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Remarks on the group of unıts of a corner ring

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Abstract

The aim of this study is to characterize rings having the following properties for a non-trivial idempotent element e of R, $U(eRe) = e + eJ(R)e = e + J(eRe)$ (and $U(eRe) = e + N(eRe)$), where U (-), N (-) and J (-) denote the group of units, the set of all nilpotent elements and the Jacobson radical of R, respectively. In the present paper, some characterizations are also obtained in terms of every element is of the form $e + u$, where $e^2 = e \in R$ and $u \in U(eRe)$.

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1. Introduction

Throughout the paper all rings considered are associative and unital. For a ring R, the Jacobson radical, the group of units and the set of all nilpotent elements are denoted by $J(R)$, $U(R)$ and $N(R)$, respectively.

One always has $1 + J(R) \subseteq U(R)$. In [1], authors defined a ring to be UJ if it satisfies the above property as two sided, that is a ring R is a UJ-ring if $1 + J(R) = U(R)$ and also they showed that the problem of lifting the UJ property from a ring R to the polynomial ring $R[x]$ is equivalent to the Köthe's problem for $\mathbb{F}_2$-algebras. They also proved that if e is an idempotent element and R is UJ-ring then the corner rings eRe and $(1 - e)R(1 - e)$ are also UJ. But the converse is true with an additional property that $eR(1-e), (1-e)Re \subseteq J(R)$.

One can see easily that $e(1 + J(R))e \subseteq eU(R)e$ for $e^2 = e \in R$ since $1 + J(R)$ is always contained in $U(R)$. So it makes sense to think about the equality as following:

$$U(eRe) = e + eJ(R)e = e + J(eRe). \quad (1)$$

Every UJ-ring satisfies this property. Also, we give examples and some characterizations and basic properties of rings having this property. For example, a ring R satisfies this property iff $U(eRe / J(eRe)) = \{e\}$, and the ring $\prod_{i \in I} R_i$ satisfies this property if and only if each ring R_i satisfies this property, for all $i \in I$. Recall that a ring is called semilocal if $R / J(R)$ is a semisimple ring. It is also shown that a semilocal ring R satisfies the (1)-property if and only if $eRe / J(eRe) \cong \mathbb{F}_2 \times \ldots \times \mathbb{F}_2$.

The behavior of this property under some classical ring constructions is studied. In particular, it is proved that if the polynomial ring $R[x]$ satisfies this property, then R satisfies this property and $J(eRe)$ is a nil ideal of eRe. It is also shown that Morita context satisfies this property.

An element is called clean if it can be written as a sum of an idempotent and a unit. A ring is called clean if each of its element is clean. Clean rings were firstly introduced by Nicholson [2]. Several people work on this subject and investigate properties of clean rings, for example see [3]. Rings for which every element is a sum of an element from the Jacobson radical and an idempotent is called J-clean. We obtain that: (1) R satisfies the (1)-property iff all clean elements of eRe are J-clean, (2) eRe is a clean ring and R satisfies the (1)-property if and only if $eRe / J(eRe)$ is a Boolean ring and idempotents lift modulo $J(eRe)$ iff eRe is a J-clean ring.

As a clean element representation of a ring, we can consider the following: Let R be a ring and a be a non-unit element of R. We say that a is of the form (*) if $a = e + u$, where $e^2 = e \in R$ and $u \in U(eRe)$. It is easy to see that an element a of a ring R is of the form (*) if

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and only if it is quasi-regular, and an element of a ring is of the form (*) is clean. Hence we also obtain that every element ($\neq 1$) of a ring R has the form $e + u$, where $e^2 = e \in R$ and $u \in U(eRe)$ if and only if R is a division ring.

2. The Results

We begin with the following well known facts / notions will be referred to several times.

