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Crossed modules bifibred over k-Algebras

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Abstract

In this paper we examine on a pair of adjoint functors $(\phi^*, \phi_*)$for a subcategory of the category of crossed modules over commutative algebras where $\phi_*: \mathrm{XMod}/P \to \mathrm{XMod}/\mathrm{Q}$, induced, and $\phi^*: \mathrm{XMod}/\mathrm{Q} \to \mathrm{XMod}/P$, pullback (co-induced), which enables us to move from crossed Q-modules to crossed P-modules by an algebra morphism $\phi : P \to \mathrm{Q}$. We show that this adjoint functor pair (ϕ^*, ϕ_*) makes $p:\ \mathrm{XMod} \to$ k-Alg into a bi- fibred category over k-Alg, the category of commutative algebras, where p is given by $p(C, R, \partial) = R$. Also, we give some examples and results on induced crossed modules in the case when ϕ is an epimorphism or the inclusion of an ideal.

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1. Introduction

The concept of fibration of categories concerns to give a general background to constructions similar to the pullback by a morphism. It seems to be a very useful notion for dealing with hierarchical structures. A functor which forgets the top level of structure is often usefully seen as a fibration or co-fibration of categories. In this work we give the main result that the forgetful functor $p:\ \mathrm{XMod} \to$k-Alg which sends $(C, R, \partial) \to R$ (base algebra) is (co)fibred. We see that the notion of (co)induced crossed modules plays a critical role in this result. Induced and co-induced crossed module constructions correspond the notion of “change of base” in the module theory. The idea is similar to the one above. That is if $\phi :\ S \to R$ is a ring homomorphism, then there is a functor ϕ^* from Mod/R to Mod/S where S acts on an R-module via ϕ. This functor has a left adjoint ϕ_* giving the well known induced module via the tensor product. Analogously induced and co-induced crossed modules give a pair of adjoint functors (ϕ^*, ϕ_*) for a subcategory of the category of crossed modules over commutative algebras where $\phi^*: \mathrm{XMod}/\mathrm{Q} \to \mathrm{XMod}/P$ and $\phi_*: \mathrm{XMod}/P \to \mathrm{XMod}/\mathrm{Q}$. Porter, [1], and Nizar, [2], have just mentioned that this adjoint functor pair (ϕ^*, ϕ_*) makes $p:\ \mathrm{XMod} \to$k-Alg into a fibred and co-fibred category over k − Alg for commutative and associative algebras, respectively. But nonetheless we prove this result with details using related definition and proposition. In particular, we give some examples and results on crossed modules induced by a morphism of commutative algebras $\phi :\ S \to R$ in the case when ϕ is an epimorphism or the inclusion of an ideal. In the applications to commutative algebras, the induced crossed modules play an important role since the free crossed modules which are related to Koszul complexes given by Porter [3] are a special case of induced crossed modules. In [3], any finitely generated free crossed module $C \to R$ of commutative algebras was shown to have $C \cong R^n/d(\Lambda^2 R^n)$, i.e. the 2^{nd} Koszul complex term module the 2-boundaries where $d:\ \Lambda^2 R^n \to R^n$ is the Koszul differential. So, we think that the induced crossed modules of commutative algebras give useful information on Koszul-like constructions.

Analogous result has appeared in [4,5,6] for the group and groupoid theoretical case and in [7] it is showed that braided regular crossed module of groupoids bi-fibred over regular groupoids. Also, fibrations of 2-crossed modules of groups is given in [8].

Conventions: Throughout this paper k is a fixed commutative ring, R is a k-algebra with identity. All k-algebras will be assumed commutative and associative, but they will not be required to have unit elements unless stated otherwise.

2. Crossed Modules of Commutative Algebras

The general concept of a crossed module originates in the work [9] of Whitehead in algebraic topology.

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A *crossed module* of algebras, (C, R, ∂), consists of an R-algebra C and a k-algebra R with an action of R on C, $(r, c) \to r \cdot c$ for $c \in C, r \in R$, and an R-algebra morphism $\partial : C \to R$ satisfying the following condition for all $c, c' \in$ C,

$$\partial(c) \cdot c' = cc' .$$

This condition is called the *Peiffer identity*. We call R, the base algebra and C, the top algebra. When we wish to emphasise the base algebra R, we call (C, R, ∂), a crossed R-module.

