}\right]I_{\mathbb{R}}(y) \tag{23}$$

where $\boldsymbol{\kappa}_1=(\mu,\sigma,\theta,\lambda)$ is parameter vector. Let $\mu=0$ and $\sigma=1$ in Eq. (22) . Then, Eq. (22) is reduce to

$$F_1(z;\boldsymbol{\kappa})=(1+\lambda)\left[1-\exp\left(1-\exp\left\{\exp(z)\right\}\right)\right]^{\theta}-\lambda\left[1-\exp\left(1-\exp\left\{\exp(z)\right\}\right)\right]^{2\theta}. \tag{24}$$

Let us consider regression model

$$\mathbf{Y} = \boldsymbol{\mu} + \sigma \mathbf{Z},$$

where $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_n)^{\mathrm{T}}$ random vector and $Y_1, Y_2, \ldots, Y_n$ are iid random variables (they are also called dependent variables) with cdf (22). $\mathbf{Z} = (Z_1, Z_2, \ldots, Z_n)^{\mathrm{T}}$ is a random error vector and $Z_1, Z_2, \ldots, Z_n$ are iid random variables with cdf (24) and $\sigma > 0$. Assume that location is linked to covariates by $\boldsymbol{\mu} = \mathbf{X}\boldsymbol{\beta}$, where $\mathbf{X}$ is $n \times (p+1)$ matrix consist of covariates(First coloumn is $\mathbf{1}$) and $\boldsymbol{\beta} = (\beta_0, \beta_1, \ldots, \beta_p)^{\mathrm{T}}$. Let $T_i = \min(Y_i, c_i)$ and c_i is censoring time for i th individual or any component. Then the log-likelihod function is written by

$$\ell(\boldsymbol{\kappa}) = \sum_{i=1}^{n} \left\{ \omega_i \log\left(f_1(t_i; \boldsymbol{\kappa}) \right) + (1 - \omega_i) \log\left(1 - F_1(t_i; \boldsymbol{\kappa}) \right) \right\}, \quad (25)$$

where ω_i is the indicator function given by

$$\omega_i = \begin{cases} 1 & , \quad t_i \le c_i \\ 0 & , \quad t_i > c_i \end{cases}$$

Let us consider the data given in page 335 in [12]. [13] carried out an experiment and obtained a data on the lifetime of specimens of solid epoxy electrical-insulation in an accelerated voltage life test. 20 specimens were put on a life test at each of three voltage levels: 52.5, 55.0, and 57.5 kV. Failure times were measured in minutes. Six lifetimes of specimens are censored at a random. Based on the data, the log-likelihood (25) is maximized and the MLEs of parameters, AIC criteria are presented in Table 4 for TCEP regression. For a comparison Weibull and TLGBXII (see [14]) regression results are also given in Table 4. From the Table 4 and Fig. 4, it can be conclude that TCEP regression can be alternative lifetime regression analysis to Weibull and TLGBXII regression.