Remark 1.1. For any idempotent element e of a ring R,

1. $J(eRe) = \{exe \in eRe: e - exeye \in U(eRe), \forall eye \in eRe\}$.
2. $N(eRe) = \{exe \in eRe : (exe)^n = 0, \text{ for some } n \in \mathbb{Z}^+\}$.
3. $U(eRe) \subseteq eU(R)e$ for $e^2 = e \in R$.
4. $U(\prod_{i\in I} eR_ie) = \prod_{i\in I} U(eR_ie)$, where

   $U(\prod_{i\in I} eR_ie) = \{\prod_{i\in I} ex_ie$ is unit where $ex_ie \in eR_ie\}$, and

   $\prod_{i\in I}(U(eR_ie)) = \{\prod_{i\in I} ex_ie : ex_ie \in eR_ie$ is unit for every $i \in I\}$

5. $J(\prod_{i\in I} eR_ie) = \prod_{i\in I} J(eR_ie)$.

We consider the following property (1):

A ring R satisfies (1) if for any idempotent element e of R, $U(eRe) = e + eJ(R)e = e + J(eRe)$.

Example 1.2. Every UJ-ring satisfies the (1)-property by [1, Proposition 2.7]. Furthermore, if R is a UJ-ring, then $eU(R)e = U(eRe)$.

Given a ring R, we define an operation $\circ$ on R, called quasi-multiplication, by $a \circ b = a + b - ab$. It is easy to see that $(R, \circ)$ is a monoid with identity element 0. An element $a \in R$ is called quasi-regular if it is invertible in $(R, \circ)$, i.e., if there exists $a' \in R$ such that $a \circ a' = 0 = a' \circ a$. In this case we say that a' is the quasi-inverse of a. If R is unital then this is equivalent to $1 - a \in U(R)$. The set of all quasi-regular elements of R will be denoted by $Q(R)$. Clearly, $(Q(R), \circ)$ is a group since this is just the group of invertible elements of the monoid $(R, \circ)$.

Theorem 1.3. Every element ($\neq 1$) of a ring R is quasi-regular if and only if R is a division ring.

Proof: Suppose that $0, 1 \neq a$ is an arbitrary element of R. Since $(1 - a)$ is a unit then there exists $1 \neq u \in U(R)$ such that $(1 - a)u = 1$. Therefore, we have $(-au) = (1 - u)$. By assumption, $1-u$ is a unit element and hence $a = -(1 - u)(1 - a)$ is a unit element. Hence, R is a division ring. The converse is clear.

Remark 1.4: If ere is a quasi-regular element of eRe then r is quasi-regular element of R by Remark 1.1.

Proposition 1.5. For a ring R and any non-trivial idempotent $e \in R$, the following conditions are equivalent:

1. $U(eRe) = e + J(eRe)$, i.e., R satisfies the (1)-property.
2. $U(eRe / J(eRe)) = \{e\}$.
3. $Q(eRe)$ is an ideal of eRe (then $Q(eRe) = J(eRe)$).
4. $erebe - ecere \in J(eRe)$ for any $ere \in eRe$ and $ebe, ece \in Q(eRe)$.
5. $ereue - evere \in J(eRe)$ for any $eue, eve \in U(eRe)$ and $ere \in eRe$.
6. $U(eRe) + U(eRe) \subseteq J(eRe)$ (then $U(eRe) + U(eRe) = J(eRe)$).

Proof: (1)⇒(2) If we take $eRe / J(eRe)$ instead of eRe in (1), then we get $U(eRe/J(eRe)) = e + J(eRe/J(eRe)) = e$, as desired.

(1)⇒(3) Let $exe \in Q(eRe)$. Then $e - exe \in U(eRe)$, and so there exists an element $eue \in U(eRe)$ such that $e - exe = eue$ which gives $exe = e - eue$, where $eue \in U(eRe) = (e + J(eRe))$. Hence there exists $eje \in J(eRe)$ such that $eue = e + eje$. Since $exe = e - eue = e - (e + eje) = eje \in J(eRe)$, we get $Q(eRe) \subseteq J(eRe)$. But by the definition we have $J(eRe) \subseteq Q(eRe)$ so we are done.

(2)⇒(1) Clearly $e + J(eRe) \subseteq U(eRe)$. For the converse, first we prove the following claim:

Claim: $[U(eRe) + J(eRe)] / J(eRe) = U(eRe / J(eRe))$: Let $exe+J(eRe) \in U(eRe / J(eRe))$. By the hypothesis, we have $U(eRe / J(eRe)) = \{e\}$ and so $exe + J(eRe) = e$ which gives $e - exe \in J(eRe)$. By Remark 1.1, one obtains $exe \in U(eRe)$ and so $exe + J(eRe) \in [U(eRe)+J(eRe)] / J(eRe)$. For the converse, let $exe+J(eRe) \in [U(eRe)+J(eRe)] / J(eRe)$.