A morphism of crossed modules from (C, R, ∂) to (C', R', ∂') is a pair (f, ϕ) of k-algebra morphisms $f : C \to C'$,

$\phi : R \to R'$ such that

(i) $\partial' f = \phi\partial$ and (ii) $f(r \cdot c) = \phi(r) \cdot f(c)$

for all $c \in C, r \in R$. Thus, one can obtain the category XMod of crossed modules of algebras. In the case of a morphism (f, ϕ) between crossed modules with the same base R, say, where ϕ is the identity on R,

$$\begin{tikzcd} C \arrow[rr, "f"] \arrow[dr, "\partial"'] & & C' \arrow[dl, "\partial'"] \\ & R & \end{tikzcd}$$

then we say that f is a morphism of crossed R-modules. This gives a subcategory XMod/R of XMod.

Examples of crossed modules

1. Any ideal, I, in R gives an inclusion map $i: I \hookrightarrow R$, which is a crossed module then we will say (I, R, i) is an ideal pair. In this case, of course, R acts on I by multiplication and the inclusion homomorphism i makes (I, R, i) into a crossed module, an "inclusion crossed modules". Conversely,

Lemma 1. If (C, R, ∂) is a crossed module, $\partial(C)$ is an ideal of R.

2. Any R-module M can be considered as an R-algebra with zero multiplication and hence the zero morphism $0 : M \to R$ is a crossed module. Again conversely,

Lemma 2. If (C, R, ∂) is a crossed module, $Ker\partial$ is an ideal in C and inherits a natural R-module structure from R-action on C. Moreover, $\partial(C)$ acts trivially on $Ker\partial$, hence $Ker\partial$ has a natural $R/\partial(C)$-module structure. As these two examples suggest, general crossed modules lie between the two extremes of ideal and modules. Both aspects are important.

3. In the category of algebras, the appropriate replacement for automorphism groups is the bi-multiplication algebra Bim(R) defined by Mac Lane [10]. (see also [11,12]). Let R be an associative (not necessarily unitary or commutative) k-algebra. Bim(R) consists of pairs (γ, δ) of R-linear mappings from R to R such that

$$\begin{aligned} \gamma(rr') &= \gamma(r) \cdot r' \, \delta(rr') \\ &= r \cdot \delta(r') \text{ and} \\ r \cdot \gamma(r' &= \delta(r) \cdot r'. \end{aligned}$$

If R is a commutative algebra and $Ann(R) = 0$ or $R^2 = R$, then, since

$$\begin{aligned} x \cdot \delta(r) &= \delta(r) \cdot x \\ &= r \cdot \gamma(x) \\ &= \gamma(x) \cdot r \\ &= \gamma(xr) \\ &= \gamma(rx) \\ &= \gamma(r) \cdot x \\ &= x \cdot \gamma(r) \end{aligned}$$

for every $x \in R$, we get $\gamma = \delta$. Thus, Bim(R) may be identified with the k-algebra *M(R)* of multipliers of R. Recall that a multiplier of R is a linear mapping $\lambda : R \to R$ such that for all $r, r' \in R$

$$\lambda(rr') = \lambda(r)\, r'.$$

Also, *M(R)* is commutative as

$$\begin{aligned} \lambda' \lambda(xr) &= \lambda'(\lambda(x) r) \\ &= \lambda(x)\, \lambda'(r) \\ &= \lambda'(r)\, \lambda(x) \\ &= \lambda\lambda'(rx) \\ &= \lambda\lambda'(xr) \end{aligned}$$

for any $x \in R$. Thus, $M(R)$ is the set of all multipliers λ such that $\lambda\gamma = \gamma\lambda$ for every multiplier γ. So automorphism crossed module corresponds to the multiplication crossed module $(R, M(R), \mu)$ where $\mu : R \to M(R)$ is defined by $\mu(r) = \lambda_r$ with $\lambda_r(r') = rr'$ for all $r, r' \in R$ and the action is given by $\lambda \cdot r = \lambda(r)$ (See [13] for details).

