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Certain results on Kenmotsu manifolds

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Abstract

In this paper, we focus on Kenmotsu manifolds. Firstly, we investigate almost quasi Ricci symmetric Kenmotsu manifolds. Then, we study Kenmotsu manifold admitting a Yamabe soliton. We find that if the soliton field V of the Yamabe soliton is orthogonal to the characteristic vector field ξ, then it is Killing and the manifold has constant scalar curvature. Also, we deal with a Kenmotsu manifold which admits a quasi-Yamabe soliton. Finally, we give an example which verify our results.

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1. Introduction

Considering the recent stage of the developments in contact geometry, there is an impression that the geometers are focused on problems in almost contact metric geometry. Many different classes of almost contact structures are defined in the literature such as Sasakian [1], Kenmotsu [2], almost cosymplectic [3], trans-Sasakian [4] and others.

In 1969, Tanno classified connected almost contact metric manifolds whose automorphism groups possess the maximum dimension [5]. For such manifolds, the sectional curvature of plane sections containing ξ is a constant c and it was proved that they can be divided into three classes: $i)$ Homogenous contact Riemannian manifolds with $c>0$. $ii)$ Global Riemannian products of a line or a circle with a Kaehler manifold of constant holomorphic sectional curvature if $c=0$. $iii)$ A warped product space $\mathbb{R}\times_f \mathbb{C}$ if $c<0$. In 1972, Kenmotsu investigated the differential geometric properties of the manifolds of class $iii)$ and obtained structure is now as known Kenmotsu structure [2]. After this work, such manifolds have been studied extensively by many mathematicians.

The notion of Yamabe soliton in Riemannian geometry was introduced by Hamilton at the same time as the Ricci flow in 1988 [6]. This notion corresponds to the self-similar solution of Hamilton's Yamabe flow. In dimension $n=2$ the Yamabe flow is equivalent to to the Ricci flow. However, In dimension $n>2$ the Yamabe and the Ricci flows do not agree, since the Yamabe flow conserve the conformal class of metric the metric but the Ricci flow does not in general. Yamabe soliton is a special solution of the Yamabe flow that moves by one parameter family of diffeomorphisms φ_t generated by a fixed vector field V on a Riemannian manifold M. Also, Yamabe flow is natural geometric deformation to metrics of constant scalar curvature. Therefore, Yamabe solitons have been studied intensively in mathematics as well as physics. For the recent studies on Yamabe solitons, we refer to ([7]-[11]).

A Riemannian manifold (M,g) is called a Yamabe soliton if there exists a real number λ and a vector field $V\in\Gamma(TM)$ such that

$$(L_V g)(X,Y)=(\lambda-r)g(X,Y), \tag{1}$$

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where $\Gamma(TM)$ denotes the set of differentiable vector fields on M, $L_V g$ denotes the Lie-derivative of the metric tensor g along vector field V, r is the scalar curvature of M and X,Y are arbitrary vector fields on M. Also, a vector field V as in the definition is called a soliton field for (M,g). A Yamabe soliton which satisfies (1) is denoted by (M,g,V,λ). If $L_V g = \rho g$, then the soliton field V is said to be conformal Killing, where ρ is a function. If ρ vanishes identically, then V is said to be Killing. Moreover, if V is zero or Killing in (1), then the Yamabe soliton reduces to a manifold of constant scalar curvature. In addition, a Yamabe soliton is called a gradient if the soliton field V is the gradient of a potential function f (i.e., $V = \nabla f$) and is called shrinking, steady or expanding depending on $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$, respectively.

On the other hand, in 2018, Chen and Deshmukh defined the notion of quasi-Yamabe soliton as a generalization of Yamabe solitons on a Riemannian manifold (M,g) as follows [12]:

$$(L_V g)(X,Y) = (\lambda - r) g(X,Y) + \mu V^*(X) V^*(Y), \tag{2}$$

where V^* stands for the dual $1-$form of V, λ is a constant and μ is a smooth function on M. The vector field V is also called soliton field for the quasi-Yamabe soliton. A quasi-Yamabe soliton is denoted by (M,g,V,λ).

Motivated by the above studies, we concentrate on Yamabe and quasi-Yamabe solitons on Kenmotsu manifolds. Also, we study almost quasi Ricci symmetric Kenmotsu manifolds.

The paper is organized as follows:

Section 1 is devoted to the introduction. In section 2, we give some basic notions which are going to be needed. In section 3, we consider almost quasi Ricci symmetric Kenmotsu manifolds. Then, we study Kenmotsu manifold admitting a Yamabe soliton. Also, we discuss quasi-Yamabe soliton on such a manifold and give some important characterizations. Finally, we give an example to support our results.

