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Finite Difference Method for Approximate Solution of a Boundary Value Problem with Interior Singular Point

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Abstract

This study is aimed at finding the approximate solutions of some boundary value problems with interior singular points. The considered problems differs from the standart boundary value problems in that they contains additional transmission conditions at the points of singularity. Naturally, such type of problems are much more complicate to solve than regular boundary value problems ones. Moreover, it is not clear how to apply the classical numerical methods to problems, involving not only boundary conditions at the end points of the considered interval, but also additional transmission conditions at some interior points. By modifying the Finite Diferrence Method we present a new numerical algoritm to find approximate solutions of the considered boundary value transmission problems. The obtained approximate solutions are compared with the exact solutions for some illustrative singular boundary value problems, involving additional transmission conditions. To justify the proposed modification some graphical illustrations of the approximate solutions are also presented.

Keywords: *Finite difference method, boundary value problems, transmission conditions,interior singular point*
2010 Mathematics Subject Classification: *34A36, 34B09, 65L10, 65L12*

1. Introduction

Many important problems of physics and engineering are modelled by initial and/or boundary value problems for linear or nonlinear differential equations. In particular, boundary value problems with interior singular point arises in classical mechanics, geophysics, electromagnetic theory, quantum physics, elasticity and other fields of natural sciences. For example, modeling heat and mass transfer problems, vibrations of loaded strings, toroidal vibrations and free vibrations of the Earth, as a rule leads to singular boundary value problems (see, for example [4], [6], [7], [9], [10], [14] and [16] references cited therein). In many cases it is impossible or very difficult to find exact analytic solutions. Even if a singular boundary value problem can be solved analytically, the exact form of the solution may even turn out to be messy to be useful. Therefore, growing interest has been given to the development of various numerical methods for solving of regular and singular problems. In the literature, there are many numerical methods (for example, the Explicit Euler Method, the Runge-Kutta Method, the Finite Difference Method, the Differential Transformation Method, the Adomian Decomposition Method, the Homotopy Perturbation Method, the Shooting Method and etc.) that are used to find the approximate solutions of different boundary value problems for ordinary and partial differential equations.[11] The Finite Difference Method (FDM) is the simple and efficient numerical method of today (see,[1], [2], [8], [15], [17]). The main advantage of this method is that it can be applied to wide class of ordinary and partial differential equations, provided that the problems under study have a complete set of boundary conditions. Moreover, unlike to many numerical methods the FDM do not require the linearization or perturbations techniques , the calculation of auxiliary parameters or auxiliary functions. This method is based on approximations that allow replacing differential equations with an algebraic system of equations, and an unknown solution is associated with grid points.
In this article, we investigate a new type boundary value problems for Sturm-Liouville equations with interior singular points and with additional transmission conditions at the points of singularity. Based on classical FDM we present a new techniques for computing the approximate solutions.Recently, some boundary value problems with additional transmission conditions were investigated in [3], [12], [13].
The organization of the rest of this paper will be as follows. In section 2 we will explain the applications of the FDM to compute the numerical solution of the boundary value problem for two order ordinary linear differential equations. In the section 3 we develop a new modification of the FDM for solving boundary value problems with interior singular point and with corresponding transmission conditions. In

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this section we also present graphical illustration of the approximate FDM-solutions and exact solutions. Concluding remarks are presented in the final section 4.

2. Outline of the Method

To ilustrate the method used in this study let us consider a boundary value problem consisting of nonhomogeneous Sturm-Liouville equation,

$$y''(x)+p(x)y'(x)+q(x)y(x)=f(x), \quad x\in[a,b] \tag{2.1}$$

and nonhomogeneous boundary conditions

$$y(a)=\alpha, \quad y(b)=\beta. \tag{2.2}$$

As a first step, the definition range [a,b] is divided into N equal ranges,

$$[x_0,x_1],[x_1,x_2],...,[x_{N-1},x_N] \quad \text{where} \quad a=x_0<x_1<...<x_N=b,$$

$$x_i=a+ih, \qquad h=\frac{b-a}{N}.$$

We seek the function values at the finite set of points $x_0,x_1,...,x_N$ that represent an numerical solution to the boundary value problems (2.1)-(2.2). The main feature of FDM is to obtain algebraic equations by replacing derivative expressions with appropriate differences. The first and second derivatives in the boundary value problem (2.1)-(2.2) are replaced by approximate value expressions as

