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On an inverse boundary-value problem for a second-order elliptic equation with non-classical boundary conditions

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Abstract

An inverse boundary value problem for a second-order elliptic equation with periodic and integral condition is investigated. The problem is considered in a rectangular domain. To investigate the solvability of the inverse problem, we perform a conversion from the original problem to some auxiliary inverse problem with trivial boundary conditions. By the contraction mapping principle we prove the existence and uniqueness of solutions of the auxiliary problem. Then we make a conversion to the stated problem again and, as a result, we obtain the solvability of the inverse problem.

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1. Introduction

Determination of differential equations according to the supplementary information about their solutions are called inverse problems for differential equations. Inverse problems arise in different scientific areas such as seismology, mineral exploration, biology, medicine, quality control of industrial products etc. so that it makes them one of the most important problems of modern mathematics. Different inverse problems for special types of partial differential equations have been studied in many works. Let us note here, first of all, A. N.Tikhonov's [1] , M. M.Lavrentiev's [2,3],V. K.Ivanov's [4] and their students' works. You can find more detailed information about it in the A. M.Denisov [5] monography.

The aim of this study is to prove the uniqueness and existence of the solution of stated inverse boundary problem for a second-order elliptic equation with periodic and integral conditions.

2. Main Results

Let us consider the equation

$$u_{tt}(x,t)+u_{xx}(x,t)=p(t)u(x,t)+f(x,t) \tag{1}$$

and state it an inverse boundary value problem in the domain $D_t=\{(x,t):0\le x\le 1, 0\le t\le T\}$.

The inverse problem has non-local conditions

$$u(x,0)=\varphi(x)+\int_0^T M_1(t)u(x,t)dt,$$

$$u_t(x,T)=\psi(x)+\int_0^T M_2(t)u(x,t)dt \quad (0\le x\le 1), \tag{2}$$

Neumann boundary condition

$$u_x(1,t)=0 \qquad (0\le t\le T), \tag{3}$$

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non- classical boundary condition

$$u_{xx}(0,t) - bu_x(0,t) + au(0,t) = 0 \quad (0 \le t \le T), \tag{4}$$

and the additional condition

$$u(x_0,t) = h(t) \quad (0 \le t \le T), \tag{5}$$

where a,b are positive constants, $x_0 \in (0,1)$ is a fixed number, $f(x,t), \varphi(x), \psi(x), M_1(t), M_2(t), h(t)$ are given functions, $u(x,t)$ and $p(t)$ are the unknown functions.

Definition. By classical solution of (1)-(5) inverse boundary value problem we shall understand the $\{u(x,t), p(t)\}$ pair of functions, if $u(x,t) \in C^2(D_T)$, $p(t) \in C[0,T]$ and relations are satisfied in the usual sense.

For the study of (1)-(5) firstly we reduce the considered problem to the equivalent problem:

$$y''(x) + \lambda y(x) = 0, \quad (0 < x < 1) \tag{6}$$

$$y'(1) = 0, \quad (a-\lambda)y(0) = by'(0), \quad a > 0, b > 0. \tag{7}$$

The following lemma is valid:

Lemma 1. Suppose that $p(t) \in C[0,T], M_1(t) \in C[0,T], M_2(t) \in C[0,T],$

$\|p(t)\|_{C[0,T]} \le R = const$

Moreover,

$$T\|M_1(t)\|_{C[0,T]} + T^2(\|M_2(t)\|_{C[0,T]} + \frac{1}{2}R) < 1. \tag{8}$$

Then the problem (6), (7) has a unique trivial solution.

Proof: It is easy to see that boundary value problem (6),(7) is equivalent to the integral equation

$$y(t) = \int_0^T (tM_2(\tau) + M_1(\tau) + p(\tau)G(t,\tau))y(\tau)d\tau, \tag{9}$$

where

$$G(t,\tau) = \begin{cases} -t, t \in [0,\tau], \\ -\tau, t \in [\tau,T]. \end{cases}$$

Let us introduce the following denotations

$$Ay(t) = \int_0^T (tM_2(\tau) + M_1(\tau) + p(\tau)G(t,\tau))y(\tau)d\tau, \tag{10}$$

and write integral equation (9) as

$$y(t) = Ay(t). \tag{11}$$

We shall investigate (11) in the space $C[0,T]$. It's obvious that the operator A is continuous in the space $C[0,T]$.

