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On some bounded operators and their characterizations in Γ-Hilbert space

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Abstract

Some bounded operators are part of this paper.Through this paper we shall obtain common properties of Some bounded operators in Γ-Hilbert space. Also, introduced 2-self-adjoint operators and it's spectrum in Γ-Hilbert Space. Characterizations of these operators are also part of this literature.

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1. Introduction and Preliminaries

Inner product plays an important role in advance Mathematics. Γ-Hilbert space opened the scope of defining Inner product in many way and in many cases where Inner product is not defined. Γ-Hilbert space plays an important role in generalization of general linear quadretic control problem in an abstract space[1] which was motivated by the work of L.Debnath and Pitor Mikusinski[2] but there is not enough literature found to study the operators of Γ-Hilbert space. The definition of Γ-Hilbert space was introduced by Bhattacharya D.K. and T.E. Aman in their paper "Γ-Hilbert space and linear quadratic control problem" in 2003[1].
Now we will extend this work by defining some operators and their characterizations in Γ-Hilbert space .At first we recall the definitions of Γ-Hilbert space.

Definition 1.1: Let E, Γ be two linear spaces over the field F. A mapping $\langle .,.,. \rangle: E \times \Gamma \times E \to \mathbb{R}$ is called a Γ-Inner product on E if

(i) $\langle .,.,. \rangle$ is linear in each variable.
(ii) $\langle u,\gamma,v \rangle = \langle v,\gamma,u \rangle \ \forall\, u,v \in E \text{ and } \gamma \in \Gamma$.
(iii) $\langle u,\gamma,u \rangle > 0 \ \forall\, \gamma \neq 0 \text{ and } u \neq 0$.

$[(E,\Gamma), \langle .,.,. \rangle]$ is called a Γ-inner product space over F.
A complete Γ-inner product space is called Γ-Hilbert Space.

Using the Γ-Inner product ,we may define three types of norm in a Γ-Hilbert Space, namely (1) γ-Norm (ii) Γ_{inf}-Norm and (iii) Γ-Norm.

Definition 1.2 : If we write $\|u\|_\gamma{}^2 = \langle u,\gamma,u \rangle$ for $u \in H$ and $\gamma \in \Gamma$ then $\|u\|_\gamma{}^2$ satisfy all the conditions of Norm,then it is called γ-Norm.

Definition 1.3 : If we define $\|u\|_{\Gamma_{inf}} = \inf\{\|u\|_\gamma : \gamma \in \Gamma\}$ .Clearly Γ_{inf}-Norm satisfy all the condition of the Norm for $u \in H$.

Definition 1.4 : If we if write $\|u\|_\Gamma$ $=\{\|u\|_\gamma : \gamma \in \Gamma\}$ then this Norm is called the Γ-Norm of the Γ-Hilbert Space.

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2. Materials and Results

2.1 Self- adjoint operator on Γ-Hilbert space:

Let A be a bounded operator on Γ-Hilbert spaceand we denote it by H_Γ.Then the operator A^*: $H_\Gamma \to H_\Gamma$ defined by

$\langle Ax, \gamma, y\rangle = \langle x, \gamma, A^*y\rangle \quad \forall x, y \in H_\Gamma$ and $\gamma \in \Gamma$

is called the adjoint operator of A.

If A=A^* then A is called self-adjoint of H_Γ.

Properties:

Theorem 2.1.1 : Let A be a bounded operator on Γ-Hilbert space H_Γ.Then the operators $T_1 = A^* A$ and $T_2 = A + A^*$ are self-adjoint.

Proof: For all $x, y \in H_\Gamma$, we have

$$\langle T_1x, \gamma, y\rangle = \langle A^*Ax, \gamma, y\rangle$$
$$=\langle Ax, \gamma, Ay\rangle$$
$$=\langle x, \gamma, T_1y\rangle \quad \text{where} \quad \gamma \in \Gamma.$$

And
$$\langle T_2x, \gamma, y\rangle = \langle (A + A^*)x, \gamma, y\rangle$$
$$= \langle x, \gamma, (A + A^*)^*y\rangle$$
$$= \langle x, \gamma, (A + A^*)y\rangle$$
$$= \langle x, \gamma, T_2y\rangle \quad \text{where } \gamma \in \Gamma .$$

So T_1 and T_2 are self –adjoint.