2. Preliminaries

In this section, we recall some notions of almost contact metric manifolds from [2], [13] and [14].

A $(2n+1)-$dimensional smooth manifold M is an almost contact metric manifold equipped with an almost contact metric structure (φ,ξ,g) such that φ is a tensor field of type $(1,1)$, ξ is a vector field (called the characteristic vector field) of type $(1,0)$, $1-$form η is a tensor field of type $(0,1)$ on M and the Riemannian metric g satisfies the following relations:

$$\varphi^2 X = -X + \eta(X)\xi, \tag{3}$$

$$\eta(\xi) = 1, \tag{4}$$

$$\varphi\xi = 0, \tag{5}$$

$$\eta \circ \varphi = 0 \tag{6}$$

and

$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{7}$$

$$g(X, \varphi Y) = -g(\varphi X, Y), \tag{8}$$

$$\eta(X) = g(X,\xi) \tag{9}$$

for any $X,Y \in \Gamma(TM)$.

If the following condition is satisfied for an almost contact metric manifold (M,φ,ξ,g) then it is called a Kenmotsu manifold

$$(\nabla_X\varphi)Y = g(\varphi X,Y)\xi - \eta(Y)\varphi X,$$

where ∇ is the Levi-Civita connection on M. For a Kenmotsu manifold we also have

$$\nabla_X\xi = X - \eta(X)\xi, \tag{10}$$

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{11}$$

$$R(X,\xi)Y = g(X,Y)\xi - \eta(Y)X, \tag{12}$$

$$S(X,\xi) = -2n\eta(X), \tag{13}$$

$$S(\xi,\xi) = -2n, \tag{14}$$

$$Q\xi = -2n\xi, \tag{15}$$

where S and R are the Ricci tensor and Riemann curvature tensor of M, respectively and Q is the Ricci operator defined by $S(X,Y) = g(QX,Y)$.

On the other hand, a Riemannian manifold (M,g) is called $\eta-$Einstein if there exists two smooth functions a and b such that its Ricci tensor S satisfies

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y) \tag{16}$$

for any $X,Y \in \Gamma(TM)$. If the function b vanishes identically in (16), then the manifold M becomes an Einstein.

Also, a non-flat semi-Riemannian manifold (M^{2n+1},g) $(n \geq 1)$ is called almost quasi Ricci symmetric manifold if its Ricci tensor field S satisfies [15]

$$(\nabla_X S)(Y,Z) = [A(X)+B(X)]S(Y,Z) - A(Y)S(X,Z) - A(Z)S(X,Y) \tag{17}$$

for any $X,Y,Z \in \Gamma(TM)$, where A and B are non-zero $1-$forms. If $A = B$ in (17), then M becomes a quasi Ricci symmetric manifold.

3. Main Results

3.1. Almost quasi Ricci symmetric manifold

In this section, we deal with almost quasi Ricci symmetric Kenmotsu manifold. We begin to this section with the following:

Theorem 1: If M is an almost quasi Ricci symmetric Kenmotsu manifold, then it becomes a quasi Ricci symmetric manifold.

Proof: Let us consider that M is an almost quasi Ricci symmetric Kenmotsu manifold. Putting $Z = \xi$ in (17), we have

$$(\nabla_X S)(Y,\xi) = [A(X)+B(X)]S(Y,\xi) - A(Y)S(X,\xi) - A(\xi)S(X,Y) \tag{18}$$

such that

$$(\nabla_X S)(Y,\xi) = \nabla_X S(Y,\xi) - S(\nabla_X Y,\xi) - S(Y,\nabla_X \xi) \tag{19}$$

for any $X,Y \in \Gamma(TM)$. From (9), (10), (13) and (19), we derive

$$(\nabla_X S)(Y,\xi) = -S(X,Y) - 2ng(X,Y). \tag{20}$$

Using the equality (13) in the right side of (18), then the equation (18) becomes

$$(\nabla_X S)(Y,\xi) = -2n[A(X)+B(X)]\eta(Y) + 2n\eta(X)A(Y) - A(\xi)S(X,Y). \tag{21}$$

Therefore, by combining (20) and (21) we get

$$-S(X,Y) - 2ng(X,Y) = -2n[A(X)+B(X)]\eta(Y) + 2n\eta(X)A(Y) - A(\xi)S(X,Y). \tag{22}$$

Taking $Y=\xi$ in (22) and making use of (4), (13) gives

$$0 = -2n[A(X)+B(X)] + 4n\eta(X)A(\xi)$$

which is equivalent to

$$A(X)+B(X) = 2\eta(X)A(\xi). \tag{23}$$

Replacing X by ξ in (23) yields

$$B(\xi) = A(\xi). \tag{24}$$

Again taking $X=\xi$ in (22) and by virtue of (4), (13) and (24) we find that

$$0 = -2n\eta(Y)A(\xi) + 2nA(Y),$$

namely

$$A(Y) = A(\xi)\eta(Y). \tag{25}$$

From (23) and (25), we obtain

$$A(X) = B(X) = A(\xi)\eta(X).$$

This completes the proof of the theorem.