Let us show that the operator A is contracting in the space $C[0,T]$. Indeed, for arbitrary $y(t), \bar{y}(t) \in C[0,T]$ we have

$$\left\|Ay(t) - A\overline{y(t)}\right\|_{C[0,T]} \le (T\|M_1(\tau)\|_{C[0,T]} + T^2\|M_2(\tau)\|_{C[0,T]} + \frac{T^2}{2}\|p(t)\|_{C[0,T]})\|y(t) - \bar{y}(t)\|_{C[0,T]} \tag{12}$$

From (12) by (8), it follows that operator A is contracting in the space $C[0,T]$. Therefore, in the space $C[0,T]$, the operator A has a single fixed point $y(t)$, which is the solution of the equation (11). Thus, integral equation (9) has unique solution in $C[0,T]$. Since, boundary value problem (6), (7) also has unique solution in $C[0,T]$. As $y(t) \equiv 0$ is the solution of (6), (7). So, the boundary value problem (6), (7) has a unique trivial solution. The proof is complete.

Besides with inverse boundary value problem (1)-(5) let's consider the following auxiliary inverse boundary value problem. It is required to determine the pair $\{u(x,t), p(t)\}$ of functions $u(x,t) \in C(D_T)$ and $p(t) \in C[0,T]$ from the relations (1)-(4) and

$$h''(t) + u(x_0,t) = p(t)h(t) + f(x_0,t) \quad (0 \le t \le T). \tag{13}$$

The following theorem is valid:

Theorem 1. Assume that $f(x,t) \in C(D_T), \varphi(x), \psi(x) \in C[0,T], M_1(t), M_2(t) \in C[0,T]$, $h(t) \in C^2[0,T], h(t) \ne 0 \ (0 \le t \le T)$ and the compatibility conditions

$$\varphi(x_0) = h(0) - \int_0^T M_1(t)h(t)dt$$

$$\psi(x_0) = h'(T) - \int_0^T M_2(t)h(t)dt \tag{14}$$

hold. Then the following assertions are valid:

1) Each classical solution $\{u(x,t), p(t)\}$ of problem (1)-(5) is also the solution of (1)-(4), (13);
2) Each solution $\{u(x,t), p(t)\}$ of problem (1)-(4), (13) satisfying

$$T\|M_1(t)\|_{C[0,T]} + T^2(\|M_2(t)\|_{C[0,T]} + \frac{1}{2}\|p(t)\|_{C[0,T]}) < 1 \tag{15}$$

is a classical solution of (1)-(5).

Proof. Let $\{u(x,t), p(t)\}$ be classical solution of (1)-(5). Taking into consideration $h(t) \in C[0,T]$ and twice differentiating (5), we find:

$$u_t(x_0,t) = h'(t), u_{tt}(x_0,t) = h''(t), \quad (0 \le t \le T). \tag{16}$$

Setting $x = x_0$ in the equation (1), we have

$$u_{tt}(x_0,t) + u_{xx}(x_0,t) = p(t)u(x_0,t) + f(x_0,t) \qquad (0 \le t \le T). \tag{17}$$

Hence by (5) and (16) we conclude that (13) is valid.

Now, assume that $\{u(x,t), p(t)\}$ is a solution of (1)-(4), (13), and the compatibility conditions (14) are satisfied. Then from (1) and (17) we get

$$\frac{d}{dt^2}(u(x_0,t)-h(t)) = a(t)(u(x_0,t)-h(t)) \quad (0 \le t \le T). \tag{18}$$

Furthermore, by (2) and the compatibility conditions (14) we have

$$u(x_0,t)-h(t)-\int_0^T M_1(t)(u(x_0,t)-h(t))dt = u(x_0,t)-\int_0^T M_1(t)u(x_0,t)dt-(h(t)-$$

$$-\int_0^T M_1(t)h(t)dt) = \varphi(x_0)-(h(t)-\int_0^T u_1(t)h(t))dt = 0$$

$$u(x_0,T)-h(T)-\int_0^T M_2(t)(u(x_0,t)-h(t))dt = u(x_0,T)-\int_0^T M_2(t)u(x_0,t)dt-(h(T)-$$

$$-\int_0^T M_2(t)h(t)dt) == \varphi(x_0)-(h(T)-\int_0^T M_2(t)h(t))dt = 0. \tag{19}$$

From (18) (19) and by virtue of Lemma 1, we conclude that the condition (5) is satisfied.

The theorem is thus proved.

Let us consider the following spectral problem

$$y''(x)+\lambda y(x) = 0 \quad 0 \le x \le 1 \tag{20}$$

$$y'(1) = 0 \ \ (a-\lambda)y(0) = by'(0) \tag{21}$$

[9], with positive the coefficients a and b. Eigenfunctions of this problem has the form

$y_k(x) = \sqrt{2}\cos(\sqrt{\lambda_k}(1-x))$, $k = 0,1,2,...$

with positive eigenvalues λ_k from characteristic equation

$$tg\sqrt{\lambda} = \frac{a-\lambda}{b\sqrt{\lambda}} \ .$$

We assign zero index to any pre-selected eigenfunction, and number all the reminded by increasing order of eigenvalues.