Note: But $A - A^*$ is not self-adjoint.

If we take $T_3 = A - A^*$ then for all $x, y \in H_\Gamma$, we have

$$\langle T_3x, \gamma, y\rangle = \langle (A - A^*)x, \gamma, y\rangle = \langle x, \gamma, (A - A^*)^*y\rangle$$
$$= \langle x, \gamma, (A^* - A)y\rangle$$
$$= \langle x, \gamma, -(A - A^*)y\rangle$$
$$= \langle x, \gamma, -T_3y\rangle$$

So T_3 is not self-adjoint.

For example, if we consider a 2×2 matrix A which is complex such that

$$A=\begin{pmatrix} i & i \\ i & 1 \end{pmatrix}.$$

Then clearly that $A - A^*$ is not self -adjoint .

Theorem 2.1.2: If the product of two self –adjoint operators in a Γ-Hilbert space is self-adjoint if and only if the operators commute.

Proof: Let A and B be self adjoint operators. Then for all $x, y \in H_\Gamma$, we have

$$\langle ABx, \gamma, y\rangle = \langle Bx, \gamma, Ay\rangle$$
$$= \langle x, \gamma, BAy\rangle \quad \text{Where} \quad \gamma \in \Gamma.$$

Thus, if $AB = BA$, then AB is self-adjoint. Conversely ,if AB is self-adjoint,then the above implies

$$AB = (AB)^* = BA.$$

Theorem 2.1.3: Let T be a self –adjoint operator on a Γ-Hilbert spaceH_Γ. Then

$$\|T\|_\gamma = \sup_{\|x\|_\gamma=1} |\langle Tx, \gamma, x\rangle| \text{ where } \gamma \in \Gamma.$$

Proof: Let M= $\sup_{\|x\|_\gamma=1} |\langle Tx, \gamma, x\rangle|$ where $\gamma \in \Gamma$.

If $\|x\|_\gamma = 1$ then

$$\begin{aligned} |\langle Tx, \gamma, x\rangle| &\le \|Tx\|\|\gamma\|\|x\| \\ &\le \|Tx\| \\ &\le \|T\|\|x\| \\ &\le \|T\|_\gamma \end{aligned}$$

Thus M $\le \|T\|_\gamma$……………………… (1)

On the other hand $x, z \in H_\Gamma$, we have –

$$\begin{aligned} \langle T(x+z), \gamma, x+z\rangle - \langle T(x-z), \gamma, x-z\rangle &= 2(\langle Tx, \gamma, z\rangle + \langle Tz, \gamma, x\rangle) \\ &= 4\, Re\langle Tx, \gamma, z\rangle \text{ [Since T is self-adjoint operator]} \end{aligned}$$

Therefore ,

$$Re\, \langle Tx, \gamma, z\rangle \le \frac{M}{4} \left(\|x+z\|_\gamma{}^2 + \|x-z\|_\gamma{}^2\right)$$

$$= \frac{M}{2} \left(\|x\|_\gamma{}^2 + \|z\|_\gamma{}^2\right) \text{ ………………(2) [by parallelogram law]}$$

Now Suppose $\|x\|_\gamma \le 1$ and $\|z\|_\gamma \le 1$ .Then it follows that $Re\, \langle Tx, \gamma, z\rangle \le M$.

If $\langle Tx, \gamma, z\rangle = re^{i\theta}$ for $r \ge 0$ and $\theta \in \mathbb{R}$, then let $x_0 = e^{-i\theta}x$, So that $\|x_0\|_\gamma = \|x\|_\gamma \le 1$.

And

$$\begin{aligned} |\langle Tx, \gamma, z\rangle| &= r \\ &= \langle Tx_0, \gamma, z\rangle \\ &= Re\, \langle Tx_0, \gamma, z\rangle \\ &\le M \end{aligned}$$

Taking Supremum over all $x, z \in H_\Gamma$ with $\|x\|_\gamma \le 1$, $\|z\|_\gamma \le 1$, we obtain

$$\|T\|_\gamma \le M \text{ ....} \quad .(3)$$

Combinding (1) and (3) we get, $\|T\|_\gamma = M$.

