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On new Simpson's type inequalities for trigonometrically convex functions with applications

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Abstract

The aim of this article is to define a special case of *h*- convex function, namely the notion of a trigonometrically convex function. Using the Hölder, Hölder-İşcan integral inequality and the power-mean, improved power-mean integral inequalities, and together with an integral identity, some new Simpson-type inequalities have been obtained for trigonometric convex functions. We also give some applications for special means.

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1. Introduction

The following definition for convex functions is well known in the mathematical analysis literature; The function $f: [a,b] \to R$ is said to be convex if the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \tag{1}$$

is valid for all $x, y \in [a,b]$ and $t \in [0,1]$ and f is said to be concave on $[a,b]$ if the inequality (1) holds in reversed direction.

Many inequalities have been established for convex functions. Convexity theory provides powerful principles and techniques to study a wide class of problems in both pure and applied mathematics. See [1,2] and the references therein.

The following integral inequality, it is well known in the literature as Simpson's inequality

Theorem 1. Let $I = [a,b]$ and $f : I \to R$ a four-time continuously differentiable mapping on (a,b) and $\|f^{(4)}\|_\infty = sup_{x\in(a,b)}|f^{(4)}(x)| < \infty$. Then the following inequality holds:

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2} + 2f(\frac{a+b}{2})\right] - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{1}{2880}\|f^{(4)}\|_\infty (b-a)^4 \tag{2}$$

There are many important studies in the literature on Simpson type integral inequalities. For recent results, generalizations and improvements about Simpson's inequality see [3-6] and the references therein.

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Definition 1. Let $\boldsymbol{h}: \boldsymbol{J} \to \boldsymbol{R}$ be a non-negative function, $\boldsymbol{h} \neq \boldsymbol{0}$. We say that $\boldsymbol{f}: \boldsymbol{I} \to \boldsymbol{R}$ is an *h*-convex function, or that $\boldsymbol{f}$ belongs to class *SX(h,I)* if $\boldsymbol{f}$ is no negative and $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{I}$ and $\boldsymbol{\alpha} \in (\boldsymbol{0}, \boldsymbol{1})$. We have

$$f(\alpha x + (1-\alpha)y) \leq h(\alpha)f(x) + h(1-\alpha)f(y) \tag{3}$$

If this inequality is reversed, then $\boldsymbol{f}$ is said to be *h*-concave, i.e. $\boldsymbol{f} \in \boldsymbol{SV}(\boldsymbol{h}, \boldsymbol{I})$ [7].

In [8], Kadakal obtained the trigonometrically convex function as follows.

Definition 2. A non- negative function $\boldsymbol{f}: \boldsymbol{I} \to \boldsymbol{R}$ is called trigonometrically convex if for every $\boldsymbol{x}, \boldsymbol{y} \in I$ and $\boldsymbol{t} \in [\boldsymbol{0}, \boldsymbol{1}]$.

$$f(tx + (1-t)y) \leq sin\left(\frac{\pi t}{2}\right) f(x) + cos\left(\frac{\pi t}{2}\right) f(y) \tag{4}$$

The class of all trigonometrically convex functions is denoted $TC(I)$ on the interval I.

Example 1. Non-negative constant functions are trigonometrically convex,

since $sin\left(\frac{\pi t}{2}\right) + cos\left(\frac{\pi t}{2}\right) \geq 1$ for all $t \in [0,1]$.

Lemma 1. i) Every non-negative convex function is a trigonometrically convex function.
ii) Every trigonometrically convex function is *h*-convex function with $h(t) = sin(\frac{\pi t}{2})$.

We note that space of trigonometrically convex functions is a convex cone in the vector space of functions $f: [a, b] \to R$.

Theorem 2. Let $f, g : [a, b] \to R$. If f and g are trigonometrically convex functions, then

(i) $f + g$ is trigonometrically convex function,

(ii) for $c \in R$ $(c \geq 0)$, $c.f$ is trigonometrically convex function.

Some results for these classes of convex functions have been obtained in recent years. In [9], Kadakal obtained of Hermite-Hadamard inequality whose second derivatives are trigonometrically convex. Also In [10], Bekar proved Hermite-Hadamard inequality which holds for trigonometrically P-convex functions.

Theorem 3. (Hölder inequality for integrals) Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If f and g are real functions defined on $[a, b]$ and if $|f|^q$, $|g|^q$ are integrable functions on $[a, b]$, $q \geq 1$ then

$$\int_a^b |f(x)g(x)|dx \leq \left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}} \left(\int_a^b |g(x)|^q \, dx\right)^{\frac{1}{q}} \tag{5}$$

with equality holding if and only if $A|f(x)|^p = B|g(x)|^q$, almost everywhere, where A and B are constants [11].

Power-mean integral inequality as a result of the Hölder integral inequality can be given as follows:

Theorem 4. (Power-mean integral inequality) Let $q \geq 1$. If f and g are real functions defined on $[a, b]$ and if $|f|$, $|f||g|^q$ are integrable functions on $[a, b]$, then

$$\int_a^b |f(x)g(x)|dx \leq \left(\int_a^b |f(x)|dx\right)^{1-\frac{1}{q}} \left(\int_a^b |f(x)||g(x)|^q\right)^{\frac{1}{q}}. \tag{6}$$

In [12], Işcan achieved the following integral inequality which gives better approach than the cllassical hölder integral inequality:

Theorem 5. (Hölder-İşcan inequality for integrals) Let $p > 1$ and $\frac{1}{p}+\frac{1}{q}=1$. If f and g are real functions defined on interval $[a,b]$ and $|f|^q$ and $|g|^q$ are integrable functions on $[a,b]$, then

$$\int_a^b |f(x)g(x)|dx \le \frac{1}{b-a}\left\{\left(\int_a^b (b-x)\,|f(x)|^p dx\right)^{\frac{1}{p}}\left(\int_a^b (b-x)\,|g(x)|^q dx\right)^{\frac{1}{q}} + \left(\int_a^b (x-a)\,|f(x)|^p dx\right)^{\frac{1}{p}}\left(\int_a^b (x-a)\,|g(x)|^q dx\right)^{\frac{1}{q}}\right\}. \tag{7}$$