Since exe is invertible, there exists $eye \in eRe$ such that $exeye = eyexe = e$. The equation $(exe + J(eRe))(eye + J(eRe)) = exeye + J(eRe) = e + J(eRe)$ implies $exe + J(eRe) \in U(eRe / J(eRe))$. Now for the converse, let $exe \in U(eRe)$. By the Claim, $exe+J(eRe) \in U(eRe)/J(eRe) =\{e\}$. Therefore $exe + eje = e$ for all $eje \in J(eRe)$ which implies $exe = e - eje \in e + J(eRe)$.

(3)⇒(4) Since $Q(eRe)$ is an ideal of eRe, we get $erebe - ece \in Q(eRe)$ for $ebe, ece \in Q(eRe)$ and $exe \in eRe$. Hence $Q(eRe) = J(eRe)$ implies $erebe\text{-}ece \in J(eRe)$.

(4)⇒(5) Setting $ece = e - eue$ and $ebe = e - eve$ for $eue, eve \in U(eRe)$, we get $ebe, ece \in Q(eRe)$. The rest follows from (4).

(5)⇒(6) If we take $ere = e$ in (5), then $eue - eve \in J(eRe)$, for any $eue, eve \in U(eRe)$ which gives $U(eRe) + U(eRe) \subseteq J(eRe)$. Hence, every $ere \in J(eRe)$ can be written as a sum of two invertible element as $ere = e + (ere - e) \in U(eRe) + U(eRe)$.

(6)⇒(1) Clearly $e + J(eRe) \subseteq U(eRe)$. Using (6), we get $U(eRe) - e \subseteq J(eRe)$, i.e. $U(eRe) \subseteq e + J(eRe)$ which completes the proof.

Remark 1.6. For $e = 1$, one has [1, Lemma 1.1].

In the following observation, we collect some basic properties of rings having the (1)- property.

Proposition 1.7. Assume that a ring R satisfies the (1)-property for any non-trivial idempotent element e of R. Then:

1. $2e \in J(eRe)$;
2. If eRe is a division ring, then $eRe = K_2 \cong F_2$ where $K_2 = \{0, e\}$;
3. $eRe / J(eRe)$ is reduced (i.e., it has no nonzero nilpotent elements) and hence abelian (i.e., every idempotent is central);
4. If $exe, eye \in eRe$ are such that $exeye \in J(eRe)$, then $eyexe \in J(eRe)$ and $exeReye, eyeRexe \in J(eRe)$;
5. Let $I \subseteq J(eRe)$ be an ideal of eRe. Then R satisfies (1)-property if and only if R / I satisfies (1)-property;
6. eRe is Dedekind finite;
7. The ring $\prod_{i \in I} R_i$ satisfies the (1)-property if and only rings R_i satisfy the (1)- property for all $i \in I$.

Proof: (1) By Proposition 1.5 (6), we have $U(eRe) + U(eRe) = J(eRe)$. So $e + e = 2e \in J(eRe)$.

(2) If eRe is a division ring, then every nonzero element of eRe has an inverse and also $U(eRe) = e + J(eRe)$, by Proposition 1.5(1). Therefore $e + J(eRe) \in eRe / J(eRe)$ is an only element which has an inverse. By Proposition 1.5(2), $U(eRe / J(eRe)) = \{e\}$.

(3) For a nilpotent element $ere + J(eRe)$ in $eRe / J(eRe)$, we shall show that $ere \in J(eRe)$. There exits $n \in N$ such that $(ere)^n + J(eRe) = J(eRe)$. Then

$$\begin{aligned} e + J(eRe) &= [((ere)^n) + J(eRe)] + (e + J(eRe)) \\ &= ((ere)^n + e) + J(eRe) \\ &= ((ere) + e)((-1)^{n-1}(ere)^{n-1} + \ldots + (-1)^2(ere)^2 - ere + 1) + J(eRe) \\ &= [((ere) + e) + J(eRe)][((1)^{n-1}(ere)^{n-1} + \ldots + (-1)^2(ere)^2 - ere + 1) + J(eRe)]. \end{aligned}$$

By Proposition 1.5(2), $(ere + e) + J(eRe) \in U(eRe / J(eRe))$. So there exists $eje \in J(eRe)$ such that $(ere + e) + eje = e$, that is $ere = -eje \in J(eRe)$. Hence it has no nonzero nilpotent elements.