2.1. Free crossed modules

Let (C, R, ∂) be a crossed module, Y be a set, and $\upsilon : Y \to C$ be a function, then (C, R, ∂) is said to be a free crossed module with basis υ or alternatively, on the function $\partial\upsilon : Y \to R$ if for any crossed R-module (A, R, δ) and a function $w : Y \to A$ such that $\delta w = \partial\upsilon$, there is a unique morphism $\phi : (C, R, \partial) \to (A, R, \delta)$ such that the diagram

$$\begin{tikzcd} C \arrow[rr, "\phi"] \arrow[dr, "\partial"'] & & A \arrow[dl, "\delta"] \\ & R & \end{tikzcd}$$

is commutative.

For our purpose, an important standard construction of free crossed R-modules is as follows:

Suppose given $f : Y \to R$. Let $E = R^{+}[Y]$, the positively graded part of the polynomial ring on Y. f induces a morphism of R-algebras, $\theta : E \to R$ defined on generators by $\theta\,(y) = f\,(y)$. We define an ideal P in E (sometimes called by analogy with the group theoretical case, the Peiffer ideal relative to f) generated by the elements

$$\{pq - \theta\,(p)\,q : p, q \in E\}$$

clearly $\theta\,(P) = 0$, so putting $C = E/P$, one obtains an induced morphism $\delta : C \to R$ which is the required free crossed R-module on f (cf. [3]). This construction will be seen later as a special case of an induced crossed module.

3. Fibrations and Co-fibrations Categories

We recall the definitions of fibration and co-fibration of categories which are also known as Grothendieck fibrations and co-fibrations.

Definition 3. Let $\Phi : \mathrm{X} \to \mathrm{B}$ be a functor. A morphism $\varphi : Y \to X$ in X over u:= $\Phi\,(\varphi)$ is called cartesian if and only if for all $v : K \to J$ in B and $\theta : Z \to X$ with $\Phi\,(\theta) = uv$ there is a unique morphism $\psi : Z \to Y$ with $\Phi\,(\psi) = v$ and $\theta = \varphi\psi$.

This is illustrated by the following diagram:

$$\begin{tikzcd} Z \arrow[r, dotted, "\psi"'] \arrow[rr, bend left, "\theta"] & Y \arrow[r, "\varphi"'] & X & \arrow[dd, "\Phi"] \\ & & & \\ K \arrow[r, "v"'] \arrow[rr, bend right, "uv"'] & J \arrow[r, "u"'] & I & \end{tikzcd}$$

It is straightforward to check that cartesian morphisms are closed under composition, and that φ is an isomorphism if and only if φ is a cartesian morphism over an isomorphism.

A morphism $\alpha : Z \to Y$ is called vertical (with respect to Φ) if and only if $\Phi\,(\alpha)$ is an identity morphism in B. In particular, for $I \in$ B we write X/I, called the fibre over I, for the subcategory of X consisting of those morphisms α with $\Phi\,(\alpha) = id_I$.

Definition 4. The functor $\Phi : \mathrm{X} \to \mathrm{B}$ is a fibration or category fibred over B if and only if for all $u : J \to I$ in B and $X \in \mathrm{X}\,/\,I$ there is a cartesian morphism $\varphi : Y \to X$ over u : such a φ is called a cartesian lifting of $\boldsymbol{X}$ along u.

In other words, in a category fibred over B, $\Phi : \mathrm{X} \to \mathrm{B}$, we can pull back objects of X along any arrow of B.

Definition 5. Let $\Phi : \mathrm{X} \to \mathrm{B}$ be a functor. A morphism $\varphi : Z \to Y$ in X over $v := \Phi(\psi)$ is called cocartesian if and only if for all $u : J \to I$ in B and $\theta : Z \to X$ with $\Phi(\theta) = uv$ there is a unique morphism $\varphi : Y \to X$ with $\Phi(\varphi) = u$ and $\theta = \varphi\psi$. This is illustrated by the following diagram:

$$\begin{tikzcd} Z \arrow[r, "\psi"'] \arrow[rr, bend left, "\theta"] & Y \arrow[r, dotted, "\varphi"'] & X & \arrow[dd, "\Phi"] \\ & & & \\ K \arrow[r, "v"'] \arrow[rr, bend right, "uv"'] & J \arrow[r, "u"'] & I & \end{tikzcd}$$

It is straightforward to check that co-cartesian morphisms are closed under composition, and that ψ is an isomorphism if and only if ψ is a co-cartesian morphism over an isomorphism.

Definition 6. The functor $\Phi : \mathrm{X} \to \mathrm{B}$ is a co-fibration or category co-fibred over B if and only if for all $v : K \to J$ in B and $Z \in \mathrm{X}/K$ there is a cocartesian morphism $\psi : Z {\to} Z'$ over v : such a ψ is called a cocartesian lifting of Z along v.