Hence prove the theorem.

Note: Above theorem does not hold if T is not a self-adjoint operator as we cannot write

$$2(\langle Tx, \gamma, z\rangle + \langle Tz, \gamma, x\rangle) = 4\, Re\, \langle Tx, \gamma, z\rangle.$$

2.2 <u>Normal operator</u>:- A bounded operator T of a Γ-Hilbert space H_Γ is called a Normal operator if It commutes with its adjoint that is $TT^* = T^*T$.

Theorem 2.2.1: A bounded operator T is Normal if and only if $\|Tx\|_\gamma = \|T^*x\|_\gamma$ for all $x \in H_\Gamma$ and $\gamma \in \Gamma$.

Proof: For all $x \in H_\Gamma$ *and* $\gamma \in \Gamma$, we have-

$$\langle T^*Tx, \gamma, x\rangle = \langle Tx, \gamma, T^*x\rangle$$

$$=\|Tx\|_\gamma{}^2$$

If T is normal, then we have-

$$\langle T^*Tx, \gamma, x\rangle = \langle TT^*x, \gamma, x\rangle$$
$$= \langle T^*x, \gamma, T^*x\rangle$$
$$= \|T^*x\|_\gamma{}^2$$

And thus $\|Tx\|_\gamma = \|T^*x\|_\gamma$.

Now assume that $\|Tx\|_\gamma = \|T^*x\|_\gamma$ for all $x \in H_\Gamma$ *and* $\gamma \in \Gamma$ .Then By preceding argument we have-

$$\langle TT^*x, \gamma, x\rangle = \langle T^*Tx, \gamma, x\rangle \text{ for all } x \in H_\Gamma \text{ and } \gamma \in \Gamma .$$

So we can write -

$$TT^* = T^*T.$$

Note: The condition $\|Tx\|_\gamma = \|T^*x\|_\gamma$ for all $x \in H_\Gamma$ *and* $\gamma \in \Gamma$ is much stronger than $\|T\|_\gamma = \|T^*\|_\gamma$.

Theorem 2.2.2 : If T is a Normal operator on H_Γ, then $\|T^n\|_\gamma = \|T\|_\gamma{}^n$ for all n $\in N$ *and* $\gamma \in \Gamma$.

Proof: From previous discussion we have- $\|T^n\|_\gamma \leq \|T\|_\gamma{}^n$ for any bounded operator T.

To show that $\|T^n\|_\gamma \geq \|T\|_\gamma{}^n$ we fix x such that $\|x\|_\gamma$=1 and use induction to show that

$$\|T^n x\|_\gamma \geq \|Tx\|_\gamma{}^n \text{ ....(i)} \quad \text{for all } n \in N .$$

Clearly (i) holds for n=1. If $Tx = 0$,then the inequality is trivially satisfied for all $n \in N$.

Assuming that $Tx \neq 0$ and that holds for n=1,2…,m. First we see that-

$$\|T^2x\|_\gamma = \|T^*Tx\|_\gamma$$
$$\geq \langle T^*Tx, \gamma, x\rangle$$
$$= \|Tx\|_\gamma{}^2 \text{ [by theorem 2.1.3 and theorem 2.2.1]}$$
$$\|T^2x\|_\gamma \geq \|Tx\|_\gamma{}^2 \text{.............(ii)}$$

Now from (ii) and the inductive assumption , we have-

$$\|T^{m+1}x\|_\gamma = \|Tx\|_\gamma \left\|T^m \frac{Tx}{\|Tx\|_\gamma}\right\|_\gamma \geq \|Tx\|_\gamma \left\|T \frac{Tx}{\|Tx\|_\gamma}\right\|_\gamma^m$$
$$= \|Tx\|_\gamma{}^{1-m} \|T^2x\|_\gamma{}^m$$
$$\geq \|Tx\|_\gamma{}^{1-m} \|Tx\|_\gamma{}^{2m}$$
$$= \|Tx\|_\gamma{}^{m+1}$$

$$\text{So,} \|T^{m+1}x\|_\gamma \geq \|Tx\|_\gamma{}^{m+1}$$

This concludes the theorem .