It is known from [6-10]:

Lemma 2. Beginning from a certain number N the following estimations holds true:

$$0 \le \sqrt{\lambda_k} - \frac{\pi}{2} - \pi(n-1) < \frac{b}{\frac{\pi}{4}+\pi(k-1)}.$$

Now let us compare the system, $\{y_k(x)\}$ without function $y_0(x)$ to a known system,

$\{v_k(x)\}$, $v_k(x) = \sqrt{2}\cos\sqrt{\mu_k}(1-x)$ where $\sqrt{\mu_k} = \frac{\pi}{2}+\pi(k-1)$, $k = 1,2,...$, which is orthonormal basis in $L_2(0,1)$.

In analogous manner [6-10] , we have

$$\sum_{k=N}^{\infty} \|y_k(x)-v_k(x)\|^2_{L_2(0,1)} < b^2 \sum_{k=N}^{\infty} \frac{2}{3(\frac{\pi}{4}+\pi(k-1))^2} \ , \tag{22}$$

which we get convergence of the series from the left hand side of this inequality.

Let us suppose that $\eta_k(x)=\sqrt{2}\sin(\sqrt{\lambda_k}(1-x))$, $\xi_k(x)=\sqrt{2}\sin(\sqrt{\mu_k}(1-x))$

then the inequalitie

$$\sum_{k=N}^{\infty}\left\|\eta_k(x)-\xi_k(x)\right\|^2_{L_2(0,1)}<b^2\frac{2}{3(\frac{\pi}{4}+\pi(k-1))^2} \tag{23}$$

hold.

It's known [9] that functions of biorthogonally conjugate system $\{z_k(x)\}$to the system $\{y_k(x)\}, k=1,2,...$ are defined by the equality

$$z_k(x)=\sqrt{2}(\cos(\sqrt{\lambda_k}(1-x))-\frac{\cos(\sqrt{\lambda_k})\cos(\sqrt{\lambda_0}(1-x))}{\cos(\sqrt{\lambda_0})})/\alpha_k \ , \tag{24}$$

where $\alpha_k=1+\frac{\cos^2(\sqrt{\lambda_k})}{b}+\frac{a\cos^2(\sqrt{\lambda_k})}{b\lambda_k}$

Here it's also known that $\{y_k(x)\}$ $k=1,2,...$ form a basis Riess in space $L_2(0,1)$.

Suppose that $g(x)\in L_2(0,1)$. Then by (22), we get

$$(\sum_{k=1}^{\infty}(\int_0^1 g(x)y_k(x)dx))^{\frac{1}{2}}\le M\left\|g(x)\right\|_{L_2(0,1)} \tag{25}$$

where

$$M=\left(\frac{N(1+N)}{2}+b\sum_{k=N}^{\infty}\frac{2}{3(\frac{\pi}{4}+\pi(k-1))^2}+2\right)^{\frac{1}{2}}$$

Similar to (25), by (23), we find:

$$(\sum_{k=1}^{\infty}(\int_0^1 g(x)\eta_k(x)dx)^2)^{\frac{1}{2}}\le M\cdot\left\|g(x)\right\|_{L_2(0,1)}$$

Since, the functions $\{y_k(x)\}$, $k=1,2,...$ are the basis of Riess in $L_2(0,1)$ space, then it's known [10] that for any function $g(x)\in L_2(0,1)$

$$g(x)=\sum_{k-1}^{\infty}g_k y_k(x)$$

holds true, where

$$g_k=\int_0^1 g(x)z_k(x) \qquad (k=1,2,...)$$

It is not difficult to see that

$$|g_k| \le \sqrt{2}(\int_0^1 g(x)y_k(x)dx + \frac{1}{\left|\cos\sqrt{\lambda_0}\right|}\cdot\frac{b\sqrt{\lambda_k}}{|a-\lambda_k|}\cdot\int_0^1 |g(x)|dx)$$

From here, by virtue of (25), we have

$$(\sum_{k=1}^{\infty} g_k^2)^{\frac{1}{2}} \le M_0 \|g(x)\|_{L_2(0,1)} \tag{27}$$

where

$$M_0 = 2\left[M + \frac{b}{\cos\sqrt{\lambda_0}}\cdot\sup_k(\frac{\lambda}{|a-\lambda_k|})(\sum_{k=1}^{\infty}\frac{1}{\lambda_k})^{\frac{1}{2}}\right].$$

Assume that $g(x)\in C[0,1], g'(x)\in L_2(0,1)$.

$$J(g) \equiv g(x) + \frac{b}{\cos\sqrt{\lambda_0}}\int_0^1 g(x)\cos\sqrt{\lambda_0}(1-x)dx \equiv 0$$