As a result of the Theorem 1, we can give the following corollary.

Corollary 1: If M is an almost quasi Ricci symmetric Kenmotsu manifold satisfying $A(\xi) \neq 1$, then it is an $\eta-$Einstein manifold.

Proof: From (22) and (25), we have

$$-S(X,Y)-2ng(X,Y)=-2nA(\xi)\eta(X)\eta(Y)-A(\xi)S(X,Y).$$

If we rearrange the last equation, we get

$$S(X,Y)=\frac{2n}{A(\xi)-1}g(X,Y)-\frac{2nA(\xi)}{A(\xi)-1}\eta(X)\eta(Y)$$

which implies that the manifold M is an $\eta-$Einstein. Therefore, the proof is completed.

3.2. Quasi-Yamabe solitons on Kenmotsu manifold

In this section, we study quasi-Yamabe solitons on a Kenmotsu manifold and give an important characterization. The first result of this section is the following:

Proposition 1: Let M be a Kenmotsu manifold admitting a Yamabe soliton as its soliton field $V.$ If the vector field V is orthogonal to $\xi,$ then V is Killing on M and the manifold M has constant scalar curvature.

Proof: Since M admits a Yamabe soliton, from (1) we write

$$g(\nabla_X V,Y)+g(\nabla_Y V,X)=(\lambda-r)g(X,Y) \tag{27}$$

for any $X,Y\in\Gamma(TM).$ Subsituting $X=Y=\xi$ in (27) gives

$$2g(\nabla_\xi V,\xi)=(\lambda-r). \tag{28}$$

On the other hand, it is easy to see that $g(\nabla_\xi V,\xi)=\nabla_\xi(g(V,\xi)).$ Using this equality in (28), we have

$$\nabla_\xi(g(V,\xi))=(\lambda-r). \tag{29}$$

Now, we assume that the vector field V is orthogonal to $\xi.$ Then, from (29) we get

$$r=\lambda \tag{30}$$

which implies that M is a manifold of constant scalar curvature. Furthermore, using (30) in (27) we obtain

$$(L_V g)(X,Y)=g(\nabla_X V,Y)+g(\nabla_Y V,X)=0. \tag{31}$$

This means that the vector field V is Killing on $M.$ Thus, the proof of the proposition is completed.

Theorem 2: Let M be a $(2n+1)-$dimensional Kenmotsu manifold satisfying the condition $R.Q=0.$ If M admits a Yamabe soliton as its soliton field $\xi,$ then the Yamabe soliton is shrinking.

Proof: Let us consider a Kenmotsu manifold M satisfies the condition $(R(X,Y).Q)Z=0,$ that is,

$$R(X,Y)QZ-Q(R(X,Y)Z)=0 \tag{32}$$

for any $X,Y,Z \in \Gamma(TM)$, where R is the Riemannian curvature of M and Q is the Ricci operator defined by $S(X,Y)=g(QX,Y)$. Replacing Y by ξ in (32), we have

$$R(X,\xi)QZ-Q(R(X,\xi)Z)=0. \tag{33}$$

With the help of (12), (13), (15) and (33), we derive

$$S(X,Z)\xi+2n\eta(Z)X+2ng(X,Z)\xi+\eta(Z)QX=0. \tag{34}$$

Taking the inner product of (34) with ξ and using (4), (13), we obtain

$$S(X,Z)=-2ng(X,Z). \tag{35}$$

Contracting over X,Z in (35), we get

$$r=-2n(2n+1). \tag{36}$$

On the other hand, from (1) and (10) one has

$$2g(X,Y)-2\eta(X)\eta(Y)=(\lambda-r)g(X,Y). \tag{37}$$

By setting $X=Y=\xi$ in (37) and from (36), we deduce that $\lambda=-2n(2n+1)$. This implies that the Yamabe soliton is shrinking. This is the desired result.

Now, we are ready to give the main theorem of this section.

Theorem 3: Let M be a Kenmotsu manifold. If M admits a quasi-Yamabe soliton as its soliton field ξ, then M is a manifold of constant scalar curvature.