Proposition 7. Let $\Phi : \mathrm{X} \to \mathrm{B}$ be a fibration of categories. Then $\psi : \mathrm{Z} \to Y$ in X over $v : K \to J$ in B is cocartesian if only if for all $\theta' : Z \to X'$ over v there is a unique morphism $\psi' : Y \to X'$ in X/J with $\theta' = \psi'\psi$.

Taking this background into account, we get following results. These results are given in [6] in the case of crossed modules of groupoids.

Proposition 8. The forgetful functor $p : \mathrm{X}Mod \to k-Alg$ which sends $(C, R, \partial) \mapsto R$ (base algebra), is fibred.

Proof. Let (C, R, ∂) be a crossed R-module and let $\phi : S \to R$ be a morphism of k-algebras. Then

$\phi^*\,(C) = \{(c, s) \mid \varphi\,(s) = \partial\,(c),\, s \in S,\, c \in C\}$ has the S-algebra structure by

$$s \cdot (c, s')\,) = (\phi\,(s) \cdot c, ss'\,)$$

for $(c, s') \in \phi^*\,(C)$, $s \in S$ and $\partial^* : \phi^*\,(C) \to S$, $\partial^*((c, s)) = s$ is a crossed S-module. It is immediate that ∂^* is a S-algebra morphism, while the Peiffer identity condition is proved as follows:

$$\begin{aligned}\partial^*(c,s)\cdot(c',s') &= s\cdot(c',s')\\ &= (\phi(s)\cdot c', ss')\\ &= (\partial(c)\cdot c', ss')\\ &=(cc', ss')\\ &= (c,s)(c',s')\end{aligned}$$

for $(c,s),(c',s')\in\phi^*(C)$. Also,
$(\phi^C,\phi):(\phi^*(C),S,\partial^*)\to(C,R,\partial)$

is a morphism of crossed modules where $(\phi^C(c,s) = c$, since

$$\begin{aligned}\phi^C(s'(c,s)) &=\phi^C(\phi(s')c,s's)\\ &= \phi(s')\cdot c\\ &= \phi(s')\cdot\phi^C(c,s)\end{aligned}$$

and clearly $\partial\phi^C = \phi\,\partial^*$.

Then for $g: T\to S$ in k-Alg and $(f,\phi g):(B,T,\partial')\to(C,R,\partial)$ in XMod with $p((f,\phi g)) = \phi g$, there is a unique morphism of crossed modules $(f^*,g):(B,T,\partial')\to(\phi^*(C),S,\partial^*)$ given by $f^*(b) = (f(b), g\,\partial'(b))$ for all $b\in B$ such that $p((f^*,g)) = g$ and $(f,\phi g) = (\phi^C,\phi) = (f^*,g)$.

Existence of

$$\begin{aligned}(f^*,g):\partial^*(f^*(b) &= \partial^*\big(f(b,g\,\partial'(b))\big) = g\,\partial'(b)\\ f^*(t\cdot b) &= (f(t\cdot b),\ g\ \partial'(t\cdot b))\\ &=((\phi\ g)(t)\,f(b), g(t)g\partial'(b))\\ &= g(t)\cdot(f(b), g\partial'(b)\\ &= g(b).\end{aligned}$$

Uniqueness of $(f^*,\ g)$: Suppose that $(f^{*\prime},g')$: $(B,T,\partial')\to(\phi^*(C),S,\partial^*)$ is a crossed module morphism with

$p((f^{*\prime},g'))= g$ and $(f,\phi g) = (\phi^C,\phi)\,(f^{*\prime},g')$.

It is clear that $g' = g$. $f^{*\prime}$ is defined as

$(f^{*\prime}(b) = (c',s')$ for some $c'\in C$, $s'\in S$. Then we have $f^{*\prime} = f^*$ as follows

$$\begin{aligned}(f^{*\prime}(b)) &= (c',s')\\ &= (\phi^C(c',s'),\partial^*(c',s'))\\ &= (\phi^C(f^{*\prime}(b)),\partial^*(f^{*\prime}(b))\,)\\ &= (f(b), g'\partial'(b)\,)\\ &= (f(b), g\partial'(b)\,)\\ &= f^*(b)\end{aligned}$$

for all $b\in B$.