(4) Let $exeye \in J(eRe)$. Then $exeye + J(eRe) = J(eRe)$. After multiplying the equation by $eye + J(eRe)$ (on the left and by $exe + J(eRe)$ on the right, we get $eyexeyexe + J(eRe) = (eyexe)^2 + J(eRe) = J(eRe)$. By (3), $eRe / J(eRe)$ is reduced and so $eyexe + J(eRe) = J(eRe)$. Hence $eyexe \in J(eRe)$. The rest follows from (3).

(5) Let $I \subseteq J(eRe)$. Then $J(eRe) / I = J(eRe / I)$. Indeed, clearly, $J(eRe) / I \subseteq J(eRe / I)$. For the converse, let $exe + I \in J(eRe / I)$. Then $(e - exeye) + I$ is a unit of eRe / I for every $eye \in eRe$. Hence $e - exeye$ is a unit for every $eye \in eRe$ which implies that $exe \in J(eRe)$. Now $exe + I \in J(eRe) + I$ and $\frac{eRe/I}{J(eRe/I)} = \frac{eRe/I}{J(eRe)/I} = eRe / J(eRe)$. The rest follows from Proposition 1.5(2).

(6) $eRe / J(eRe)$ is Dedekind finite since it is reduced. Let $exeye = e$ for $exe, eye \in eRe$. Then $exeye + J(eRe) = e + J(eRe)$ but $eRe / J(eRe)$ is Dedekind finite so $eyexe + J(eRe) = e + J(eRe)$ that is $eyexe$ is invertible. Clearly $eyexe$ is an idempotent so $eyexe = e$.

(7) This follows from Remark 1.1.

Remark 1.8. For $e = 1$, one has [1, Proposition 1.3]

Proposition 1.9. A semilocal ring R satisfies the (1)-property if and only if $eRe / J(eRe) \cong F_2 \times \ldots \times F_2$.

Proof. Since $eRe / J(eRe)$ is semisimple by the definition and reduced by Proposition 1.7(3), so $eRe / J(eRe)$ is a finite direct product of division ring. But it is isomorphic to F_2 by Proposition 1.7(2).

We focus on some algebraic constructions of rings having the (1)-property.

Proposition 1.10. If a ring R satisfies the (1)-property and S is a subring of R such that $U(eSe) = U(eRe) \cap eSe$, then S also satisfies the (1)-property.

Proof. Since $U(eSe) = U(eRe) \cap eSe$, we also have $J(eRe) \cap eSe \subseteq J(eSe)$. Thus, using $U(eRe) = e + J(eRe)$ we get

$e + J(eSe) \subseteq U(eSe) = U(eRe) \cap eSe = (e + J(eRe)) \cap eSe = e + (J(eRe) \cap eSe) \subseteq e + J(eSe)$.

Therefore $U(eSe) = e + J(eSe)$. We consider the following property (*2*):

A ring R satisfies (*2*) if for any non-trivial idempotent element e of R, $U(eRe) = e + N(eRe)$.

Lemma 1.11. Let e be a non-trivial idempotent and eRe be a ring such that its unit element is only e. Then $U(eRe[X]) = \{e\}$, where $eRe[X]$ denotes the polynomial ring in the set X of commuting indeterminates.

Proof. Since being a unit in $eRe[X]$ is a local property, i.e., depends only on finitely many indeterminates, we may assume that X is a finite set. By assumption $U(eRe) = \{e\}$, so eRe does not contain non-trivial nilpotent elements, i.e., it is a reduced ring. [1, Corollary 1.7] characterizes reduced rings as rings such that $U(eRe[x]) = U(eRe)$ and the thesis follows easily.

Let us recall that a ring R is 2-primal if its prime radical $B(R)$ coincides with the set of all its nilpotent elements.