Then we have

$$g_k = \frac{\sqrt{2}}{\alpha_k}\int_0^1 g(x)(\cos(\sqrt{\lambda_k}(1-x)) - \frac{\cos\sqrt{\lambda_k}}{\cos\sqrt{\lambda_0}}\cos(\sqrt{\lambda_0}(1-x)))dx =$$
$$= \frac{\sqrt{2}}{\alpha_k}\cdot\frac{a}{b\lambda_k}g(0)\cos\sqrt{\lambda_k} + \frac{\sqrt{2}}{\alpha_k}\cdot\frac{1}{\sqrt{\lambda_k}}\int_0^1 g'(x)\sin(\sqrt{\lambda_k}(1-x)dx \tag{28}$$

Hence by (26) we have

$$(\sum_{k=1}^{\infty}(\lambda_k g_k)^2)^{\frac{1}{2}} \le m_0|g(0)| + 2M\|g'(x)\|_{L_2(0,1)}, \tag{29}$$

where

$$m_0 = \frac{2a}{b}(\sum_{k=1}^{\infty}\frac{1}{\lambda_k})^{\frac{1}{2}}$$

Now suppose that $g(x)\in C^1[0,1]$, $g'''(x)\in L_2(0,1)$, $J(g)=0$, $g'(1)=0$ .Then from (28) we get

$$g_k = \frac{\sqrt{2}}{\alpha_k}\cdot\frac{1}{a-\lambda_k}\cdot\frac{1}{\sqrt{\lambda_k}}(ag(0) - bg'(0)\sin\sqrt{\lambda_k} + \frac{\sqrt{2}}{\alpha_k}\cdot\frac{1}{\lambda_k\sqrt{\lambda_k}}\cdot$$
$$\cdot\int_0^1 g''(x)d\sin(\sqrt{\lambda_k}(1-x)) = -\frac{\sqrt{2}}{\alpha_k}\cdot\frac{1}{a-\lambda_k}\cdot\frac{a}{\lambda_k\sqrt{\lambda_k}}g''(0) -$$
$$-\frac{\sqrt{2}}{\alpha_k}\cdot\frac{1}{\lambda_k\sqrt{\lambda_k}}\int_0^1 g'''(x)\sin(\sqrt{\lambda_k}(1-x))dx\ . \tag{30}$$

From here we find:

$$(\sum_{k=1}^{\infty}(\lambda_k\sqrt{\lambda_k}\,|f_k|^2)^{\frac{1}{2}} \le m_1 a|g''(0)| + 2M\|g'''(x)\|_{L_2(0,1)} \tag{31}$$

where

$$m_1 = 2\sup_k \frac{\lambda_k}{|a-\lambda_k|}\left(\sum_{k=1}^{\infty}\frac{1}{\lambda_k}\right)^{\frac{1}{2}}.$$

Now suppose that $g(x)\in C^2[0,1]$, $g'''(x)\in L_2(0,1)$, $J(g)=0$, $g'(1)=0$, $g''(0)-bg(0)+ag(0)=0$. Then from (29) we get

$$g_k = -\frac{\sqrt{2}}{\alpha_k}\cdot\frac{1}{a-\lambda_k}\cdot\frac{a}{\lambda_k\sqrt{\lambda_k}}g''(0) - \frac{\sqrt{2}}{\alpha_k}\cdot\frac{1}{\lambda_k\sqrt{\lambda_k}}\int_0^1 g'''(x)\sin(\sqrt{\lambda_k}(1-x))dx \ . \tag{32}$$

From (32) we find:

$$\left(\sum_{k=1}^{\infty}(\lambda_k\sqrt{\lambda_k}\,|g_k|)^2\right)^{\frac{1}{2}} \le m_1 a|g''(0)| + 2M\|g'''(x)\|_{L_2(0,1)}. \tag{33}$$

Denoted by $B_{2,T}^{\frac{3}{2}}$, the set of all the functions $u(x,t)$ of the form

$$u(x,t)=\sum_{k=1}^{\infty}u_k(t)y_k(x) \ ,$$

is considered D_T, where each function $u_k(t)$ is continuous on $[0,T]$ and

$$\left\{\sum_{k=1}^{\infty}(\lambda_k\sqrt{\lambda_k}\|u_k(t)\|_{C[0,T]})^2\right\}^{\frac{1}{2}} < +\infty .$$

We define the norm on this set as follows:

$$\|u(x,t)\|_{B_{2,T}^{\frac{3}{2}}} = \left\{\sum_{k=1}^{\infty}(\lambda_k\sqrt{\lambda_k}\|u_k(t)\|_{C[0,T]})^2\right\}^{\frac{1}{2}}.$$

Denoted by $E_T^{\frac{3}{2}}$ the space that consist of the topological product

$$B_{2,T}^{\frac{3}{2}}\times C[0,T].$$

The norm of the element $z=\{u,p\}$ is defined by the formula

$$\|z\|_{E_T^{\frac{3}{2}}} = \|u(x,t)\|_{B_{2,T}^{\frac{3}{2}}} + \|p(t)\|_{C[0,T]}.$$

It is known that $B_{2,T}^{\frac{3}{2}}$ and $E_T^{\frac{3}{2}}$ are Banach spaces.