Proof: Since η is the dual $1-$form of the characteristic vector field ξ, the equation (2) becomes

$$(L_\xi g)(X,Y)=(\lambda-r)g(X,Y)+\mu\eta(X)\eta(Y) \tag{38}$$

for any $X,Y \in \Gamma(TM)$. Also, it follows from the definition of the Lie-derivative and from (10), one has

$$\begin{aligned}(L_\xi g)(X,Y)&=g(\nabla_X\xi,Y)+g(\nabla_Y\xi,X)\\&=2g(X,Y)-2\eta(X)\eta(Y).\end{aligned} \tag{39}$$

By combining (38) and (39) we have

$$2g(X,Y)-2\eta(X)\eta(Y)=(\lambda-r)g(X,Y)+\mu\eta(X)\eta(Y). \tag{40}$$

If we put $X=Y=\xi$ in equation (40), then we get

$$\lambda-r+\mu=0. \tag{41}$$

On the other hand, let $\{e_1, e_2, ..., e_{2n+1} = \xi\}$ be an orthonormal basis of the tangent space $T_p M$, at each point $p \in M$. By setting $X = Y = e_i$ in (40) and taking summation over i $(i = 1, 2, ..., 2n+1)$, we deduce

$$4n = (\lambda - r)(2n+1) + \mu. \tag{42}$$

Using the equalities (41) and (42), we obtain

$$\mu = -2. \tag{43}$$

Therefore, from (41) and (43) we have $r = \lambda - 2$ which means that the manifold M has constant scalar curvature. Thus, we get the requested result.

Example 1: [16] We consider the three-dimensional Riemannian manifold

$$M = \{(x, y, z) \in \mathbb{R}^3,\ (x, y, z) \neq (0,0,0)\},$$

and the linearly independent vector fields

$$e_1 = z\frac{\partial}{\partial x}, \qquad e_2 = z\frac{\partial}{\partial y}, \qquad e_3 = -z\frac{\partial}{\partial z},$$

where (x, y, z) are the Cartesian coordinates in $\mathbb{R}^3$. Let g be the Riemannian metric defined by

$$g(e_i, e_i) = 1$$
$$g(e_i, e_j) = 0, \qquad i \neq j$$

and is given by

$$g = \frac{1}{z^2}\{dx \otimes dx + dy \otimes dy + dz \otimes dz\}.$$

Also, let η, φ be the $1-$form and $(1,1)-$tensor field, respectively defined by

$$\eta(Z, e_3) = 1, \quad \varphi(e_1) = -e_2, \quad \varphi(e_2) = e_1, \quad \varphi(e_3) = 0$$

for any $Z \in \Gamma(TM)$. Therefore, $(M, \varphi, \xi, \eta, g)$ becomes an almost contact metric manifold with the characteristic vector field ξ.

On the other hand, by direct calculations we have

$$[e_1, e_2] = 0, \qquad [e_1, e_3] = e_1, \qquad [e_2, e_3] = e_2. \tag{44}$$

Using Koszul's formula for the Riemannian metric g, we get:

$$\nabla_{e_1} e_3 = e_1, \quad \nabla_{e_2} e_3 = e_2, \quad \nabla_{e_3} e_3 = 0, \quad \nabla_{e_1} e_1 = \nabla_{e_2} e_2 = -e_3, \quad \nabla_{e_1} e_2 = \nabla_{e_2} e_1 = \nabla_{e_3} e_1 = \nabla_{e_3} e_2 = 0. \tag{45}$$

Thus, (M,φ,ξ,η,g) is a three-dimensional Kenmotsu manifold. Using the equations (44) and (45), we obtain

$$R(e_2,e_1)e_1=-e_2,\quad R(e_3,e_1)e_1=-e_3,\quad R(e_1,e_3)e_3=-e_1,\quad R(e_2,e_3)e_3=-e_2,$$
$$R(e_1,e_2)e_2=-e_1,\quad R(e_3,e_2)e_2=-e_3,\quad R(e_2,e_3)e_1=0,\quad R(e_1,e_2)e_3=0,\quad R(e_3,e_1)e_2=0$$

which yields

$$S(e_1,e_1)=-2,\qquad S(e_2,e_2)=-2,\qquad S(e_3,e_3)=-2,\qquad S(e_i,e_j)=0$$

for all $i,j=1,2,3\ (i\neq j)$. Hence, we have

$$r=S(e_1,e_1)+S(e_2,e_2)+S(e_3,e_3)=-6.$$

In this case, the manifold M is a quasi-Yamabe soliton with the soliton field $e_3=\xi$ which satisfies the equation (2) for $\lambda=-4$ and $\mu=-2.$ This result verifies the Theorem 3.

Also, the manifold M is a Yamabe soliton with the soliton field e_1 or e_2 which satisfies the equation (1) for $\lambda=-6$ and this result verifies the Proposition 1.

Conflicts of interest

There is no conflict of interest.

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