Proposition 1.12. Let R be a 2-primal having the (*2*)-property. Then, for any set X of commuting indeterminates, the polynomial ring $R[X]$ satisfies the (1)-property.

Proof. It is known that $B(eRe[X]) = B(eRe)[X]$ (cf.[4, Theorem 10.19]). Thus the assumptions imposed on R and Lemma 1.11 imply that the ring $eRe[X] / B(eRe[X]) \cong (eRe / B(eRe))[X]$ has trivial units. Now, by Proposition 1.7(5), $R[X]$ satisfies the (1)-property.

Proposition 1.13. If the polynomial ring $R[x]$ satisfies the (1)-property, then R satisfies the (1)-property and $J(eRe)$ is a nil ideal of eRe.

Proof. It is known that $J\ eRe[x]) = I[x]$ for some nil ideal I of eRe. Thus, as $R[x]$ satisfies the (1)-property, we have $e + J(eRe) \subseteq U(eRe[x]) = e + J(eRe[x]) = e + I[x]$. This implies that $J(eRe) = I$ is nil. As $R[x]$ satisfies the (1)-property, then $\{e\} = U(eRe[x] / J(eRe[x])) = U((eRe / J(eRe)[X])$. Hence also $U(eRe / J(eRe)) = \{e\}$, i.e. R satisfies the (1)-property.

A Morita context is a 4-tuple $\begin{pmatrix} R & M \\ N & S \end{pmatrix}$, where R, S are rings, ${}_RM_S$ and ${}_SN_R$ are bimodules, and there exist context products $M \times N \to R$ and $N \times M \to S$ written multiplicatively as $(x, y) \to xy$ and $(y, x) \to yx$, such that $\begin{pmatrix} R & M \\ N & S \end{pmatrix}$ is an associative ring with the obvious matrix operations. A Morita context $\begin{pmatrix} R & M \\ N & S \end{pmatrix}$ is called trivial if the context products are trivial, i.e., $MN = 0$ and $NM = 0$. A trivial Morita context is also called the ring of a Morita context with zero pairings in [5]. A trivial Morita context $\begin{pmatrix} R & M \\ N & S \end{pmatrix}$ with $N = 0$ is commonly called a formal triangular matrix ring. Given a ring R and a bimodule V over R, we can easily see that $\{ \begin{pmatrix} a & v \\ 0 & a \end{pmatrix} : a \in R, v \in V \}$ is a subring of the formal triangular matrix ring $\begin{pmatrix} R & V \\ 0 & R \end{pmatrix}$, and this subring is the just trivial extension of R by V. By [6, Lemma 2], trivial Morita contexts (in particular, formal triangular matrix rings) are special cases of trivial extensions.

Theorem 1.14. Let (eRe, eVf, fWe, fSf) be a Morita context and $T := \begin{pmatrix} eRe & eVf \\ fWe & fSf \end{pmatrix}$ where e and f are non-trivial idempotents. The following conditions are equivalent:

1. T satisfies the (1)- property for an idempotent $E = \begin{pmatrix} e_R & 0 \\ 0 & f_S \end{pmatrix}$, where e and f are non-trivial idempotents.
2. R satisfies the (1)- property for an idempotent e, S satisfies the (1)- property for an idempotent f and $eVfWe \subseteq J(eRe)$, $WV \subseteq J(fSf)$.
3. R satisfies the (1)- property for an idempotent e, S satisfies the (1)- property for an idempotent f and $T / J(ETE) \cong eRe / J(eRe) \oplus fSf / J(fSf)$.

Proof: (1) ⇒ (2) Suppose T satisfies the (1)- property for an idempotent $E=\begin{pmatrix} e_R & 0 \\ 0 & f_S \end{pmatrix}$ where e and f are non-trivial idempotents. Hence $ETE / J(ETE)$ has no nonzero nilpotent elements. Let $\begin{pmatrix} 0 & eVf \\ 0 & 0 \end{pmatrix}$,

$\begin{pmatrix} 0 & 0 \\ fWe & 0 \end{pmatrix} \subseteq J(ETE) = \begin{pmatrix} J(eRe) & B \\ C & J(fSf) \end{pmatrix}$, where $B = \{evf : fWevf \subseteq J(fSf)\} = \{evf : evfWe \subseteq J(eRe)\}$, and $C = \{fwe : fweVf \subseteq J(fSf)\} = \{fwe : eVfwe \subseteq J(eRe)$.