Theorem 2.2.3: Let H_Γ be a Γ-Hilbert space and A $\in BL(H_\Gamma)$ where A be a bounded linear operator on H_Γ .Then
A is unitary if and only if $\|A(x)\|_\gamma = \|Ax\|_\gamma$ for all $x \in H_\Gamma$, $\gamma \in \Gamma$ and A is Surjective. In that Case, $\|A^{-1}(x)\|_\gamma = \|x\|_\gamma$ for all $x \in H_\Gamma, \gamma \in \Gamma$ and also $\|A\|_\gamma = 1 = \|A^{-1}\|_\gamma$.

2.3. <u>Positive operators:</u> This is an important sub-class of self-adjoint operators to which we now turn.

Definition 2.3.1 : A self-adjoint operator A on a Γ-Hilbert space H_Γ over K is said to be positive if $\langle A(x), \gamma, x\rangle \geq 0 \; \forall \; x \in H_\Gamma$ and $\gamma \in \Gamma$.

Then we write $A \geq 0$. If A and B are self-adjoint operators and $A - B \geq 0$, then we write $A \geq B$ or $B \leq A$. The relation $\geq$ on the set of all self-adjoint operators on H_Γ is a partial order.

Example 2.3.2: Let K be a positive continuous function defined on $[a,b] \times \Gamma \times [a,b]$.The integral operator T of H_Γ on $L^2([a,b])$ defined by $(Tx)(s) = \int_a^b K(s,t)x(t)dt$ is positive.
Indeed we have, $\langle Tx, \gamma, x\rangle = \int_a^b \int_a^b K(x,t)\, x(t)\, \gamma \; \overline{x(t)} \;\; dt\, ds$

$$= \int_a^b \int_a^b K(x,t)\, |x(t)|^2\, \gamma\, dt\, ds$$

Hence $\langle Tx, \gamma, x\rangle \geq 0$ for all $x \in L^2([a,b])$ and $\gamma \in \Gamma$.

Properties 2.3.3: Let A and B be two operators on H_Γ . Then-

(i) $A + B$ is a positive operator on H_Γ .

(ii) The composition operator AB may not be a positive operator.

We will prove property (ii) by an example
Example 2.3.4: Let $H_\Gamma = K^2$ where K^2 is scalar field of real number or complex number of two Dimension and

$$A\big(x(1), \gamma, x(2)\big) = (x(1) + x(2), \gamma, x(1) + 2x(2)),$$
$$B\big(x(1), \gamma, x(2)\big) = (x(1) + x(2), \gamma, x(1) + x(2))$$

Where $\gamma \in \Gamma$.

Then
$AB\big(x(1), \gamma, x(2)\big) = (2x(1) + 2x(2), \gamma, 3x(1) + 3x(2)$ for all $(x(1), \gamma, x(2)) \in K^2$
Here note that A and B are positive operators. But AB is not a positive operator since it is not
Self-adjoint operator if $x = (-4,3)$ then $\langle AB(x), \gamma, x\rangle = -1$. So we can conclude that is A and B are Positive operators and AB=BA then AB is a Positive Operator.

(iii) Each orthogonal Projection is a positive operator.
Proof: Let Y be a closed subspace of H_Γ and let P denote the orthogonal projection onto Y . For i=1,2, consider $x_i \in H_\Gamma$, $x_i = y_i + z_i$ with $y_i \in Y$ and $z_i \in Y^\perp$, so that P$(x_i) = y_i$. Then

$$\langle P(x_1), \gamma, x_2\rangle = \langle y_1, \gamma, y_2 + z_2\rangle \quad \text{Where } \gamma \in \Gamma .$$

$= \langle y_1, \gamma, y_2 \rangle$
$= \langle y_1 + z_1, \gamma, y_2 \rangle$
$= \langle x_1, \gamma, P(x_2) \rangle$, *So that* P is self-adjoint.

Since $\langle P(x_1), \gamma, x_1 \rangle = \langle y_1, \gamma, y_1 \rangle \geq 0$ for all $x_1 \in H_\Gamma$ and $\gamma \in \Gamma$. Clearly
P is a positive operator.