$$y'(x)\approx\frac{y(x+h)-y(x-h)}{2h} \tag{2.3}$$

and

$$y''(x)\approx\frac{y(x+h)-2y(x)+y(x-h)}{h^2}. \tag{2.4}$$

Taking in view (2.3) and (2.4) in the FDM we replace the first and second derivatives of the solution y(x) at each grid points x_i by

$$y'(x)\approx\frac{y_{i+1}-y_{i-1}}{2h}$$

and

$$y''(x)\approx\frac{y_{i+1}-2y_i+y_{i-1}}{h^2}$$

where $\quad y_i=y(x_i)$.

Substituting (2.3) and (2.4) in the boundary value problem (2.1)-(2.2) we have the finite difference solution $y_0,y_1,...,y_N$ of the following linear system of algebraic equations

$$\left(1-\frac{1}{2}hp_1\right)y_0+\left(-2+h^2q_1\right)y_1+\left(1+\frac{1}{2}hp_1\right)y_2=h^2f(x_1)$$

$$\left(1-\frac{1}{2}hp_2\right)y_1+\left(-2+h^2q_2\right)y_2+\left(1+\frac{1}{2}hp_2\right)y_3=h^2f(x_2)$$

$$\dots$$

$$\left(1-\frac{1}{2}hp_{N-2}\right)y_{N-3}+\left(-2+h^2q_{N-2}\right)y_{N-2}+\left(1+\frac{1}{2}hp_{N-2}\right)y_{N-1}=h^2f(x_{N-2})$$

$$\left(1-\frac{1}{2}hp_{N-1}\right)y_{N-2}+\left(-2+h^2q_{N-1}\right)y_{N-1}+\left(1+\frac{1}{2}hp_{N-1}\right)y_N=h^2f(x_{N-1})$$

where

$$y_0=\alpha, \quad y_N=\beta.$$

This system can be written in the matrix form

$$My=B$$

where M is the $(N-1)\times(N-1)$ matrix, given by

$$M = \begin{pmatrix} -2+h^2q_1 & 1+\frac{1}{2}hp_1 & 0 & \cdots & 0 & 0 & 0 \\ 1-\frac{1}{2}hp_2 & -2+h^2q_2 & 1+\frac{1}{2}hp_2 & \cdots & 0 & 0 & 0 \\ 0 & 1-\frac{1}{2}hp_3 & -2+h^2q_3 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -2+h^2q_{N-3} & 1+\frac{1}{2}hp_{N-3} & 0 \\ 0 & 0 & 0 & \cdots & 1-\frac{1}{2}hp_{N-2} & -2+h^2q_{N-2} & 1+\frac{1}{2}hp_{N-2} \\ 0 & 0 & 0 & \cdots & 0 & 1-\frac{1}{2}hp_{N-1} & -2+h^2q_{N-1} \end{pmatrix}$$

and Y and B are vectors of $N-1$ components, given by

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{N-3} \\ y_{N-2} \\ y_{N-1} \end{pmatrix}, \qquad B = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_{N-3} \\ b_{N-2} \\ b_{N-1} \end{pmatrix}.$$

where

$$b_i = \begin{cases} h^2 f_1 - \left(1-\frac{1}{2}hp_1\right), & i=1 \\ h^2 f_i, & i=2,3,...,N-2 \\ h^2 f_{N-1} - \left(1+\frac{1}{2}hp_{N-1}\right)\beta, & i=N-1 \end{cases}$$

The system $My=B$ is tridiagonal and can be solved efficiently in O(CN) operations by the Crout or Cholesky algoritm.[5]