In [13], a different representation of the Hölder-İşcan inequality is given as follows:

Theorem 6. (Improved power-Mean Integral Inequality) Let $q \ge 1$ and $\frac{1}{p}+\frac{1}{q}=1$. If f and g are real functions defined on interval $[a,b]$ and $|f|$ and $|f||g|^q$ are integrable functions on $[a,b]$, then

$$\int_a^b |f(x)g(x)|dx \le \frac{1}{b-a}\left\{\left(\int_a^b (b-x)\,|f(x)|dx\right)^{1-\frac{1}{q}}\left(\int_a^b (b-x)\,|f(x)||g(x)|^q dx\right)^{\frac{1}{q}} + \left(\int_a^b (x-a)\,|f(x)|dx\right)^{1-\frac{1}{q}}\left(\int_a^b (x-a)\,|f(x)||g(x)|^q dx\right)^{\frac{1}{q}}\right\}. \tag{8}$$

2. Simpson's Type Inequalities for Trigonometrıcally Convex

In this section, in order to prove our main theorems by subracting Simpson type integral inequalities for trigonometrically convex funtions, we will use the following lemma. This lemma can be easily obtained by taking partial integration in the lemma [14].

Lemma 2 Let $I \subseteq R$, $f: I \to R$ be a absolutely countinous mapping I^0 where $a,b \in I$ with $a<b$, then the following equality holds*:*

$$\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx = (b-a)\int_0^1 p(t)f'(tb+(1-t)a)\,dt \tag{9}$$

where

$$p(t)=\begin{cases} t-\frac{1}{6}\,, & t\in\left[0,\frac{1}{2}\right) \\ t-\frac{5}{6}\,, & t\in\left[\frac{1}{2},1\right] \end{cases}.$$

Theorem 7. Let $f: I \subset [0,\infty) \to R$ be a differentiable mapping I^0 such that $f' \in L_1[a,b]$, where $a,b \in I$, with $a<b$. If $|f'|$ is trigonometrically convex function on $[a,b]$, then the following inequality holds for $t \in [0,1]$:

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \le (b-a)\left(\frac{24\left(1+\sqrt{2}-\sqrt{6}\right)+2\pi}{3\pi^2}\right)A(|f'(a)|,|f'(b)|) \tag{10}$$

where A is the arithmetic mean, $\frac{1}{p}+\frac{1}{q}=1$.

Proof: By using the Lemma 2, trigonometrically convexity of the function $|f'|$ and the inequality

$$|f'(tb+(1-t)a)| \leq sin\left(\frac{\pi t}{2}\right)|f'(b)| + cos\left(\frac{\pi t}{2}\right)|f'(a)|,$$

we obtain

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq (b-a)\left|\int_0^1 p(t)f'(tb+(1-t)a)\,dt\right|$$

$$\leq (b-a)\int_0^{1/2}\left|t-\frac{1}{6}\right||f'(tb+(1-t)a)|\,dt + (b-a)\int_{1/2}^1\left|t-\frac{5}{6}\right||f'(tb+(1-t)a)|\,dt$$

$$\leq (b-a)\int_0^{1/2}\left|t-\frac{1}{6}\right|\left[cos\left(\frac{\pi t}{2}\right)|f'(a)| + sin\left(\frac{\pi t}{2}\right)|f'(b)|\right]dt$$

$$+(b-a)\int_{1/2}^1\left|t-\frac{5}{6}\right|\left[cos\left(\frac{\pi t}{2}\right)|f'(a)| + sin\left(\frac{\pi t}{2}\right)|f'(b)|\right]dt$$

$$=(b-a)\int_0^{1/6}\left(\frac{1}{6}-t\right)\left[cos\left(\frac{\pi t}{2}\right)|f'(a)| + sin\left(\frac{\pi t}{2}\right)|f'(b)|\right]dt$$

$$+(b-a)\int_{1/6}^{1/2}\left(t-\frac{1}{6}\right)\left[cos\left(\frac{\pi t}{2}\right)|f'(a)| + sin\left(\frac{\pi t}{2}\right)|f'(b)|\right]dt$$

$$+(b-a)\int_{1/2}^{5/6}\left(\frac{5}{6}-t\right)\left[cos\left(\frac{\pi t}{2}\right)|f'(a)| + sin\left(\frac{\pi t}{2}\right)|f'(b)|\right]dt$$

$$+(b-a)\int_{5/6}^{1}\left(t-\frac{5}{6}\right)\left[cos\left(\frac{\pi t}{2}\right)|f'(a)| + sin\left(\frac{\pi t}{2}\right)|f'(b)|\right]dt$$

$$=(b-a)\left[\int_0^{1/6}\left(\frac{1}{6}-t\right)cos\left(\frac{\pi t}{2}\right)dt + \int_{1/6}^{1/2}\left(t-\frac{1}{6}\right)cos\left(\frac{\pi t}{2}\right)dt + \int_{1/2}^{5/6}\left(\frac{5}{6}-t\right)cos\left(\frac{\pi t}{2}\right)dt\right.$$

$$\left.+\int_{5/6}^{1}\left(t-\frac{5}{6}\right)cos\left(\frac{\pi t}{2}\right)dt\right]|f'(a)| + (b-a)\left[\int_0^{1/6}\left(\frac{1}{6}-t\right)sin\left(\frac{\pi t}{2}\right)dt + \int_{1/6}^{1/2}\left(t-\frac{1}{6}\right)sin\left(\frac{\pi t}{2}\right)dt\right.$$

$$\left.+\int_{1/2}^{5/6}\left(\frac{5}{6}-t\right)sin\left(\frac{\pi t}{2}\right)dt + \int_{5/6}^{1}\left(t-\frac{5}{6}\right)sin\left(\frac{\pi t}{2}\right)dt\right]|f'(b)|$$