We shall seek for the first component $u(x,t)$ of $\{u(x,t),p(t)\}$ in the form

$$u(x,t)=\sum_{k=1}^{\infty}u_k(t)y_k(x), \tag{34}$$

where

$$u_k(t)=\int_0^1 u(x,t)z_k(x)dx \quad (k=1,2,...)\ ,$$

and

$$y_k(x)=\sqrt{2}\cos(\sqrt{\lambda_k}(1-x)).$$

By applying method of separation of variables, from (1), (2) we have

$$u_k''(t)-\lambda_k u_k(t)=F_k(t,u,p) \qquad (k=1,2,...;\ 0\le t\le T)\ , \tag{35}$$

$$u_k(0)=\varphi_k+\int_0^T M_1(t)u_k(t)dt\ , \tag{36}$$

$$u_k'(T)=\psi_k+\int_0^T M_2(t)u_k(t)dt \quad (k=1,2,...) \tag{37}$$

where

$$F_k(t,u,p)=f_k(t)+p(t)u_k(t),\qquad f_k(t)=\int_0^1 f(x,t)y_k(x)dx\ ,$$

$$\varphi_k=\int_0^1\varphi(x)z_k(x)dx\ ,\ \psi_k=\int_0^1\psi(x)z_k(x)dx \quad (k=1,2,...)$$

Solving problem (35)- (37) we obtain

$$u_k(t)=\frac{ch(\lambda_k(T-t)}{ch(\lambda_k T)}\left(\varphi_k+\int_0^T M_1(t)u_k(t)dt\right)+\frac{sh(\lambda_k(T-t)}{\lambda_k ch(\lambda_k T)}\left(\psi_k+\int_0^T M_2(t)u_k(t)dt\right)+$$

$$+\frac{1}{\lambda_k}\int_0^T G_k(t,\tau)F_k(\tau,u,p)d\tau,(k=1,2,...) \tag{38}$$

Where

$$G_k(t,\tau)=\begin{cases}\dfrac{sh(\sqrt{\lambda_k}(T-(t+\tau)))-sh(\sqrt{\lambda_k}(T+t-\tau))}{2ch(\sqrt{\lambda_k}T)},t\in[0,\tau]\\[2ex] \dfrac{sh(\sqrt{\lambda_k}(T-(t+\tau)))-sh(\sqrt{\lambda_k}(T-(t-\tau)))}{2ch(\sqrt{\lambda_k}T)},t\in[\tau,T]\end{cases}$$

After substitution the expression (38) in (34), by the definition of the component $u(x,p)$ of problem, we get (1)-(4), (13)

$$u(x,t)=\sum_{k=1}^{\infty}\left[\frac{ch(\lambda_k(T-t)}{ch(\lambda_k T)}\left(\varphi_k+\int_0^T M_1(t)u_k(t)dt\right)+\right.$$

$$\left.+\frac{sh(\lambda_k(T-t)}{\lambda_k ch(\lambda_k T)}\left(\psi_k+\int_0^T M_2(t)u_k(t)dt\right)+\frac{1}{\lambda_k}\int_0^T G_k(t,\tau)F_k(\tau,u,p)d\tau\right]y_k(x). \tag{39}$$

Now, from (13), by (34) we have

$$p(t)=[h(t)]^{-1}\left\{h''(t)-f(x_0,t)-\sqrt{2}\sum_{k=1}^{\infty}\lambda_k u_k(t)\cos\sqrt{\lambda_k}(1-x_0)\right\}\ . \tag{40}$$

In order to get the equation for the second component $p(t)$ of solution $\{u(x,t),p(t)\}$ of problem (1)-(4), (13) we substitute the expression (38) in (40):

$$p(t)=[h(t)]^{-1}\left\{h''(t)-f(x_0,t)-\sqrt{2}\sum_{k=1}^{\infty}\lambda_k\left[\frac{ch(\lambda_k(T-t)}{ch(\lambda_k T)}\left(\varphi_k+\int_0^T M_1(t)u_k(t)dt\right)+\right.\right.$$

$$\left.\left.+\frac{sh(\lambda_k(T-t)}{\lambda_k ch(\lambda_k T)}\left(\psi_k+\int_0^T M_2(t)u_k(t)dt\right)+\frac{1}{\lambda_k}\int_0^T G_k(t,\tau)F_k(\tau,u,p)d\tau\right]\cos\sqrt{\lambda_k}(1-x_0)\right\} \tag{41}$$

Thus, the solution of problem (1)-(4), (13) was reduced to the solution of the system (39), (41) respectively to unknown function $u(x,t)$ and $p(t)$

Now let us consider the operator $\Phi(u,p)=\{\Phi_1(u,p),\Phi_2(u,p)\}$ in the space $E_T^{3/2}$, where

$$\Phi_1(u,p)=\tilde{u}(x,v)\equiv\sum_{k=1}^{\infty}\tilde{u}_k(v)\cdot y_k(x),\ \ \Phi_2(u,p)=\tilde{p}(v),\ \text{a}\ \tilde{u}_k(v)\ \ (k=1,2,...)$$

and $\tilde{p}(v)$ are equal to the right-hand side parts of (38) and (41) respectively.