3. Numerical Applications

Example 3.1. *Let us consider the following boundary value problem on the range* $[-1,1]$ *with interior singular point* $x=0$*, given by the non homogeneous Sturm-Liouville equation*

$$y''+4y=8e^{2x}, \quad x\in[-1,0)\cup(0,1] \tag{3.1}$$

together with the non homogeneous boundary conditions

$$y(-1)=0, \qquad y(1)=1 \tag{3.2}$$

and with additional transmission conditions at the interior point of singularity $x=0$*, given by*

$$3y(-0)=y(+0), \quad y'(-0)=2y'(+0) \tag{3.3}$$

At first we shall investigate this problem without interior singularity , i.e. without transmission conditions (3.3)*. The exact solution of the BVP* (3.1), (3.2) *is*

$$y=\left(\frac{1-e^2-e^{-2}}{2cos2}\right)cos2x+\left(\frac{1-e^2+e^{-2}}{2sin2}\right)sin2x+e^{2x}.$$

Consider the uniform cartesian grid for $N = 100$*, i.e.* $h = \dfrac{1-(-1)}{100} = 0,02 \quad x_i = -1 + ih, \quad i = 0,1,\ldots,99,100$ *where in particular* $x_0 = -1,\ x_{100} = 1$.

Then using the FDM at a typical grid point x_i*, we obtain*

$$y_{i-1} + \left(-2+4h^2\right) y_i + y_{i+1} = 8h^2 2e^{2x_i} \tag{3.4}$$

for $i = 1,2,\ldots,99$.

In an tridiagonal matrix-vector form, this linear algebraic system can be written as

$$\begin{pmatrix} -2+4h^2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -2+4h^2 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -2+4h^2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots & -2+4h^2 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & -2+4h^2 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & -2+4h^2 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{97} \\ y_{98} \\ y_{99} \end{pmatrix} = \begin{pmatrix} 8h^2e^{2x_1} \\ 8h^2e^{2x_2} \\ 8h^2e^{2x_3} \\ \vdots \\ 8h^2e^{2x_{97}} \\ 8h^2e^{2x_{98}} \\ 8h^2e^{2x_{99}} - 1 \end{pmatrix}$$

The solution of this system of algebraic equations can be found by using MATLAB/Octave.An error analysis of the obtained results are given in the following table

Table 1: An error analysis

	N=20				N=100		
x	Exact Solutions	FDM-Solution	EROR	x	Exact Solutions	FDM Solution	EROR
1	1,7331	1,7227	0,0104	1	0,3479	0,34782	0,00008
2	3,4103	3,3896	0,0207	2	0,6957	0,69553	0,00017
3	4,9677	4,9372	0,0305	3	1,0429	1,0426	0,0003
4	6,3468	6,3069	0,0399	4	1,3888	1,3885	0,0003
5	7,497	7,4485	0,0485	5	1,7331	1,7327	0,0004
10	8,839	8,7655	0,0735	10	3,4103	3,4095	0,0008
11	8,221	8,1472	0,0738	50	8,839	8,8361	0,0029
12	7,3729	7,3007	0,0722	60	7,3729	7,37	0,0029
13	6,3503	6,2815	0,0688	70	5,2202	5,2177	0,0025
14	5,2202	5,1568	0,0634	80	2,9556	2,9537	0,0019
15	4,0601	4,0039	0,0562	90	1,2868	1,2858	0,001
16	2,9556	2,9082	0,0474	96	0,89672	0,89631	0,00041
17	1,9989	1,9619	0,037	97	0,89239	0,89208	0,00031
18	1,2868	1,2615	0,0253	98	0,90761	0,9074	0,00021
19	0,91977	0,90692	0,01285	99	0,9432	0,9431	0,0001

In Table 1 the exact solution is compared with the numerical FDM-solutions for the BVP (3.1)-(3.2).

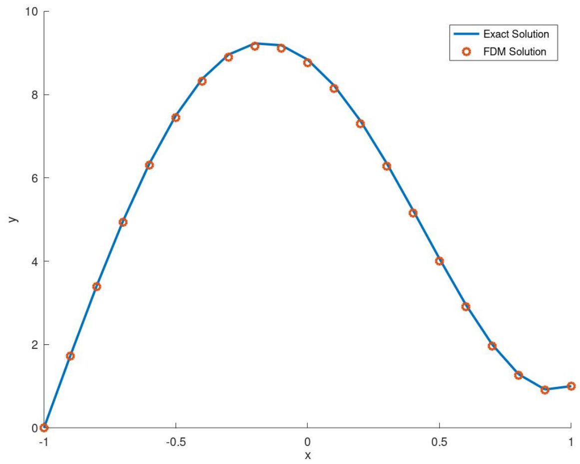

Figure 3.1: Comparison of the FDM-solution with the exact solution for N=20.