$$=(b-a)\left(\frac{4\sqrt{2}(1-\sqrt{3})}{\pi^2}+\frac{12+\pi}{3\pi^2}\right)|f'(a)| + (b-a)\left(\frac{4\sqrt{2}(1-\sqrt{3})}{\pi^2}+\frac{12+\pi}{3\pi^2}\right)|f'(b)|$$

$$=(b-a)\left(\frac{24(1+\sqrt{2}-\sqrt{6})+2\pi}{3\pi^2}\right)A(|f'(a)|,|f'(b)|)$$

where

$$\int_0^{1/6}\left(\frac{1}{6}-t\right)cos\left(\frac{\pi t}{2}\right)dt=\frac{12-12\cos\left(\frac{\pi}{12}\right)}{3\pi^2},$$

$$\int_{1/6}^{1/2}\left(t-\frac{1}{6}\right)cos\left(\frac{\pi t}{2}\right)dt=\frac{\sqrt{2}\pi+6\sqrt{2}-12cos(\frac{\pi}{12})}{3\pi^2},$$

$$\int_{1/2}^{5/6}\left(\frac{5}{6}-t\right)cos\left(\frac{\pi t}{2}\right)dt=\frac{-\sqrt{2}\pi+6\sqrt{2}-12\cos\left(\frac{5\pi}{12}\right)}{3\pi^2},$$

$$\int_{5/6}^{1}\left(t-\frac{5}{6}\right)cos\left(\frac{\pi t}{2}\right)dt=\frac{\pi-12cos\left(\frac{5\pi}{12}\right)}{3\pi^2},$$

$$\int_0^{1/6}\left(\frac{1}{6}-t\right)sin\left(\frac{\pi t}{2}\right)dt=\frac{\pi-12\sin\left(\frac{\pi}{12}\right)}{3\pi^2},$$

$$\int_{1/6}^{1/2}\left(t-\frac{1}{6}\right)sin\left(\frac{\pi t}{2}\right)dt=\frac{-\sqrt{2}\pi+6\sqrt{2}-12sin\left(\frac{\pi}{12}\right)}{3\pi^2},$$

$$\int_{1/2}^{5/6}\left(\frac{5}{6}-t\right)sin\left(\frac{\pi t}{2}\right)dt=\frac{\sqrt{2}\pi+6\sqrt{2}-12\sin\left(\frac{5\pi}{12}\right)}{3\pi^2},$$

$$\int_{5/6}^{1}\left(t-\frac{5}{6}\right)sin\left(\frac{\pi t}{2}\right)dt=\frac{12-12sin\left(\frac{5\pi}{12}\right)}{3\pi^2}.$$

This completes the proof of the Theorem.

An immediate consequence of Theorem 7 is the following Corollary:

Corollary 1. Let $f: I\subset[0,\infty)\to R$ be a differentiable mapping I^0 such that $f'\in L_1[a,b]$ where $a,b\in I$ with $a<b$. If $f(a)=f\left(\frac{a+b}{2}\right)=f(b)$ and $|f'|$ is trigonometrically convex function on $[a,b]$, then we have,

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-f(\frac{a+b}{2})\right|\le(b-a)\left(\frac{24\left(1+\sqrt{2}-\sqrt{6}\right)+2\pi}{3\pi^2}\right)A(|f'(a)|,|f'(b)|).$$

Theorem 8. Let $f: I\subset[0,\infty)\to R$ be a differentiable mapping I^0 such that $f'\in L_1[a,b]$ where $a,b\in I$ with $a<b$. If $|f'|^q$ is trigonometrically convex function on $[a,b]$ and $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. Then the following inequality holds for $t\in[0,1]$.

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\le(b-a)\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}\left[\left(\frac{\sqrt{2}}{\pi}|f'(a)|^q+\frac{2-\sqrt{2}}{\pi}|f'(b)|^q\right)^{\frac{1}{q}}+\left(\frac{2-\sqrt{2}}{\pi}|f'(a)|^q+\frac{\sqrt{2}}{\pi}|f'(b)|^q\right)^{\frac{1}{q}}\right]\quad(11)$$

Proof: By using the Lemma 2, trigonometrically convexity of the function $|f'|^q$, the inequality

$$|f'(tb+(1-t)a)|^q \le sin\left(\frac{\pi t}{2}\right)|f'(b)|^q + cos\left(\frac{\pi t}{2}\right)|f'(a)|^q$$

and well-known Hölder integral inequality, we have

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\le (b-a)\left|\int_0^1 p(t)f'(tb+(1-t)a)\,dt\right|$$

$$\le (b-a)\int_0^{1/2}\left|t-\frac{1}{6}\right||f'(tb+(1-t)a)|\,dt + (b-a)\int_{1/2}^1\left|t-\frac{5}{6}\right||f'(tb+(1-t)a)|\,dt$$

$$\le (b-a)\left(\int_0^{1/2}\left|t-\frac{1}{6}\right|^p dt\right)^{1/p}\left(\int_0^{1/2}|f'(tb+(1-t)a)|^q\,dt\right)^{1/q}$$

$$+(b-a)\left(\int_{1/2}^1\left|t-\frac{5}{6}\right|^p dt\right)^{1/p}\left(\int_{1/2}^1|f'(tb+(1-t)a)|^q\,dt\right)^{1/q}$$

$$= (b-a)\left(\int_0^{1/2}\left|t-\frac{1}{6}\right|^p dt\right)^{1/p}\left(\int_0^{1/2} sin\left(\frac{\pi t}{2}\right)|f'(b)|^q dt + \int_0^{1/2} cos\left(\frac{\pi t}{2}\right)|f'(a)|^q dt\right)^{1/q}$$

$$+(b-a)\left(\int_{1/2}^1\left|t-\frac{5}{6}\right|^p dt\right)^{1/p}\left(\int_{1/2}^1 sin\left(\frac{\pi t}{2}\right)|f'(b)|^q dt + \int_{1/2}^1 cos\left(\frac{\pi t}{2}\right)|f'(a)|^q dt\right)^{1/q}$$