It is obvious that

$$\frac{ch(\lambda_k(T-t))}{ch(\lambda_k T)}<1,\ \ \frac{sh(\lambda_k T)}{ch(\lambda_k T)}<1,\ \ \frac{sh(\lambda_k(T+t-\tau)}{ch(\lambda_k T)}<1\quad (t\in[0,\tau]),$$

$$\frac{sh(\lambda_k(T-(t+\tau)}{ch(\lambda_k T)}<1\ ,\ \ \frac{sh(\lambda_k(T-(t-\tau))}{ch(\lambda_k T)}<1\quad (t\in[\tau,T]).$$

Using these relations, and simple transformations we find

$$\left(\sum_{k=1}^{\infty}(\lambda_k\sqrt{\lambda_k}\|\tilde{u}_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}\le\sqrt{6}\left(\sum_{k=1}^{\infty}(\lambda_k\sqrt{\lambda_k}|\varphi_k|^2\right)^{\frac{1}{2}}+\sqrt{6}\left(\sum_{k=1}^{\infty}(\lambda_k|\psi_k|^2\right)^{\frac{1}{2}}+$$

$$+\sqrt{6}T\|M_1(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k\sqrt{\lambda_k}\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}+\sqrt{6}T\|M_2(T)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}+$$

$$+\sqrt{6T}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k|f_k(\tau)|)^2d\tau\right)^{\frac{1}{2}}+\sqrt{6}T\|p(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k\sqrt{\lambda_k}\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}} \tag{42}$$

$$\|\tilde{p}(t)\|_{C[0,T]}\le\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\{\|h''(t)-f(x_0,t)\|_{C[0,T]}+\left(\sum_{k=1}^{\infty}\lambda_k^{-1}\right)^{\frac{1}{2}}\left[\left(\sum_{k=1}^{\infty}(\lambda_k\sqrt{\lambda_k}|\varphi_k|^2\right)^{\frac{1}{2}}+\right.\right.$$

$$+\left(\sum_{k=1}^{\infty}(\lambda_k|\psi_k|^2\right)^{\frac{1}{2}}+T\|M_1(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k\sqrt{\lambda_k}\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}+T\|M_2(t)\|_{C[0,T]}\,.$$

$$\cdot(\sum_{k=1}^{\infty}(\lambda_k\sqrt{\lambda_k}\|u_k(t)\|_{C[0,T]})^2)^{\frac{1}{2}}+\sqrt{T}(\int_0^T\sum_{k=1}^{\infty}(\lambda_k|f_k(\tau)|)^2d\tau)^{\frac{1}{2}}+T\|p(t)\|_{C[0,T]}\cdot$$

$$\cdot(\sum_{k=1}^{\infty}(\lambda_k\sqrt{\lambda_k}\|u_k(t)\|_{C[0,T]})^2)^{\frac{1}{2}}\Bigg]\Bigg\}\ . \tag{43}$$

Suppose that the data of (1)-(4), (13) satisfy the following conditions:

1. $\varphi(x)\in C^2[0,1], \varphi'''(x)\in L_2(0,1), J(\varphi)=0, \varphi'(1)=0, \varphi''(0)-b\varphi'(0)+a\varphi(0)=0$,
2. $\psi(x)\in C^1[0,1], \varphi''(x)\in L_2(0,1), J(\psi)=0, \psi'(1)=0$,
3. $f(x,t), f_x(x,t), f''_{xx}(x,t)\in L_2(D_T)$,
   $J(f)=0,\ f_x(1,t)=0\quad(0\le t\le T)$
4. $a>0, b>0, M_1(t), M_2(t)\in C[0,T], h(t)\in C^2[0,T], h(t)\ne 0\ \ (0\le t\le T)$