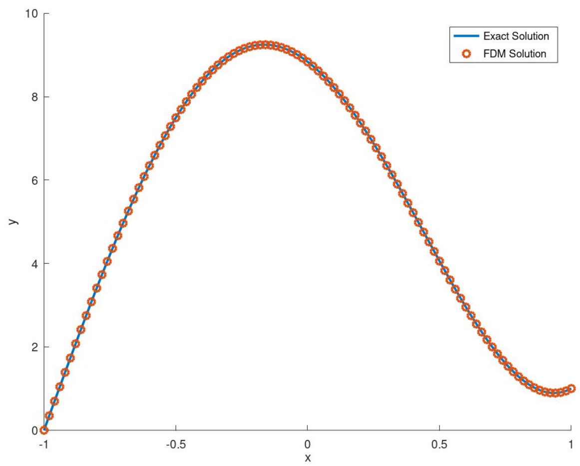

Figure 3.2: Comparison of the FDM-solution with the exact solution for N=100.

Example 3.2. *Now we shall investigate the main problem, i.e the BVP* (3.1), (3.2) *under additional transmission conditions* (3.3). *We can find that the exact solution of this problem has the following form*

$$y = \begin{cases} \left(\dfrac{e^2(tan(2)-4)-sec(2)(1+2e^4)}{7e^2}\right)cos2x + \left(\dfrac{2csc(2)(3-e^4-e^2(3sin(2)+2cos(2))+7e^2}{7e^2}\right)sin2x + e^{2x}, & x \in [-1,0), \\ \left(\dfrac{e^2(2+3tan(2))-sec(2)(3+6e^4)}{7e^2}\right)cos2x + \left(\dfrac{csc(2)(3-e^4-e^2(3sin(2)+2cos(2)))}{7e^2}\right)sin2x + e^{2x}, & x \in (0,1]. \end{cases}$$

If we select N = 99 and apply the transmission conditions (3.3) *then we have two addiational algebriac equations, given by*

$$3y_{49} - y_{50} = 0 \tag{3.5}$$

and

$$y_{48} - y_{49} - 2y_{50} + 2y_{51} = 0. \tag{3.6}$$

The solution of the algebraic system of equations (3.4), (3.5), (3.6) *can be found by using MATLAB/Octave.*

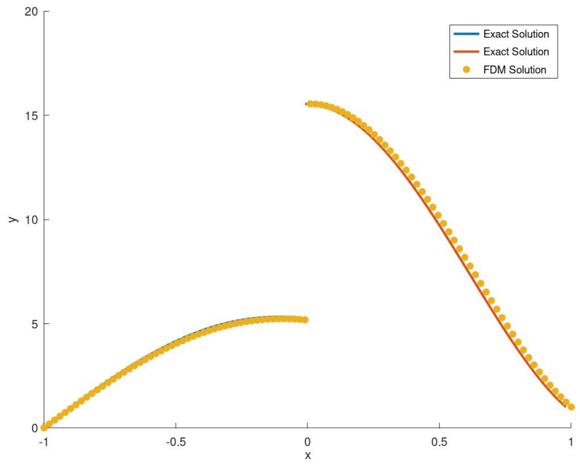

Figure 3.3: Comparison of the exact solution and the FDM-solution for the boundary value problem (3.1)-(3.2) under additional transmission conditions (3.3).

Example 3.3. *Let us consider the following boundary value problem on the range $[-1,1]$ with interior singular point $x=0$, given by the homogeneous singular Sturm-Liouville equation*

$$xy'' - (x+1)y' + y = 0, \quad x \in [-1,0) \cup (0,1] \tag{3.7}$$

together with non homogeneous boundary conditions

$$y(-1) = -2, \qquad y(3) = 4 \tag{3.8}$$

and with additional transmission conditions at the interior point of singularity $x=1$ given by