$$= (b-a)\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{1/p}\left[\left(\frac{2-\sqrt{2}}{\pi}|f'(b)|^q + \frac{\sqrt{2}}{\pi}|f'(a)|^q\right)^{\frac{1}{q}} + \left(\frac{\sqrt{2}}{\pi}|f'(b)|^q + \frac{2-\sqrt{2}}{\pi}|f'(a)|^q\right)^{1/q}\right]$$

where

$$\int_0^{1/2} sin\left(\frac{\pi t}{2}\right)dt = \int_{1/2}^1 cos\left(\frac{\pi t}{2}\right)dt = \frac{2-\sqrt{2}}{\pi},$$

$$\int_{1/2}^1 sin\left(\frac{\pi t}{2}\right)dt = \int_0^{1/2} cos\left(\frac{\pi t}{2}\right)dt = \frac{\sqrt{2}}{\pi}$$

and

$$\int_0^{1/6}\left(\frac{1}{6}-t\right)^p dt + \int_{1/6}^{1/2}\left(t-\frac{1}{6}\right)^p dt = \int_{1/2}^{5/6}\left(\frac{5}{6}-t\right)^p dt + \int_{5/6}^1\left(t-\frac{5}{6}\right)^p dt = \frac{1+2^{p+1}}{6^{p+1}(p+1)}$$

which completes the proof.

Corollary 2. Let $f: I \subset [0,\infty) \to R$ be a differentiable mapping I^0 such that $f' \in L_1[a,b]$ where $a,b \in I$ with $a < b$. If $|f'(a)| = |f'(b)|$ and $|f'|^2$ is trigonometrically convex function on $[a,b]$ and $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then we get the following the inequality:

$$\left|\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{3\sqrt{\pi}}.|f'(a)|.$$

Corollary 3. Under the assumption of Corollary 2 with $f(a) = f\left(\frac{a+b}{2}\right) = f(b)$, then we have

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{b-a}{3\sqrt{\pi}}.|f'(a)|$$

Theorem 9. Let $f: I \subset [0,\infty) \to R$ be a differentiable mapping I^0 such that $f' \in L_1[a,b]$ where $a,b \in I$ with $a < b$. If $|f'|^q$ is trigonometrically convex function on $[a,b]$ and $q > 1$, then the following inequality holds:

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq 2(b-a)\left(\frac{2^{p+2}+3p+5}{6^{p+2}(p+1)(p+2)}\right)^{\frac{1}{p}}\left(\frac{\pi - 2\sqrt{2}}{\pi^2}|f'(b)|^q + \frac{4-2\sqrt{2}}{\pi^2}|f'(a)|^q\right)^{\frac{1}{q}} \\
&+2(b-a)\left(\frac{2^{p+1}(3p+4)+1}{6^{p+2}(p+1)(p+2)}\right)^{\frac{1}{p}}\left(\frac{2\sqrt{2} - \frac{\sqrt{2}}{2}\pi}{\pi^2}|f'(b)|^q + \frac{\frac{\sqrt{2}}{2}\pi + 2\sqrt{2} - 4}{\pi^2}|f'(a)|^q\right)^{\frac{1}{q}} \\
&+2(b-a)\left(\frac{2^{p+1}(3p+4)+1}{6^{p+2}(p+1)(p+2)}\right)^{\frac{1}{p}}\left(\frac{\frac{\sqrt{2}}{2}\pi + 2\sqrt{2} - 4}{\pi^2}|f'(b)|^q + \frac{2\sqrt{2} - \frac{\sqrt{2}}{2}\pi}{\pi^2}|f'(a)|^q\right)^{\frac{1}{q}} \\
&+2(b-a)\left(\frac{2^{p+2}+3p+5}{6^{p+2}(p+1)(p+2)}\right)^{\frac{1}{p}}\left(\frac{4-2\sqrt{2}}{\pi^2}|f'(b)|^q + \frac{\pi - 2\sqrt{2}}{\pi^2}|f'(a)|^q\right)^{\frac{1}{q}} \qquad (12)
\end{aligned}$$

where, with $\frac{1}{p} + \frac{1}{q} = 1$.

Proof: Since $|f'|^q$ trigonometrically convex function

$$|f'(tb + (1-t)a)|^q \leq sin\left(\frac{\pi t}{2}\right)|f'(b)|^q + cos\left(\frac{\pi t}{2}\right)|f'(a)|^q$$

from Lemma 2 and using the Hölder-İşcan integral inequality, we have the inequality

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq (b-a)\left|\int_0^1 p(t)f'(tb + (1-t)a)\,dt\right| \\
&\leq (b-a)\int_0^{1/2}\left|t - \frac{1}{6}\right||f'(tb + (1-t)a)|\,dt + (b-a)\int_{1/2}^1\left|t - \frac{5}{6}\right||f'(tb + (1-t)a)|\,dt
\end{aligned}$$