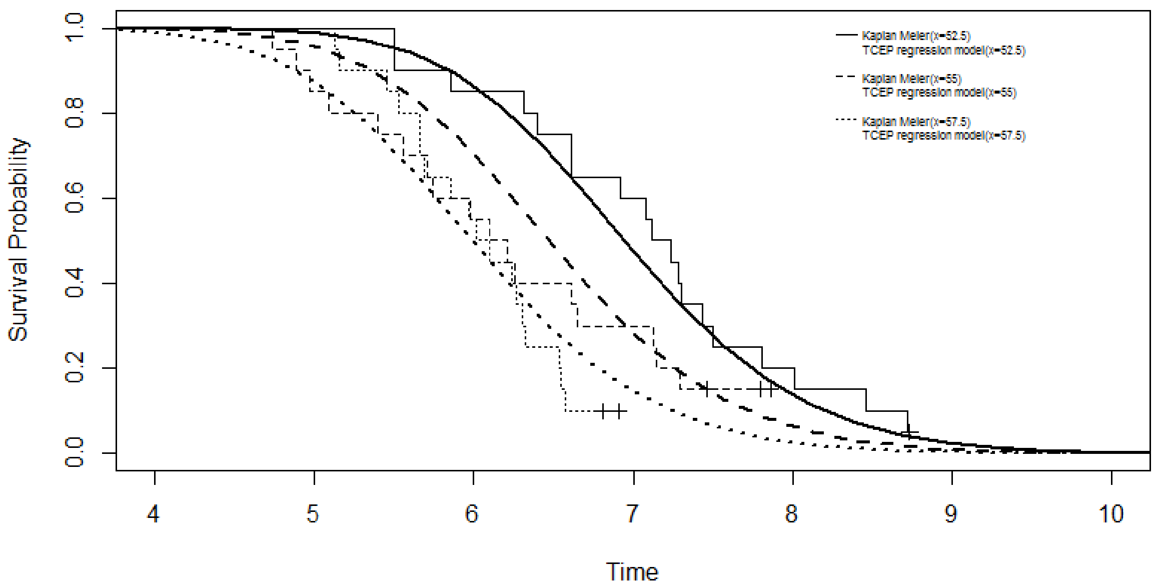

Figure 4. Fitted survival functions and the empirical survivals

Table 4. Lifetime regression analysis results: MLEs of the parameters, standard errors in second line, p-values in [·]) and the AIC statistics.

Model		θ	β	β_0	β_1	σ	$-\ell$	AIC
TCEP	$\lambda=0.6184$	126.8443		11.3612	-0.19132	9.9420	78.3754	166.7
	0.5023	257.2796		6.6557	0.0598	5.8034		
				[0.0878]	[0.0014]			
TLGBXII	$\gamma=0.6860$	7.7247	0.7089	14.4513	-0.1790	0.8024	78.2	168.4
	0.685	7.027	0.984	4.876	0.074	0.827		
Weibull				22.000	-0.274	0.845	83.7	173.4
				3.046	0.055	0.09		

In the lifetime regression analysis, 1000 initial parameters are generated uniformly and the best ten solutions (in the likelihood manner) are presented in Table 5. The estimates are treated MLEs which gives maximum likelihood. In Table 5, Method "1" indicates Nelder-Mead whereas Method "2" indicates BFGS.

Table 5. First 10 best solutions with initial values for TCEP regression

Trial	**Method**	$-\ell$	$\hat\beta_0$	$\hat\beta_1$	$\hat\sigma$	$\hat\theta$	$\hat\lambda$	se $(\hat\beta_0)$	se $(\hat\beta_1)$	se $(\hat\sigma)$	se $(\hat\theta)$	se $(\hat\lambda)$	$\beta_0^{(0)}$	$\beta_1^{(0)}$	$\sigma^{(0)}$	$\theta^{(0)}$	$\lambda^{(0)}$
359	1	-78.375	11.361	-0.191	9.942	126.844	0.619	6.656	0.060	5.803	257.279	0.502	26.936	-0.357	0.845	0.620	0.366
971	1	-78.455	12.328	-0.192	8.719	86.939	0.464	3.994	0.059	2.617	70.926	0.640	21.215	-0.225	0.840	0.894	0.685
262	2	-78.483	13.115	-0.198	8.318	72.457	0.509	4.963	0.059	3.688	101.625	0.570	27.251	-0.048	0.670	0.905	0.892
309	2	-78.494	13.225	-0.199	8.221	69.651	0.510	4.904	0.059	3.610	96.054	0.565	16.533	-0.096	1.146	0.541	0.103
631	2	-78.497	13.205	-0.198	8.216	69.078	0.517	4.868	0.059	3.579	93.937	0.556	15.586	-0.073	1.352	0.733	0.910
487	2	-78.513	12.499	-0.186	8.287	70.057	0.523	4.481	0.059	3.122	79.607	0.532	27.940	-0.330	1.276	0.640	0.161
727	2	-78.521	13.492	-0.200	8.003	63.720	0.509	4.872	0.059	3.541	87.711	0.558	22.355	-0.118	1.386	0.138	0.374
218	1	-78.528	12.926	-0.193	8.255	67.090	0.587	4.058	0.059	2.715	60.461	0.473	26.250	-0.356	0.991	0.727	0.879
781	2	-78.535	13.363	-0.197	7.946	61.167	0.521	4.547	0.059	3.182	73.457	0.541	19.195	0.260	1.329	0.730	0.276
890	1	-78.537	12.225	-0.196	8.658	102.473	0.090	4.362	0.085	2.547	138.953	3.308	23.208	-0.295	0.839	0.903	0.297

Conflicts of interest

There is no conflict of interest.

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