2.4. 2 -Self adjoint operator on Γ-Hilbert space

Definition 2.4.1:Let $T_\gamma \in BL(H_\Gamma)$. We say that T_γ is a 2-self adjoint operator defined on H_Γ if and Only if $T_\gamma{}^2 = T_\gamma{}^{*2}$. The class of a 2-self adjoint operator defined on H_Γ is denoted by 2-$Se(H_\Gamma)$.

Example2.4.2: Let $T_\gamma: H_\Gamma \rightarrow H_\Gamma$ and H_Γ is any complex Γ-Hilbert space, which is defined as follows
$T_\gamma x = 5ix$ for all $x \in H_\Gamma$. Then $T_\gamma \in 2 - Se(H_\Gamma)$.

It is clear that if T_γ is self adjoint operator then $T_\gamma \in 2 - Se(H_\Gamma)$. However T_γ in this example is not Self-adjoint operator.

Note: From definition we have $T_\gamma \in 2 - Se(H)$ if and only if $T_\gamma{}^* \in 2 - Se(H)$.

Proposition2.4.3: Let $T_\gamma, S_\gamma \in$ BL(H_Γ), if $T_\gamma, S_\gamma \in$ 2-$Se(H_\Gamma)$ then the following statements are true:

(i) If $T_\gamma S_\gamma = S_\gamma T_\gamma$ *then* $T_\gamma S_\gamma$ as well as $S_\gamma T_\gamma \in$ 2-$Se(H_\Gamma)$.

(ii) If $(T_\gamma + S_\gamma) \in$ 2-$Se(H_\Gamma)$ if and only if $Im(S_\gamma T_\gamma) = -Im(T_\gamma S_\gamma)$

Proof: (i) We have
$$\begin{aligned}(T_\gamma S_\gamma)^2 &= T_\gamma{}^2 S_\gamma{}^2 \\ &= T_\gamma{}^{*2} S_\gamma{}^{*2} \\ &= T_\gamma{}^{2*} S_\gamma{}^{2*} \\ &= (S_\gamma{}^2 T_\gamma{}^2)^* \\ &= (S_\gamma T_\gamma)^{2*} \\ &= (T_\gamma S_\gamma)^{*2}\end{aligned}$$

Which implies that $T_\gamma S_\gamma$ and $S_\gamma T_\gamma$ are in 2-$Se(H_\Gamma)$.

(ii) Suppose that $T_\gamma + S_\gamma \in$ 2-Se(H_Γ) then

$(T_\gamma + S_\gamma)^2 = (T_\gamma{}^* + S_\gamma{}^*)^2$ and $(T_\gamma + S_\gamma)^2 = T_\gamma{}^2 + T_\gamma S_\gamma + S_\gamma T_\gamma + S_\gamma{}^2$

Also, $(T_\gamma{}^* + S_\gamma{}^*)^2 = T_\gamma{}^{*2} + T_\gamma{}^* S_\gamma{}^* + S_\gamma{}^* T_\gamma{}^* + S_\gamma{}^{*2}$
$= T_\gamma{}^{*2} + (T_\gamma S_\gamma)^* + (S_\gamma T_\gamma)^* + S_\gamma{}^{*2}$

Which implies that $T_\gamma S_\gamma + S_\gamma T_\gamma = (S_\gamma T_\gamma)^* + (T_\gamma S_\gamma)^*$.
Hence $Im(S_\gamma T_\gamma) = -Im(T_\gamma S_\gamma)$.
Now if $Im(S_\gamma T_\gamma) = -Im(T_\gamma S_\gamma)$ then $(S_\gamma T_\gamma) - (S_\gamma T_\gamma)^* = -(T_\gamma S_\gamma) + (T_\gamma S_\gamma)^*$.

So,
$$\begin{aligned}(T_\gamma + S_\gamma)^2 &= T_\gamma{}^2 + T_\gamma S_\gamma + S_\gamma T_\gamma + S_\gamma{}^2 \\ &= T_\gamma{}^{*2} + (S_\gamma T_\gamma)^* + (T_\gamma S_\gamma)^* + S_\gamma{}^{*2} \\ &= (T_\gamma + S_\gamma)^{*2}\end{aligned}$$

And $\quad T_\gamma + S_\gamma \in 2\text{-}Se(H_\Gamma)$

Corollary2.4.4: Let$T_\gamma \in BL(H_\Gamma)$ be a self-adjoint operator on H_Γ , if λ is real or pure imaginary number then $\lambda T_\gamma \in 2\text{-}Se(H_\Gamma)$.