Clearly, $B = eVf$ and $C = fWe$. We have also obtain that $B = eVf$, $C = fWe$, $eVfWe \subseteq J(eRe)$, $fWeVf \subseteq J(fSf)$ and $ETE / J(ETE) \cong eRe / J(eRe) \oplus fSf / J(fSf)$. Since T satisfies the (1)-property for an idempotent E, we get $U(ETE / J(ETE)) = \{E\}$.

We also know that $U(ETE / J(ETE)) \cong U(eRe / J(eRe)) \oplus U(fSf / J(fSf))$ which implies R satisfies the (1)-property for an idempotent e, and S satisfies the (1)-property for an idempotent f.

(2)⇒(3) It is clear by the fact that $B = eVf$ and $C = fWe$.

(3)⇒(1) It is a consequence of Proposition 1.5.

Recall that an element $r \in R$ is clean (J-clean) provided there exist an idempotent $e \in R$ and an element $t \in U(R)$ ($t \in J(R)$) such that $r = e + t$. A ring R is clean (J-clean) if every element of R has such clean (J-clean) decomposition. It is known that every J-clean ring is clean (in fact if $-r = e + j$ is a J-clean decomposition of $-r \in R$, then $r = (1 - e) + (-1 - j)$ is a clean decomposition of r.

Proposition 1.15. For a ring R, the following conditions are equivalent:
1. R satisfies the (1)-property.
2. All clean elements of eRe are J-clean.

Proof: (1)⇒(2) Assume that $r \in R$ is a clean element of eRe and $r = f + u$ is its clean *decomposition for* $f \in Id(eRe)$ *and* $u \in U(eRe)$. *Then* $r = f + u + e - e + f - f = (2f + u - e) + (e - f)$.

Claim 1. *$(e - f)$ is an idempotent element of eRe:* Indeed, $(e - f)^2 = e - f$ since $ef = f$ and $fe = f$.

Claim 2. $2f \in J(eRe)$: By Proposition 1.7, $2f \in J(fRf) = fJ(R)f = exeJ(R)exe \subseteq eJ(R)e = J(eRe)$. Also by Proposition 1.5, $u - e \in J(eRe)$ and so one obtains $2f + u - e \in J(eRe)$.

(2)⇒(1) Let $e^2 = e \in R$ and $u \in U(eRe)$. Then u is a clean element of eRe and, by the hypothesis, u is J-clean. Let $u = f + j$ be a J-clean decomposition of u. Since $e = fu^{-1} + ju^{-1}$, we obtain that $fu^{-1} = eju^{-1}$ is a unit of eRe. Hence, $e = f$. This means that $u = f + j = e + j$ *and* $U(eRe) = e + J(eRe)$ as desired.

Theorem 1.16. For a ring R and for any idempotent $e \in R$, the following conditions are equivalent:
1. eRe is a clean ring and R satisfies the (1)-property.
2. $eRe / J(eRe)$ is a Boolean ring and idempotents lift modulo $J(eRe)$.
3. eRe is a J-clean ring and R satisfies the (1)-property.
4. eRe is a J-clean ring.

Proof: (1)⇒(2) By the assumptions, $eRe / J(eRe)$ is a clean ring such that $U(eRe) = \{e\}$. In particular, $2e = 0$ in $(eRe / J(eRe))$ and every element $r \in eRe / J(eRe)$ is of the form $r = f + e$, for a suitable idempotent f. Hence $r^2 = r$, i.e., $eRe / J(eRe)$ is Boolean. By (cf.[7, Lemma 17]), idempotents lift modulo every ideal I of a clean ring R, which gives (2).

(2)⇒(3) Suppose (2) holds and let $a \in eRe$. Then $a + J(eRe) \in eRe / J(eRe)$ is idempotent. Hence, there exists an idempotent $f \in eRe$ such that $a - f \in J(eRe)$, i.e. a is a J-clean element. This shows that eRe is J-clean. If $u \in U(eRe)$, then $u + J(eRe)$ is a unit in a Boolean ring $eRe / J(eRe)$. Thus $u - e \in J(eRe)$, so satisfies the (1)-property.