Then from (42) and (43), by (30) and (32), we find

$$\left(\sum_{k=1}^{\infty}(\lambda_k\sqrt{\lambda_k}\|\tilde{u}_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}\le$$

$$\le\sqrt{6}\left[m_1a|\varphi''(0)|+2M\|\varphi'''(x)\|_{L_2(0,T)}\right]+\sqrt{6}\left(m_1(a|\psi(0)|+b|\psi'(0)|)+\sqrt{b}M\|\psi''(x)\|_{L_2(0,1))}\right)+$$

$$+\sqrt{b}T(\|M_1(t)\|_{C[0,T]}++\|M_2(t)\|_{C[0,T]})\left(\sum_{k=1}^{\infty}(\lambda_k\sqrt{\lambda_k}\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}+$$

$$+\sqrt{6T}\left[m_1(a\|f(0,t)\|_{C[0,T]}+b\|f_x(0,t)\|_{C[0,T]})+2M\|f_{xx}(x,t\|_{L_2(D_T)})\right]+$$

$$+\sqrt{6}T\|p(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k\sqrt{\lambda_k}\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}, \tag{44}$$

$$\|\tilde{p}(t)\|_{C[0,T]}\le\left\|[h(t)]^{-1}\right\|_{C[0,T]}\Bigg\{\|h''(t)-f(x_0,t)\|_{C[0,T]}+$$

$$+\left(\sum_{k=1}^{\infty}\frac{1}{\lambda_k}\right)^{\frac{1}{2}}\Bigg[m_1a|\varphi''(0)|+2M\|\varphi'''(x)\|_{L_2(0,T)}+m_1(a|\psi(0)|+b|\psi'(0)|)+$$

$$+\sqrt{b}M\|\psi''(x)\|_{L_2(0,1))}+T\left(\|M_1(t)\|_{C[0,T]}+\|M_2(t)\|_{C[0,T]}\right)\left(\sum_{k=1}^{\infty}(\lambda_k\sqrt{\lambda_k}\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}+$$

$$+\sqrt{T}\left[m_1(a\|f(0,t)\|_{C[0,T]}+b\|f_x(0,t)\|_{C[0,T]})+2M\|f_{xx}(x,t\|_{L_2(D_T)})\right]+$$

$$+T\|p(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k\sqrt{\lambda_k}\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}\Bigg]\Bigg\} \tag{45}$$

Let us introduce the denotations

$$A_1(T)=\sqrt{6}\left[m_1a|\varphi''(0)|+2M\|\varphi'''(x)\|_{L_2(0,1)}\right]+\sqrt{6}\left[m_1(a|\psi(0)|+b|\psi'(0)|)+\sqrt{b}M\|\psi''(x)\|_{L_2(0,1)}\right]+$$
$$+\sqrt{bT}\left[m_1(a\|f(0,t)\|_{C[0,T]}+b\|f_x(0,t)\|_{C[0,T]})+\sqrt{b}M\|f_{xx}(x,t)\|_{L_2(D_T)}\right]$$

$$B_1(T)=\sqrt{6}T\,,\ D_1(T)=\sqrt{6}T(\|M_1(t)\|_{C[0,T]}+\|M_2(t)\|_{C[0,T]})\,,$$

$$A_2(T)=\left\|[h(t)]^{-1}\right\|_{C[0,T]}\Big\{\|h''(t)-f(x_0,t)\|_{C[0,T]}+$$

$$+\left(\sum_{k=1}^{\infty}\frac{1}{\lambda_k}\right)^{\frac{1}{2}}\Big[m_1a|\varphi''(0)|+2M\|\varphi'''(x)\|_{L_2(0,1)}+m_1(a|\psi(0)|+b|\psi'(0)|)+2M\|\psi''(x)\|_{L_2(0,1)}+$$

$$+\sqrt{T}\big[m_1(a\|f(0,t)\|_{C[0,T]}+b\|f_x(0,t)\|_{C[0,T]})+2M\|f_{xx}(x,t)\|_{L_2(D_T)}\big]\Big]\Big\}$$

$$B_2(T)=\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\frac{1}{\lambda_k}\right)^{\frac{1}{2}}T\,,$$

$$D_2(T)=\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\frac{1}{\lambda_k}\right)^{\frac{1}{2}}T(\|M_1(t)\|_{C[0,T]}+\|M_2(t)\|_{C[0,T]}).$$

and write the estimations (44) and (45) as follows

$$\|\tilde{u}(x,t)\|_{B^{3/2}_{2,T}}\le A_1(T)+B_1(T)\|p(t)\|_{C[0,T]}\|u(x,t)\|_{B^{3/2}_{2,T}}+D_1(T)\|u(x,t)\|_{B^{3/2}_{2,T}}\,, \tag{46}$$

$$\|\tilde{p}(t)\|_{C[0,T]}\le A_2(T)+B_2(T)\|p(t)\|_{C[0,T]}\|u(x,t)\|_{B^{3/2}_{2,T}}+D_2(T)\|u(x,t)\|_{B^{3/2}_{2,T}}\,. \tag{47}$$

From the inequalities (46) and (47) we conclude:

$$\|u(x,t)\|_{B^{\frac{3}{2}}_{2,T}}+\|\tilde{p}(t)\|_{C[0,T]}\le A(t)+B(T)\|p(t)\|_{C[0,T]}\cdot\|u(x,t)\|_{B^{3/2}_{2,T}}+D(T)\|u(x,t)\|_{B^{3/2}_{2,T}}\,, \tag{48}$$

where

$$A(T)=A_1(T)+A_2(T)\ ,\ B(T)=B_1(T)+B_2(T)\,,\ D(T)=D_1(T)+D_2(T)\,.$$

The following theorem can be proved.