2.5. Spectrum of 2-self adjoint operator:

In this section, we study the spectrum of 2-self-adjoint operator defined on Γ-Hilbert space.

We denote the spectrum of 2-self adjoint operator of a Γ-Hilbert space by $\sigma(T_\gamma)$ which is a subset of ℝ.

Theorem 2.5.1: Let $T_\gamma \in 2\text{-}Se(H_\Gamma)$ then $\sigma(T_\gamma) \subseteq \mathbb{R}$ or $\sigma(T_\gamma) \subseteq i\,\mathbb{R}$,where $i\,\mathbb{R} = \{ix : x \in \mathbb{R}\}$.

Proof: Suppose $\lambda \in \sigma(T_\gamma)$ and $\lambda = a + ib$ where a and b are real numbers.

Then by Spectrul mapping theorem we have-

$$\lambda^2 \in \sigma({T_\gamma}^2).$$

Therefore $\lambda^2 = a^2 + 2iab - b^2$ is real number which implies that

$$2iab = 0$$

$$\text{So, } ab = 0$$

Hence $\lambda \in \mathbb{R}$ or $\lambda \in i\mathbb{R}$

Which leads $\sigma(T_\gamma) \subseteq \mathbb{R}$ or $\sigma(T_\gamma) \subseteq i\,\mathbb{R}$.

Proposition 2.5.2: Let $T_\gamma \in 2\text{-}Se(H_\Gamma)$. If $\lambda \in \sigma({T_\gamma}^2)$ then λ is a real number.

Proof: Let $\lambda \in \sigma({T_\gamma}^2)$ then there exist $x(\neq 0) \in H_\Gamma$ Such that $T^2x = \lambda x$, therefore

$$\begin{aligned} \langle \lambda x, \gamma, x\rangle &= \langle T^2x, \gamma, x\rangle \\ &= \langle x, \gamma, T^{*2}x\rangle \\ &= \langle x, \gamma, T^2x\rangle \\ &= \langle x, \gamma, \lambda x\rangle \\ &= \bar{\lambda}\langle x, \gamma, x\rangle \end{aligned}$$

Which implies

$$(\lambda - \bar{\lambda})\langle x, \gamma, x\rangle = 0 \text{ and } \lambda = \bar{\lambda}.$$

Theorem 2.5.3: Let$T_\gamma \in 2\text{-}Se(H_\Gamma)$, if T_γ is invertable operator then ${T_\gamma}^{-1} \in 2\text{-}Se(H_\Gamma)$.

Proof:
$$\begin{aligned} ({T_\gamma}^{-1})^{\mathbf{2}} &= ({T_\gamma}^2)^{-1} \\ &= ({T_\gamma}^{*2})^{-1} \\ &= (({T_\gamma}^*)^{-1})^2 \\ &= (({T_\gamma}^{-1})^*)^2 \end{aligned}$$

Then ${T_\gamma}^{-1} \in 2\text{-}Se(H_\Gamma)$.

Corollary 2.5.4: If $T_\gamma - \lambda \in 2\text{-}Se(H_\Gamma)$ for all $\lambda \neq \sigma(T_\gamma)$ *and* $\gamma \in \Gamma$, then $(T_\gamma - \lambda)^{-1} \in 2\text{-}Se(H_\Gamma)$.

Proposition 2.5.5: If$T_\gamma \in 2\text{-}Se(H_\Gamma), \gamma \in \Gamma$ and ${T_\gamma}^2$ *or* ${T_\gamma}^{*2}$ is onto then

$$(i) Range(T_\gamma) = Range\,({T_\gamma}^*)$$

(ii) T_γ and $T_\gamma{}^*$ are invertible operators.

3. Conclusion

Here we work with two linear spaces. As a result of this study any one can introduce a new linear finite dimensional operator and their characterizations in Γ-Hilbert space .Further we will experiment on more new operators and inequalities of Γ-Hilbert space and extend our work on this topic.

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Conflicts of interest

The authors state that there is no financial interests or non financial interests in the subject matter or materials discussed in the manuscript.

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