$$\leq 2(b-a)\left(\int_0^{1/2}\left(\frac{1}{2}-t\right)\left|t-\frac{1}{6}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^{1/2}\left(\frac{1}{2}-t\right)|f'(tb+(1-t)a)|^q\, dt\right)^{\frac{1}{q}}$$
$$+2(b-a)\left(\int_0^{1/2} t\left|t-\frac{1}{6}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^{1/2} t|f'(tb+(1-t)a)|^q\, dt\right)^{\frac{1}{q}}$$
$$+2(b-a)\left(\int_{1/2}^{1}(1-t)\left|t-\frac{5}{6}\right|^p dt\right)^{\frac{1}{p}}\left(\int_{1/2}^{1}(1-t)|f'(tb+(1-t)a)|^q\, dt\right)^{\frac{1}{q}}$$
$$+2(b-a)\left(\int_{1/2}^{1}\left(t-\frac{1}{2}\right)\left|t-\frac{5}{6}\right|^p dt\right)^{\frac{1}{p}}\left(\int_{1/2}^{1}\left(t-\frac{1}{2}\right)|f'(tb+(1-t)a)|^q\, dt\right)^{\frac{1}{q}}$$
$$\leq 2(b-a)\left(\int_0^{1/2}\left(\frac{1}{2}-t\right)\left|t-\frac{1}{6}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^{1/2}\left(\frac{1}{2}-t\right)\sin\left(\frac{\pi t}{2}\right)|f'(b)|^q\, dt+\int_0^{1/2}\left(\frac{1}{2}-t\right)\cos\left(\frac{\pi t}{2}\right)|f'(a)|^q\, dt\right)^{\frac{1}{q}}$$
$$+2(b-a)\left(\int_0^{1/2} t\left|t-\frac{1}{6}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^{1/2} t\sin\left(\frac{\pi t}{2}\right)|f'(b)|^q\, dt+\int_0^{1/2} t\cos\left(\frac{\pi t}{2}\right)|f'(a)|^q\, dt\right)^{\frac{1}{q}}$$
$$+2(b-a)\left(\int_{1/2}^{1}(1-t)\left|t-\frac{5}{6}\right|^p dt\right)^{\frac{1}{p}}\left(\int_{1/2}^{1}(1-t)\sin\left(\frac{\pi t}{2}\right)|f'(b)|^q\, dt+\int_{1/2}^{1}(1-t)\cos\left(\frac{\pi t}{2}\right)|f'(a)|^q\, dt\right)^{\frac{1}{q}}$$
$$+2(b-a)\left(\int_{1/2}^{1}\left(t-\frac{1}{2}\right)\left|t-\frac{5}{6}\right|^p dt\right)^{\frac{1}{p}}\left(\int_{1/2}^{1}\left(t-\frac{1}{2}\right)\sin\left(\frac{\pi t}{2}\right)|f'(b)|^q\, dt+\int_{1/2}^{1}\left(t-\frac{1}{2}\right)\cos\left(\frac{\pi t}{2}\right)|f'(a)|^q\, dt\right)^{\frac{1}{q}}$$
$$=2(b-a)\left(\frac{2^{p+2}+3p+5}{6^{p+2}(p+1)(p+2)}\right)^{\frac{1}{p}}\left(\frac{\pi-2\sqrt{2}}{\pi^2}|f'(b)|^q+\frac{4-2\sqrt{2}}{\pi^2}|f'(a)|^q\right)^{\frac{1}{q}}$$
$$+2(b-a)\left(\frac{2^{p+1}(3p+4)+1}{6^{p+2}(p+1)(p+2)}\right)^{\frac{1}{p}}\left(\frac{2\sqrt{2}-\frac{\sqrt{2}}{2}\pi}{\pi^2}|f'(b)|^q+\frac{\frac{\sqrt{2}}{2}\pi+2\sqrt{2}-4}{\pi^2}|f'(a)|^q\right)^{\frac{1}{q}}$$
$$+2(b-a)\left(\frac{2^{p+1}(3p+4)+1}{6^{p+2}(p+1)(p+2)}\right)^{\frac{1}{p}}\left(\frac{\frac{\sqrt{2}}{2}\pi+2\sqrt{2}-4}{\pi^2}|f'(b)|^q+\frac{2\sqrt{2}-\frac{\sqrt{2}}{2}\pi}{\pi^2}|f'(a)|^q\right)^{\frac{1}{q}}$$
$$+2(b-a)\left(\frac{2^{p+2}+3p+5}{6^{p+2}(p+1)(p+2)}\right)^{\frac{1}{p}}\left(\frac{4-2\sqrt{2}}{\pi^2}|f'(b)|^q+\frac{\pi-2\sqrt{2}}{\pi^2}|f'(a)|^q\right)^{\frac{1}{q}}$$

where

$$\int_0^{1/2}\left(\frac{1}{2}-t\right)\left|t-\frac{1}{6}\right|^p dt=\int_{1/2}^{1}\left(t-\frac{1}{2}\right)\left|t-\frac{5}{6}\right|^p dt=\frac{2^{p+2}+3p+5}{6^{p+2}(p+1)(p+2)}$$
$$\int_0^{1/2} t\left|t-\frac{1}{6}\right|^p dt=\int_{1/2}^{1}(1-t)\left|t-\frac{5}{6}\right|^p dt=\frac{2^{p+1}(3p+4)+1}{6^{p+2}(p+1)(p+2)}$$

$$\int_0^{1/2}\left(\frac{1}{2}-t\right)\sin\left(\frac{\pi t}{2}\right)dt=\int_{1/2}^{1}\left(t-\frac{1}{2}\right)\cos\left(\frac{\pi t}{2}\right)dt=\frac{\pi-2\sqrt{2}}{\pi^2}$$

$$\int_0^{1/2}\left(\frac{1}{2}-t\right)\cos\left(\frac{\pi t}{2}\right)dt=\int_{1/2}^{1}\left(t-\frac{1}{2}\right)\sin\left(\frac{\pi t}{2}\right)dt=\frac{4-2\sqrt{2}}{\pi^2}$$

$$\int_0^{1/2} t\sin\left(\frac{\pi t}{2}\right)dt=\int_{1/2}^{1}(1-t)\cos\left(\frac{\pi t}{2}\right)dt=\frac{2\sqrt{2}-\frac{\sqrt{2}}{2}\pi}{\pi^2}$$

$$\int_0^{1/2} t\cos\left(\frac{\pi t}{2}\right)dt=\int_{1/2}^{1}(1-t)\sin\left(\frac{\pi t}{2}\right)dt=\frac{\frac{\sqrt{2}}{2}\pi+2\sqrt{2}-4}{\pi^2}$$

The proof is completed.