Theorem 2. If conditions (1)-(4) and the condition

$$(B(T)(A(T)+2)+D(T))(A(T)+2)<1 \tag{49}$$

hold. Then problem (1)-(4), (13) has a unique solution in the ball $K=K_R\ (\|z\|_{E_T^{3/2}}\le R=A(t)+2)$ of the space $E_T^{3/2}$.

Proof. In the space $E_T^{3/2}$, we consider the equation

$$z=\Phi z \tag{50}$$

where $z=\{u,p\}$, the components $\Phi_i(u,p)\ \ (i=1,2,...)$ of operator $\Phi(u,p)$ is defined by the right sides of equations (39), (41).

Consider the operator $\Phi(u,p)$ in $K=K_R$ of the space $E_T^{3/2}$. Similar to (48) we get that for any $z,z_1,z_2\in K_R$ the following inequalities hold:

$$\|\Phi z\|_{E_T^{3/2}}\le A(T)+B(T)\|p(t)\|_{C[0,T]}\|u(x,t)\|_{B^{3/2}_{2,T}}+D(T)\|u(x,t)\|_{B^{3/2}_{2,T}}\le A(T)+B(T)\cdot$$

$$\cdot(A(T)+2)^2+D(T)(A(T)+2)=A(T)+(B(T)(A(T)+2)+D(T))(A(T)+2) \tag{51}$$

$$\|\Phi z_1-\Phi z_2\|_{E_T^{3/2}} \le B(T)R(\|P_1(t)-P_2(t)\|_{C[0,T]}+\|u_1(x,t)-u_2(x,t)\|_{B_{2,T}^{3/2}})+$$
$$+D(T)\|u_1(x,t)-u_2(x,t)\|_{B_{2,T}^{3/2}}) \qquad (52)$$

Then by (49) from (51) and (52), it follows that the operator Φ acts in the sphere $K=K_R$ and is contracting. Therefore in the sphere $K=K_R$ operator Φ has only unique fixed point $\{u,p\}$, which is the solution of the equation (50), i.e. $\{u,p\}$, is the unique solution of the system (39), (41) in the sphere $K=K_R$.

Then the function $u(x,t)$, as an element of $B_{2,T}^{3/2}$, is continuous and has continuous derivatives $u_x(x,t)$ and $u_{xx}(x,t)$ in D_T.

Furthermore, from (38) it is obvious that $u_k''(t)\in C[0,T]$ and

$$\left(\sum_{K=1}^{\infty}(\sqrt{\lambda_k}\|u_k''(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}} \le \sqrt{3}\left(\sum_{k=1}^{\infty}(\lambda_k\sqrt{\lambda_k}\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}+\sqrt{3}\left(\sum_{k=1}^{\infty}\sqrt{\lambda_k}\|f_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}+$$
$$+\sqrt{3}\|p(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k\sqrt{\lambda_k}\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}$$

or considering (29) we find:

$$\left(\sum_{K=1}^{\infty}(\sqrt{\lambda_k}\|u_k''(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}} \le \sqrt{3}(m_0\|f(0,t)\|_{C[0,T]}+2M\left\|\|f_x(x,t)\|_{C[0,T]}\right\|_{L_2(0,1)})+$$
$$+\sqrt{3}(1+\|p(t)\|_{C[0,T]})\left(\sum_{k=1}^{\infty}(\lambda_k\sqrt{\lambda_k}\|u_k''(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}$$

This implies that $u_{tt}(x,t)$ is continuous in D_T .It is easy to verify that equation (1) and conditions (2), (3), (4) and (13) are satisfied in the usual sense.

Consequently, $\{u(x,t),p(t)\}$ is the solution of (1)-(4), (13). The proof is complete.

With the aid of Theorem 1 the following theorem was proved.

Theorem 3. Suppose that all conditions of Theorem 2

$$T\|M_1(t)\|_{C[0,T]}+T^2(\|M_2(t)\|_{C[0,T]}+\frac{1}{2}(A(T)+2))<1$$

and the compatibility condition (14) hold. If

$$\varphi(x_0)=h(0)-\int_0^T M_1(t)h(t)dt,$$

$$\varphi(x_0)=h'(T)-\int_0^T M_2(t)h(t)dt.$$

then problem (1)-(5) has a unique classical solution in $K=K_R$ of $E_T^{3/2}$.

Conflicts of interest

There is no conflict of interest.

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