Thus, we get a cartesian morphism $(\phi^C,\phi):(\phi^*(C),S,\partial^*)\to(C,R,\partial)$, for $\phi: S\to R$ in *k-Alg* and (C,R,∂) in XMod/R, as required.

This is illustrated by the following diagram:

$$\begin{tikzcd}
(B,T,\partial') \arrow[drr, bend left, "{(f,\phi g)}"] \arrow[dr, dashed, "{(f^*,g)}"'] \arrow[dd, "p"'] & & \\
& (\phi^*(C),S,\partial^*) \arrow[r, "{(\phi^C,\phi)}"'] \arrow[dd, "p"] & (C,R,\partial) \arrow[dd, "p"] \\
T \arrow[drr, "\phi g"] \arrow[dr, "g"'] & & \\
& S \arrow[r, "\phi"'] & R.
\end{tikzcd}$$

We note that $(\phi^*(C), S, \partial^*)$ is usually called “pullback crossed module” and it can be given by a pullback diagram:

$$\begin{tikzcd}
B \arrow[drr, bend left, "f"] \arrow[dr, dashed, "f^*"] \arrow[ddr, bend right, "\mu"'] & & \\
& \phi^*(C) \arrow[r, "\phi^C"'] \arrow[d, "\partial^*"] & C \arrow[d, "\partial"] \\
& S \arrow[r, "\phi"'] & R.
\end{tikzcd}$$

We will give some examples of pullback crossed modules of commutative algebras, some of them is given in [2] for non-commutative algebra case.

Examples of pullback crossed modules

1. Given crossed module $i: I\hookrightarrow R$ where i is an inclusion of an ideal. The pullback crossed module is $(\phi^*(I),S,i^*)\cong(\phi^{-1}(I),S,i^*)$

where,

$$\begin{aligned}\phi^*(I) &= \{(i,s)\mid\phi(s) = i(i), s\in S, i\in I\}\\ &\cong\{s\in S\mid\phi(s) = i\in I\} = \phi^{-1}(I)\\ &\trianglelefteq S\end{aligned}$$

The pullback diagram is

$$\begin{tikzcd}
\phi^{-1}(I) \arrow[r] \arrow[d, "i^*"'] & I \arrow[d, "i"] \\
S \arrow[r, "\phi"'] & R.
\end{tikzcd}$$

Particularly if $I = \{0\}$, then

$\phi^*(\{0\})\cong Ker\phi$

and so $(\mathrm{Ker}\phi, S, i^*)$ is a pullback crossed module. Kernels are thus particular cases of pullbacks.

Also, if ϕ is onto and $I = R$, then $\phi^*(R) \cong S$

2. Given a crossed module $0: M \to R, 0(m) = 0$, where M is any R-module, so it is also an R-algebra with zero multiplication.

Then $\phi^*(M, R, 0) = \phi^*(M), S, 0^*)$ where

$\phi^*(M) = \{(m, s) \in M \times S \mid \phi(s) = 0(m) = 0\} \cong M \times Ker\phi$

The corresponding pullback diagram is

$$\begin{tikzcd} M \times Ker\phi \arrow[r] \arrow[d, "0^*"'] & M \arrow[d, "0"] \\ S \arrow[r, "\phi"'] & R. \end{tikzcd}$$

So, if φ is injective, then $M \cong \phi^*(M)$. If $M = \{0\}$, then $\phi^*(M) \cong Ker\phi$.

A pullback crossed module $(\phi^*(C), S, \partial^*)$ for $\phi: S \to R$ in k-Alg gives a functor

$$\phi^*: \mathrm{XMod}/R \to \mathrm{XMod}/S$$

which has appeared in the work of Porter [1] as "restriction along $\phi: S \to R$".

Proposition 9. For each morphism $(\psi, id_R): (C, R, \partial) \to (C', R', \partial')$ in XMod/R, the morphism

$(\phi^*\psi, id_S): \phi^*(C), S, \partial^*) \to \phi^*(C'), S, \partial'^*)$ in XMod/S is the unique morphism satisfying the equality $(\psi, id_R)(\phi^C, \phi) = (\phi^{C'}, \phi)(\phi^*\psi, id_S)$.