Theorem 10. Let $f: I\subset[0,\infty)\to R$ be a differentiable mapping I^0 such that $f'\in L_1[a,b]$ where $a,b\in I$ with $a<b$. If $|f'|^q$ is trigonometrically convex function on $[a,b]$ and $q\geq 1$ with $\frac{1}{p}+\frac{1}{q}=1$. Then the following inequality holds for $t\in[0,1]$:

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq(b-a)\left(\frac{5}{72}\right)^{1-\frac{1}{q}}\left\{\left[\left(\frac{\pi(1-\sqrt{2})+6\sqrt{2}-24sin\frac{\pi}{12}}{3\pi^2}\right)|f'b)|^q+\left(\frac{\sqrt{2}(6+\pi)+12-24cos\frac{\pi}{12}}{3\pi^2}\right)|f'a)|^q\right]^{\frac{1}{q}}+\right.$$

$$\left.\left[\left(\frac{\sqrt{2}(6+\pi)+12-24sin\frac{5\pi}{12}}{3\pi^2}\right)|f'b)|^q+\left(\frac{\pi(1-\sqrt{2})+6\sqrt{2}-24cos\frac{5\pi}{12}}{3\pi^2}\right)|f'a)|^q\right]^{\frac{1}{q}}\right\} \quad (13)$$

Proof: From the Lemma 2, power-mean integral inequality and trigonometrically convexity of the function $|f'|^q$, we have

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq(b-a)\left|\int_0^1 p(t)f'(tb+(1-t)a)\,dt\right|$$

$$\leq(b-a)\int_0^{1/2}\left|t-\frac{1}{6}\right||f'(tb+(1-t)a)|\,dt+(b-a)\int_{1/2}^{1}\left|t-\frac{5}{6}\right||f'(tb+(1-t)a)|\,dt$$

$$\leq(b-a)\left(\int_0^{1/2}\left|t-\frac{1}{6}\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^{1/2}\left|t-\frac{1}{6}\right||f'(tb+(1-t)a)|^q\,dt\right)^{1/q}$$

$$+(b-a)\left(\int_{1/2}^{1}\left|t-\frac{5}{6}\right|dt\right)^{1-\frac{1}{q}}\left(\int_{1/2}^{1}\left|t-\frac{5}{6}\right||f'(tb+(1-t)a)|^q\,dt\right)^{\frac{1}{q}}$$

$$= (b-a)\left(\int_0^{1/2}\left|t-\frac{1}{6}\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^{1/6}\left(\frac{1}{6}-t\right)|f'(tb+(1-t)a)|^q dt+\int_{1/6}^{1/2}\left(t-\frac{1}{6}\right)|f'(tb+(1-t)a)|^q dt\right)$$

$$+(b-a)\left(\int_{1/2}^{1}\left|t-\frac{5}{6}\right|dt\right)^{1-\frac{1}{q}}\left(\int_{1/2}^{5/6}\left(\frac{5}{6}-t\right)|f'(tb+(1-t)a)|^q dt+\int_{5/6}^{1}\left(t-\frac{5}{6}\right)|f'(tb+(1-t)a)|^q dt\right)$$

$$\leq (b-a)\left(\int_0^{1/2}\left|t-\frac{1}{6}\right|dt\right)^{1-\frac{1}{q}}\left(\begin{array}{c}\int_0^{1/6}\left(\frac{1}{6}-t\right)\left(sin\left(\frac{\pi t}{2}\right)|f'(b)|^q+cos\left(\frac{\pi t}{2}\right)|f'(a)|^q\right)dt\\+\int_{1/6}^{1/2}\left(t-\frac{1}{6}\right)\left(sin\left(\frac{\pi t}{2}\right)|f'(b)|^q+cos\left(\frac{\pi t}{2}\right)|f'(a)|^q\right)dt\end{array}\right)$$

$$+(b-a)\left(\int_{1/2}^{1}\left|t-\frac{5}{6}\right|dt\right)^{1-\frac{1}{q}}\left(\begin{array}{c}\int_{1/2}^{5/6}\left(\frac{5}{6}-t\right)\left(sin\left(\frac{\pi t}{2}\right)|f'(b)|^q+cos\left(\frac{\pi t}{2}\right)|f'(a)|^q\right)dt\\+\int_{5/6}^{1}\left(t-\frac{5}{6}\right)\left(sin\left(\frac{\pi t}{2}\right)|f'(b)|^q+cos\left(\frac{\pi t}{2}\right)|f'(a)|^q\right)dt\end{array}\right)$$

$$= (b-a)\left(\frac{5}{72}\right)^{1-\frac{1}{q}}\left(\left(\frac{2\sqrt{2}-8sin\frac{\pi}{12}}{\pi^2}+\frac{1-\sqrt{2}}{3\pi}\right)|f'(b)|^q+\left(\frac{2\sqrt{2}-8cos\frac{\pi}{12}}{\pi^2}+\frac{12+\sqrt{2}\pi}{3\pi^2}\right)|f'(a)|^q\right)^{\frac{1}{q}}$$

$$+(b-a)\left(\frac{5}{72}\right)^{1-\frac{1}{q}}\left(\left(\frac{2\sqrt{2}-8sin\frac{5\pi}{12}}{\pi^2}+\frac{12+\sqrt{2}\pi}{3\pi^2}\right)|f'(b)|^q+\left(\frac{2\sqrt{2}-8cos\frac{5\pi}{12}}{\pi^2}+\frac{1-\sqrt{2}}{3\pi}\right)|f'(a)|^q\right)^{\frac{1}{q}}$$

where

$$\int_0^{1/2}\left|t-\frac{1}{6}\right|dt=\int_{1/2}^{1}\left|t-\frac{5}{6}\right|dt=\frac{5}{72},$$

when the above inequalities are adjusted, the desired proof of the Theorem 10 is completed.

For $q=1$ we use the proof methods of Theorem 7, it follows the above methods step by step.

Corollary 4. Under the assumption of Theorem 10 with 2 with $q=1$, we get the conclusion of Thereom 7.