$$y(1+0) = 3y(1-0), \tag{3.9}$$

$$2y'(1+0) = y'(1-0). \tag{3.10}$$

At first we shall investigate this problem without interior singularity , i.e without transmission conditions. The exact solution of the boundary value problem (3.7)-(3.8) is

$$y = x - 2e^{x+1} + \frac{1}{2}e^4(x+1) + 1.$$

Consider the uniform cartesian grid for $N=100$, i.e. $h = \dfrac{3-(-1)}{100} = 0,04, \quad x_i = -1+ih, \quad i = 0,1,\dots,99,100,$ where in particular $x_0 = -2,\ x_{100} = 4$. Then using the FDM at a typical grid point x_i, we obtain

$$((2+h)x_i + h)y_{i-1} + (2h^2 - 4x_i)y_i + ((2-h)x_i - h)y_{i+1} = 8h^2 2e^{2x_i} \tag{3.11}$$

for $i = 1,2,\dots,99$.

In an tridiagonal matrix-vector form, this linear algebraic system can be written as MY=B, where

$$M = \begin{pmatrix} 2h^2-4x_1 & 2h^2-4x_1 & 0 & \cdots & 0 & 0 & 0 \\ (2+h)x_2+h & 2h^2-4x_2 & (2-h)x_2-h & \cdots & 0 & 0 & 0 \\ 0 & (2+h)x_3+h & 2h^2-4x_3 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots & 2h^2-4x_{97} & (2-h)x_{97}-h & 0 \\ 0 & 0 & 0 & \cdots & (2+h)x_{98}+h & 2h^2-4x_{98} & (2-h)x_{98}-h \\ 0 & 0 & 0 & \cdots & 0 & (2+h)x_{99}+h & 2h^2-4x_{99} \end{pmatrix},$$

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{97} \\ y_{98} \\ y_{99} \end{pmatrix}, \quad B = \begin{pmatrix} 2((2+h)x_1+h) \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 4((h-2)x_{99}+h) \end{pmatrix}.$$

The solution of this system of algebraic equations can be found by using MATLAB/Octave. An error analysis of the obtained results are given in the following table.

Table 2: An error analysis

	N=25				N=200		
x	Exact Solutions	FDM Solution	EROR	x	Exact Solutions	FDM Solution	EROR
1	2,1808	2,2772	-0,0964	1	-1,4744	-1,4745	0,0001
2	6,3014	6,4943	-0,1929	2	-0,94966	-0,94987	0,0002
3	10,351	10,641	-0,29	9	2,6994	2,6984	0,0010
4	14,318	14,705	-0,387	10	3,217	3,2159	0,0011
5	18,188	18,673	-0,485	30	13,335	13,332	0,0030
9	32,309	33,19	-0,881	50	22,863	22,857	0,0060
10	35,372	36,345	-0,973	97	40,983	40,973	0,0100
14	44,603	45,861	-1,258	98	41,268	41,258	0,0100
15	45,871	47,17	-1,299	99	41,547	41,537	0,0100
16	46,574	47,896	-1,322	100	41,82	41,81	0,0100
17	46,613	47,936	-1,323	120	45,871	45,86	0,0110
18	45,873	47,171	-1,298	130	46,65	46,638	0,0120
19	44,219	45,459	-1,24	150	44,726	44,711	0,0150
20	41,492	42,637	-1,145	170	36,289	36,27	0,0190
21	37,507	38,511	-1,004	196	10,131	10,106	0,0250
22	32,044	32,852	-0,808	197	8,6612	8,6354	0,0258
23	24,848	25,393	-0,545	198	7,1497	7,1235	0,0262
24	15,617	15,821	-0,204	199	5,5963	5,5697	0,0266

In Table 2 the exact solution is compared with the numerical solutions of FDM for the boundary value problem (3.7)-(3.8).

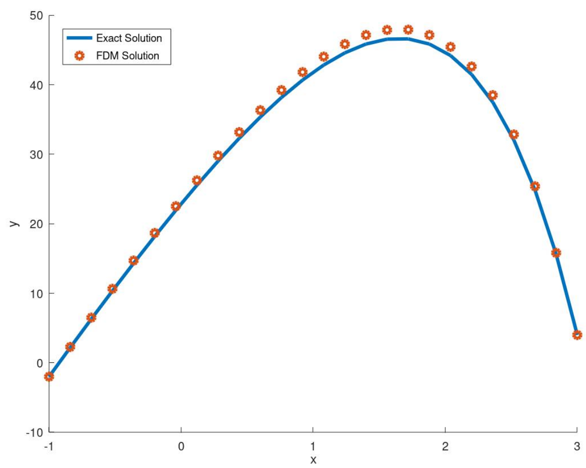

Figure 3.4: Comparison of FDM-solution with the exact solution for N=25.