Proof. Since $\phi^{C'}((\phi^*\psi)(c, s)) = \phi^{C'}(\psi(c), s) = \psi(c) = \psi\,\phi^C(c, s)$ *for* $(c, s) \in \phi^*(C)$ and $id_R\phi = \phi id_S$, we get the equation as required.

$$\begin{tikzcd} & & \phi^*(C') \arrow[d, "\partial'^*"] \arrow[drr, "\phi^{C'}"] & & \\ \phi^*(C) \arrow[urr, "\phi^*\psi"] \arrow[d, "\partial^*"'] \arrow[drr, "\phi^C"] & & S \arrow[drr, "\phi"] & & C' \arrow[d, "\partial'"] \\ S \arrow[urr, "id_S"] \arrow[dr, "\phi"'] & & C \arrow[urr, "\psi"'] \arrow[d, "\partial"] & & R \\ & & R \arrow[urr, "id_R"'] & & \end{tikzcd}$$

Proposition 10. If $\phi_1: S \to R, \phi_2: T \to S$ are two morphisms of k-algebras, then $(\phi_1\phi_2)^*$ and ${\phi_2}^*{\phi_1}^*$ are naturally isomorphic.

Proof. Given any crossed R-module (C, R, ∂), we define $\alpha_C: {\phi_2}^*{\phi_1}^*(C) \to (\phi_1\phi_2)^*(C)$ as $\alpha_C\left((c, \phi_2(t)), t\right) = (c, t)$ for all $\left((c, \phi_2(t)), t\right) \in {\phi_2}^*{\phi_1}^*(C)$. Clearly α_C is well defined and a k-algebra morphism. Also, since

$$\begin{aligned} \alpha_C(t' \cdot (c, \phi_2(t)), t) &= \alpha_C(\phi_2(t')) \cdot ((c, \phi_2(t)), t't) \\ &= \alpha_C\left((\phi_1(\phi_2(t')) \cdot c, \phi_2(t')\phi_2(t), t't\right) \\ &= \alpha_C\left((\phi_1(\phi_2(t')) \cdot c, \phi_2(t't), t't\right) \\ &= \left(\phi_1(\phi_2(t')) \cdot c, t't\right) \\ &= t' \cdot (c, t) \\ &= t' \cdot \alpha_C((c, \phi_2(t)), t) \end{aligned}$$

and

$\overline{\partial^*}\alpha_C\left((c, \phi_2(t)), t\right) = \overline{\partial^*}(c, t) = t = id_T\overline{\overline{\partial^*}}\left((c, \phi_2(t)), t\right)$, (α_C, id_R) is a crossed module morphism. It is clear that (α_C, id_T) is an isomorphism.

$$\begin{tikzcd}
& & & C \arrow[d, "\partial"] & \\
& & \phi_1^*(C) \arrow[ur, "\phi_1^C"] \arrow[d, "\partial^*"'] & R & \\
\phi_2^*(\phi_1^*(C)) \arrow[urr, "\phi_2^{\phi_1^*(C)}"] \arrow[d, "\overline{\overline{\partial}}^*"] \arrow[rrrr, bend right, "\alpha_C"'] & & S \arrow[ur, "\phi_1"'] & & (\phi_1\phi_2)^*(C) \arrow[uul, "(\phi_1\phi_2)^C"'] \arrow[d, "\overline{\partial}^*"] \\
T \arrow[urr, "\phi_2"'] \arrow[rrrr, bend right, "id_T"'] & & & & T \arrow[uul, "\phi_1\phi_2"']
\end{tikzcd}$$

In addition, for each crossed R-module morphism $(\psi, id_R): (C, R, \partial) \to (D, R, \varphi)$ and $((c, \phi_2(t)), t) \in \phi_2{}^*\phi_1{}^*(C)$, we get

$$\begin{aligned}
(((\phi_1\phi_2)^*\psi)\alpha_C)((c, \phi_2(t)), t) &= ((\phi_1\phi_2)^*\psi)(c, t) \\
&= (\psi(c), t) = \alpha_D((\psi(c), \phi_2(t)), t) \\
&= \alpha_D(\phi_2{}^*\phi_1{}^*\psi)((c, \phi_2(t)), t)
\end{aligned}$$

i.e., the diagram

$$\begin{tikzcd}
(\phi_2^*\phi_1^*(C), T, \overline{\overline{\partial}}^*) \arrow[r, "(\phi_2^*\phi_1^*\psi, id_T)"] \arrow[d, "(\alpha_C, id_T)"'] & (\phi_2^*\phi_1^*(D), T, \overline{\overline{\partial}}^*) \arrow[d, "(\alpha_D, id_T)"] \\
((\phi_1\phi_2)^*(C), T, \overline{\partial^*}) \arrow[r, "((\phi_1\phi_2)^*\psi, id_T)"'] & ((\phi_1\phi_2)^*(D), T, \overline{\partial^*})
\end{tikzcd}$$

is commutative and this completes the proof.