Theorem 11. Let $f: I \subset R \to R$ be a differentiable mapping I^0 such that $f' \in L_1[a,b]$ where $a,b \in I$ with $a<b$. If $|f'|^q$ is trigonometrically convex function on $[a,b]$ and $q \geq 1$ with $\frac{1}{p}+\frac{1}{q}=1$. Then the following inequality holds for $t \in [0,1]$:

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq 2(b-a)\left(\frac{1}{81}\right)^{1-\frac{1}{q}}\left(\begin{array}{c}\frac{96\cos\left(\frac{\pi}{12}\right)-8\pi\sin\left(\frac{\pi}{12}\right)+\pi\left(\frac{\pi}{2}-2\sqrt{2}\right)-48-24\sqrt{2}}{3\pi^3}|f'(b)|^q\\ \frac{-96\sin\left(\frac{\pi}{12}\right)-8\pi\cos\left(\frac{\pi}{12}\right)+\pi\left(8-2\sqrt{2}\right)+24\sqrt{2}}{3\pi^3}|f'(a)|^q\end{array}\right)^{\frac{1}{q}}$$

$$+2(b-a)\left(\frac{29}{1296}\right)^{1-\frac{1}{q}}\left(\begin{array}{c}\frac{-96\cos\left(\frac{\pi}{12}\right)-4\pi\sin\left(\frac{\pi}{12}\right)-\pi\left(\frac{\sqrt{2}}{2}\pi-5\sqrt{2}\right)+24\sqrt{2}+48}{3\pi^3}|f'(b)|^q\\ \frac{96\sin\left(\frac{\pi}{12}\right)-4\pi\cos\left(\frac{\pi}{12}\right)+\pi\left(\frac{\sqrt{2}}{2}\pi+5\sqrt{2}-2\right)-24\sqrt{2}}{3\pi^3}|f'(a)|^q\end{array}\right)^{\frac{1}{q}}$$

$$+2(b-a)\left(\frac{29}{1296}\right)^{1-\frac{1}{q}}\left(\begin{array}{c}\frac{96\cos\left(\frac{5\pi}{12}\right)-4\pi\sin\left(\frac{5\pi}{12}\right)+\pi\left(\frac{\sqrt{2}}{2}\pi+5\sqrt{2}-2\right)-24\sqrt{2}}{3\pi^3}|f'(b)|^q\\ \frac{-96\sin\left(\frac{5\pi}{12}\right)-4\pi\cos\left(\frac{5\pi}{12}\right)-\pi\left(\frac{\sqrt{2}}{2}\pi-5\sqrt{2}\right)+48+24\sqrt{2}}{3\pi^3}|f'(a)|^q\end{array}\right)^{\frac{1}{q}}$$

$$+2(b-a)\left(\frac{1}{81}\right)^{1-\frac{1}{q}}\left(\begin{array}{c}\frac{-96\cos\left(\frac{5\pi}{12}\right)-8\pi\sin\left(\frac{5\pi}{12}\right)-\pi\left(2\sqrt{2}-8\right)+24\sqrt{2}}{3\pi^3}|f'(b)|^q\\ \frac{96\sin\left(\frac{5\pi}{12}\right)-8\pi\cos\left(\frac{5\pi}{12}\right)+\pi\left(\frac{\pi}{2}-2\sqrt{2}\right)-48-24\sqrt{2}}{3\pi^3}|f'(a)|^q\end{array}\right)^{\frac{1}{q}} \quad (14)$$

Proof: From Lemma 2, improved power-mean integral inequality and definition of trigonometrically of the function $|f'|^q$, we have

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq (b-a)\left|\int_0^1 p(t)f'(tb+(1-t)a)\,dt\right|$$

$$\leq (b-a)\int_0^{1/2}\left|t-\frac{1}{6}\right||f'(tb+(1-t)a)|\,dt+(b-a)\int_{1/2}^1\left|t-\frac{5}{6}\right||f'(tb+(1-t)a)|\,dt$$

$$\leq 2(b-a)\left(\int_0^{1/2}\left(\frac{1}{2}-t\right)\left|t-\frac{1}{6}\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^{1/2}\left(\frac{1}{2}-t\right)\left|t-\frac{1}{6}\right|\left(sin\left(\frac{\pi t}{2}\right)|f'(b)|^q+cos\left(\frac{\pi t}{2}\right)|f'(a)|^q\right)dt\right)^{1/q}$$

$$+2(b-a)\left(\int_0^{1/2}t\left|t-\frac{1}{6}\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^{1/2}t\left|t-\frac{1}{6}\right|\left(sin\left(\frac{\pi t}{2}\right)|f'(b)|^q+cos\left(\frac{\pi t}{2}\right)|f'(a)|^q\right)dt\right)^{1/q}$$

$$+2(b-a)\left(\int_{1/2}^1(1-t)\left|t-\frac{5}{6}\right|dt\right)^{1-\frac{1}{q}}\left(\int_{1/2}^1(1-t)\left|t-\frac{5}{6}\right|\left(sin\left(\frac{\pi t}{2}\right)|f'(b)|^q+cos\left(\frac{\pi t}{2}\right)|f'(a)|^q\right)dt\right)^{\frac{1}{q}}$$

$$+2(b-a)\left(\int_{1/2}^1\left(t-\frac{1}{2}\right)\left|t-\frac{5}{6}\right|dt\right)^{1-\frac{1}{q}}\left(\int_{1/2}^1\left(t-\frac{1}{2}\right)\left|t-\frac{5}{6}\right|\left(sin\left(\frac{\pi t}{2}\right)|f'(b)|^q+cos\left(\frac{\pi t}{2}\right)|f'(a)|^q\right)dt\right)^{\frac{1}{q}}$$

$$=2(b-a)\left(\frac{1}{81}\right)^{1-\frac{1}{q}}\left(\begin{array}{c}\frac{96\cos\left(\frac{\pi}{12}\right)-8\pi\sin\left(\frac{\pi}{12}\right)+\pi\left(\frac{\pi}{2}-2\sqrt{2}\right)-48-24\sqrt{2}}{3\pi^3}|f'(b)|^q\\ \frac{-96\sin\left(\frac{\pi}{12}\right)-8\pi\cos\left(\frac{\pi}{12}\right)+\pi\left(8-2\sqrt{2}\right)+24\sqrt{2}}{3\pi^3}|f'(a)|^q\end{array}\right)^{\frac{1}{q}}$$