(3)⇒(4) Trivial.

(4)⇒(1) This follows from Proposition 1.15.

Let us continue on a clean decomposition for a non-unit element of a ring.

Lemma 1.17. For a non-unit element a of ring R, the following conditions are equivalent:
1. $a = e + u$ (and $eu = ue$), where $e^2 = e \in R$ and $u \in U(eRe)$;
2. $a \in 1 + U(R)$ (and $u \in Z(U(R))$).

Proof: Assume that a non-unit element a of ring R is of the form $e + u$, where $e^2 = e \in R$ and $u \in U(eRe)$. It is easy to check (or well known) that $U(eRe) = eRe \cap ((1 - e) + U(R))$, where $e^2 = e \in R$. Hence $u = (1 - e) + v$ for some $v \in U(R)$. Now, $a = e + (1 - e) + v = 1 + v \in 1 + U(R)$, as desired.

Now assume that a non-unit element a of ring R is of the form $e + u$ and $eu = ue$, where $e^2 = e \in R$ and $u \in U(eRe)$. Then $au = eu + u^2$ and $ua = ue + u^2$ which imply $au = ua$. Since $a \in 1 + U(R)$, we write $a = 1 +$

v for some $v \in U(R)$. Hence $au = u(1+v)$ and $ua = (1+v)u$ that implies $uv = vu$.

Corollary 1.18. An element a of ring R is of the form (*) if and only if it is quasi-regular.

Example 1.19. 1. An element a of ring R is of the form (*) is clean.

2. It is well known that the units, idempotents, and quasi-regular elements of any ring are clean, but units and idempotents are not of the form (*).

3.Any element Z_2 is of the form (*). Consider its matrix ring $M(Z_2)$. The set of units of $M(Z_2)$ is $U(M(Z_2)) = \{ \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\1 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\1 & 1\end{pmatrix}, \begin{pmatrix}1 & 1\\1 & 0\end{pmatrix}, \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix} \}$.

Consider the non-unit element $a = \begin{pmatrix}1 & 1\\0 & 0\end{pmatrix} \in M(Z_2)$. Then, there is no unit in $U(M(Z_2))$ such that a $\in 1+ U(M(Z_2))$.

Remark 1.20. We should remind the reader that $1 \in R$ never has the form $e + u$, where $e^2 = e \in R$ and $u \in U(eRe)$. To see this,

1. If $e = 1$ then we conclude that $u = 0$. By assumption, $0 = u \in U(1R1) = U(R)$ is a unit element that is impossible.

2. If $e = 0$ then we conclude that $u = 1$. By assumption, $1 - u \in U(0R0) = 0$ that is impossible.

3. If $e \neq 0, 1$ then $1- e = u$ and we conclude that $1 - e$ is a unit element of eRe while is a zero-divisor of eRe and that is a contradiction.

Corollary 1.21. Let R be a ring in which every element ($\neq 1$) of R has the form $e + u$, where $e^2 = e \in R$ and $u \in U(eRe)$. Then $Id(R) = \{0, 1\}$.

Proof: We claim that R has the only trivial idempotents. To see this, let $e \neq 0, 1$ be an idempotent of R. By Lemma 1.17, since e is a non-unit element, there exists $u \in U(R)$ in which $e = 1 + u$. Hence, we have $-u = 1 - e$. It is clear that $1 - e$ is a zero divisor thus u is a zero-divisor that is a contradiction. Therefore, R has the only trivial idempotents.

Corollary 1.22. Every element ($\neq 1$) of a ring R has the form $e + u$, where $e^2 = e \in R$ and $u \in U(eRe)$ if and only if R is a division ring.

Proof. By Corollary 1.21, R has only trivial idempotents. Hence every element of R has either the form $0 + u$ where $u \in U(0R0)$ (that is impossible) or the form $1 + u$ where $u \in U(1R1) = U(R)$. Therefore, every element ($\neq 1$) of R has the form $1 + u$ where $u \in U(R)$. Hence every element ($\neq 1$) of R is quasi-regular and we conclude that R is a division ring by Lemma 1.3.

The converse is clear.

Conflicts of interest

The authors state that did not have a conflict of interests.

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