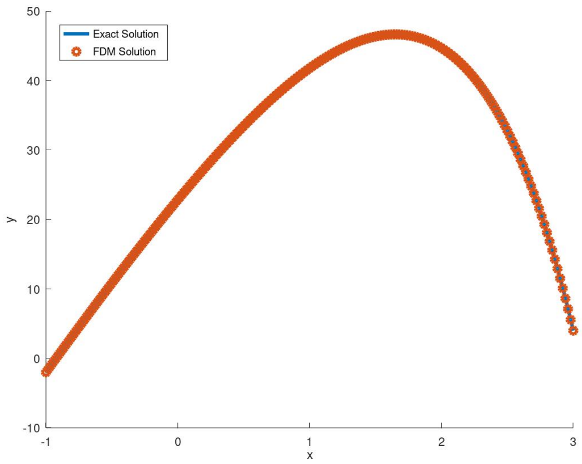

Figure 3.5: Comparison of FDM-solution with the exact solution for N=200.

Example 3.4. *Now we shall investigate the main problem, i.e. the boundary value problem (3.7)-(3.8) under additional transmission conditions (3.9)-(3.10) at the interior singular point $x = 1$,*
We can find that the exact solution of this problem has the following form

$$y = \begin{cases} \left(\dfrac{-4-20e_2+4e^4}{5e^2-22}\right)(x+1)-2e^{x+1}, & x \in [-1,1), \\ \left(\dfrac{22+3e^4}{22-5e^2}\right)(x+1)+\left(\dfrac{20+12e^2}{5e^3-22e}\right)e^x\ , & x \in (1,3]. \end{cases}$$

If we select N = 49 and apply the transmission conditions (3.9)-(3.10) then we have the following additional algebraic equations

$$y_{24} - y_{25} = 0 \tag{3.12}$$

and

$$y_{23} - y_{24} - 2y_{25} + 2y_{26} = 0. \tag{3.13}$$

The solution of the algebraic system of equations (3.7)-(3.8) ,(3.12)-(3.13) can be found by using MATLAB/Octave.

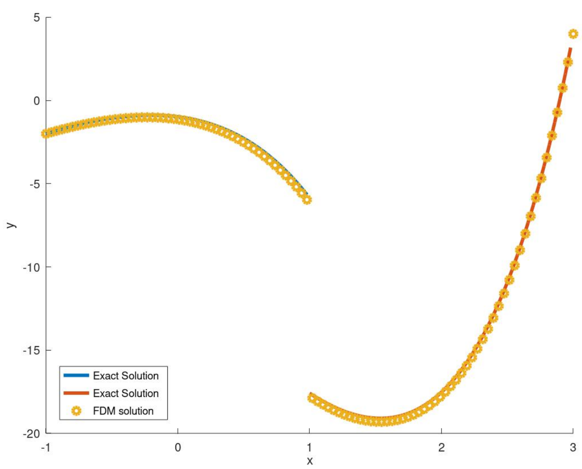

Figure 3.6: Comparison of the exact solution and the FDM-solution for the boundary value problem (3.7)-(3.8) under additional transmission conditions (3.12)-(3.13) for N=49.

4. Conclusion

In the present work we have developed a new modification of FDM to compute an approximate solutions of some singular boundary value problems, the main feature of which is the nature of the transmission conditions imposed. Naturally, such type of problems are much more complicate to solve than classical boundary value problems ones. To justify the proposed modification of FDM we have studied some illustrative problems, involving additional transmission conditions. The obtained approximate FDM- solutions are compared with the exact solutions graphically. The graphs were sketched by using "MATLAB / Octave". The obtained results showed that the classical FDM can be modified for solving singular boundary value problems involving additional transmission conditions.

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