$$+2(b-a)\left(\frac{29}{1296}\right)^{1-\frac{1}{q}}\left(\begin{array}{c}\frac{-96\cos\left(\frac{\pi}{12}\right)-4\pi\sin\left(\frac{\pi}{12}\right)-\pi\left(\frac{\sqrt{2}\pi}{2}-5\sqrt{2}\right)+24\sqrt{2}+48}{3\pi^3}|f'(b)|^q\\ \frac{96\sin\left(\frac{\pi}{12}\right)-4\pi\cos\left(\frac{\pi}{12}\right)+\pi\left(\frac{\sqrt{2}}{2}\pi+5\sqrt{2}-2\right)-24\sqrt{2}}{3\pi^3}|f'(a)|^q\end{array}\right)^{\frac{1}{q}}$$

$$+2(b-a)\left(\frac{29}{1296}\right)^{1-\frac{1}{q}}\left(\begin{array}{c}\frac{96\cos\left(\frac{5\pi}{12}\right)-4\pi\sin\left(\frac{5\pi}{12}\right)+\pi\left(\frac{\sqrt{2}}{2}\pi+5\sqrt{2}-2\right)-24\sqrt{2}}{3\pi^3}|f'(b)|^q\\ \frac{-96\sin\left(\frac{5\pi}{12}\right)-4\pi\cos\left(\frac{5\pi}{12}\right)-\pi\left(\frac{\sqrt{2}}{2}\pi-5\sqrt{2}\right)+48+24\sqrt{2}}{3\pi^3}|f'(a)|^q\end{array}\right)^{\frac{1}{q}}$$

$$+2(b-a)\left(\frac{1}{81}\right)^{1-\frac{1}{q}}\left(\begin{array}{c}\frac{-96\cos\left(\frac{5\pi}{12}\right)-8\pi\sin\left(\frac{5\pi}{12}\right)-\pi\left(2\sqrt{2}-8\right)+24\sqrt{2}}{3\pi^3}|f'(b)|^q\\ \frac{96\sin\left(\frac{5\pi}{12}\right)-8\pi\cos\left(\frac{5\pi}{12}\right)+\pi\left(\frac{\pi}{2}-2\sqrt{2}\right)-48-24\sqrt{2}}{3\pi^3}|f'(a)|^q\end{array}\right)^{\frac{1}{q}}$$

where

$$\int_0^{1/2}\left(\frac{1}{2}-t\right)\left|t-\frac{1}{6}\right|dt=\int_{1/2}^{1}\left(t-\frac{1}{2}\right)\left|t-\frac{5}{6}\right|dt=\frac{1}{81}\quad\text{and}\quad\int_0^{1/2}t\left|t-\frac{1}{6}\right|dt=\int_{1/2}^{1}(1-t)\left|t-\frac{5}{6}\right|dt=\frac{29}{1296}$$

and

$$\int_0^{1/2}\left(\frac{1}{2}-t\right)\left|t-\frac{1}{6}\right|\sin(\frac{\pi t}{2})\,dt=\frac{96\cos\left(\frac{\pi}{12}\right)-8\pi\sin\left(\frac{\pi}{12}\right)+\pi\left(\frac{\pi}{2}-2\sqrt{2}\right)-48-24\sqrt{2}}{3\pi^3}$$

$$\int_0^{1/2}\left(\frac{1}{2}-t\right)\left|t-\frac{1}{6}\right|\cos\left(\frac{\pi t}{2}\right)dt=\frac{-96\sin\left(\frac{\pi}{12}\right)-8\pi\cos\left(\frac{\pi}{12}\right)+\pi\left(8-2\sqrt{2}\right)+24\sqrt{2}}{3\pi^3}$$

$$\int_0^{1/2}t\left|t-\frac{1}{6}\right|\sin\left(\frac{\pi t}{2}\right)dt=\frac{-96\cos\left(\frac{\pi}{12}\right)-4\pi\sin\left(\frac{\pi}{12}\right)-\pi\left(\frac{\sqrt{2}\pi}{2}-5\sqrt{2}\right)+24\sqrt{2}+48}{3\pi^3}$$

$$\int_0^{1/2}t\left|t-\frac{1}{6}\right|\cos\left(\frac{\pi t}{2}\right)dt=\frac{96\sin\left(\frac{\pi}{12}\right)-4\pi\cos\left(\frac{\pi}{12}\right)+\pi\left(\frac{\sqrt{2}}{2}\pi+5\sqrt{2}-2\right)-24\sqrt{2}}{3\pi^3}$$

$$\int_{1/2}^{1}(1-t)\left|t-\frac{5}{6}\right|\sin(\frac{\pi t}{2})\,dt=\frac{96\cos\left(\frac{5\pi}{12}\right)-4\pi\sin\left(\frac{5\pi}{12}\right)+\pi\left(\frac{\sqrt{2}}{2}\pi+5\sqrt{2}-2\right)-24\sqrt{2}}{3\pi^3}$$

$$\int_{1/2}^{1}(1-t)\left|t-\frac{5}{6}\right|\cos(\frac{\pi t}{2})\,dt=\frac{-96\sin\left(\frac{5\pi}{12}\right)-4\pi\cos\left(\frac{5\pi}{12}\right)-\pi\left(\frac{\sqrt{2}}{2}\pi-5\sqrt{2}\right)+48+24\sqrt{2}}{3\pi^3}$$

$$\int_{1/2}^{1}\left(t-\frac{1}{2}\right)\left|t-\frac{5}{6}\right|\sin(\frac{\pi t}{2})\,dt=\frac{-96\cos\left(\frac{5\pi}{12}\right)-8\pi\sin\left(\frac{5\pi}{12}\right)-\pi\left(2\sqrt{2}-8\right)+24\sqrt{2}}{3\pi^3}$$

$$\int_{1/2}^{1}\left(t-\frac{1}{2}\right)\left|t-\frac{5}{6}\right|\cos(\frac{\pi t}{2})\,dt=\frac{96\sin\left(\frac{5\pi}{12}\right)-8\pi\cos\left(\frac{5\pi}{12}\right)+\pi\left(\frac{\pi}{2}-2\sqrt{2}\right)-48-24\sqrt{2}}{3\pi^3}.$$

Therefore, the proof is completed.

Corollary 5. If we choose $q=1$ in Theorem 11, the inequality (14) becomes the inequality (10).

Conflicts of interest

There is no conflict of interest among the authors of the article

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