Proposition 11. If p : XMod → k - Alg is fibred, $\phi: S \to R$ in k-Alg, and a reindexing functor

ϕ^*: XMod$/R$ → XMod / S is chosen, then there is a bijection

$$\mathrm{XMod}_\phi((B, S, \mu), (C, R, \partial)) \cong \mathrm{XMod}/S((B, S, \mu), \phi^*(C), S, \partial^*))$$

natural in $(B, S, \mu) \in$ XMod$/S$, $(C, R, \partial) \in$ XMod$/R$ where $\mathrm{XMod}_\phi((B, S, \mu), (C, R, \partial))$ consists of those morphisms $\alpha \in \mathrm{XMod}((B, S, \mu), (C, R, \partial))$ with $p(\alpha) = \phi$.

Proof. Define

F: $\mathrm{XMod}_\phi((B, S, \mu), (C, R, \partial)) \to \mathrm{XMod}/S((B, S, \mu), \phi^*(C), S, \partial^*))$

as $F(f, \phi) = (f^*, id_S)$ such that $f^*(b) = (f(b), \mu(b))$. Assume that for $(f, \phi), (g, \phi) \in \mathrm{XMod}_\phi((B, S, \mu), (C, R, \partial))$, $F(f, \phi) = F(g, \phi)$. Then we get $(f^*, id_S) = (g^*, id_S)$ and so $(f(b), \mu(b)) = (g(b), \mu(b))$ i.e., $f = g$. Thus, F is one to one. Suppose that $(f^*, id_S) \in \mathrm{XMod}/S((B, S, \mu), \phi^*(C), S, \partial^*))$. Then there is $(\phi^C f^*, \phi) \in \mathrm{XMod}_\varphi((B, S, \mu), (C, R, \partial))$ where $\varphi^C(c, s) = c$ such that $F(\varphi^C f^*, \varphi) = (f^*, id_S)$.

It is clear from $F(\varphi^C f^*, \varphi) = ((\varphi^C (f^*)^*, id_S)$ and

$$\begin{aligned}
(\phi^C f^*)^*(b) &= \big((\phi^C f^*)(b), \mu(b)\big) \\
&= \Big(\phi^C\big(f(b), \mu(b)\big), \mu(b)\Big) \\
&= \big(f(b), \mu(b)\big) \\
&= f^*(b)
\end{aligned}$$

$$\begin{tikzcd}
\phi^*(C) \arrow[dd, "\partial^*"'] \arrow[rrd, bend left, "\phi^C"] & & \\
& B \arrow[ul, "f^*"'] \arrow[r, "f"'] \arrow[dd, "\mu"'] & C \arrow[dd, "\partial"] \\
S \arrow[dr, equal, "id_S"'] & & \\
& S \arrow[r, "\phi"'] & R.
\end{tikzcd}$$

So, F is a bijection.

Moreover, since for crossed module morphism $(\psi, id_S): (B', S, \mu) \to (B, S, \mu')$, $b' \in B'$

$$\begin{aligned}
(f^*\psi)(b') &= (f(\psi(b')), \mu(\psi(b'))) \\
&= \big((f\psi)(b'), (\mu\psi)(b')\big) \\
&= (f\psi^*)(b')
\end{aligned}$$

and

$$\begin{aligned}
\big(- \circ (\psi, id_S)\big)F(f, \phi) &= \big(- \circ (\psi, id_S)\big)(f^*, id_S) \\
&= (f^*\psi, id_S) \\
&= ((f\psi)^*, id_S) \\
&= F'(f\psi, \phi) \\
&= \Big(F'\big(- \circ (\psi, id_S)\big)\Big)(f, \phi),
\end{aligned}$$

we get following commutative diagram

$$\begin{tikzcd}
\mathbf{XMod}_\phi((B, S, \mu), (C, R, \partial)) \arrow[r, "F"] \arrow[d, "- \circ (\psi, id_S)"'] & \mathbf{XMod}/S((B, S, \mu), (\phi^*(C), S, \partial^*)) \arrow[d, "- \circ (\psi, id_S)"] \\
\mathbf{XMod}_\phi((B', S, \mu'), (C, R, \partial)) \arrow[r, "F'"'] & \mathbf{XMod}/S((B', S, \mu'), (\phi^*(C), S, \partial^*))
\end{tikzcd}$$

Thus, F is natural in (B, S, μ). Furthermore, for $(\psi, id_R): (C, R, \partial') \to (C', R, \partial')$ we have

$$\begin{aligned}
(\psi f)^*(b) &= \big((\psi f)(b), \mu(b)\big) = \big(\psi(f(b)), \mu(b)\big) \\
&= (\phi^*\psi)\big(f(b), \mu(b)\big) \\
&= (\phi^*\psi)f^*(b)
\end{aligned}$$

for $b \in B$, and

$$\begin{aligned}((\phi^*\psi, id_S)\circ -)F(f,\phi) &= (\phi^*\psi, id_S)(f^*, id_S)\\ &= \big((\phi^*\psi)f^*, id_S\big)\\ &= ((\psi f)^*, id_S)\\ &= F''(\psi f, \phi)\\ &= \big(F''((\psi, id_R)\circ -)\big)(f,\phi).\end{aligned}$$

which means the diagram

$$\begin{tikzcd}
\mathbf{XMod}_{\phi}((B,S,\mu),(C,R,\partial)) \arrow[r, "F"] \arrow[d, "(\psi,id_R)\circ -"'] & \mathbf{XMod}/S((B,S,\mu),(\phi^*(C),S,\partial^*)) \arrow[d, "(\phi^*\psi,id_S)\circ -"] \\
\mathbf{XMod}_{\phi}((B,S,\mu),(C',R,\partial')) \arrow[r, "F''"'] & \mathbf{XMod}/S((B,S,\mu),(\phi^*(C'),S,\partial'^*)).
\end{tikzcd}$$

is commutative and F is natural in (C,R,∂).

We now give the dual of Proposition [8].

Proposition 12. The forgetful functor $p\colon \mathrm{XMod} \to k$-Alg is co-fibred.

Proof. Given a k-algebra morphism $\phi\colon S \to R$ and a crossed module $\partial\colon D \to S$, and let the set $F(D\times R)$ be a free algebra generated by the elements of $D \times R$. Let P be the ideal generated by all the relations of the three following types:

$$\begin{aligned}(d_1,r)+(d_2,r) &= (d_1+d_2,r)\\ (s\cdot d,r) &= (d,\phi(s)r)\\ (d_1,r_1)(d_2,r_2) &= (d_2,r_1(\phi\partial d_1)r_2)\end{aligned}$$

for any $d, d_1, d_2 \in D$, and $r \in R, s \in S$. We define

$\phi_*(D) = F(D\times R)/P.$

This is an R-algebra with

$r'\cdot(d\otimes r) = d\otimes r'r$

for $d \in D, r, r' \in R$ and $\partial_*\colon \phi_*(D) \to R, \partial_*(d\otimes r) = \phi\partial(d)r$ is an R-algebra morphism and since

$$\begin{aligned}\partial_*(d\otimes r)\cdot(d_1\otimes r_1) &= \big((\phi\partial d)r\big)\cdot(d_1\otimes r_1)\\ &= (d_1\otimes\phi(\partial d)rr_1)\\ &= (\partial d\cdot d_1\otimes rr_1)\\ &= (dd_1\otimes rr_1)\\ &= (d\otimes r)(d_1\otimes r_1),\end{aligned}$$

for $d\otimes r, d_1\otimes r_1 \in D\otimes_S R$, ∂_* is a crossed R-module. Also, since R has a unit, if $\phi'\colon D \to \phi_*(D)$ is defined by $\phi'(d) = (d\otimes 1)$, then

$$\begin{aligned}\phi'(s\cdot d) &= (s\cdot d\otimes 1)\\ &= (d\otimes\phi(s))\\ &= \phi(s)\cdot(d\otimes 1)\\ &= \phi(s)\cdot\phi'(d)\end{aligned}$$

for $\in D$, $s \in S$, and clearly $\partial_*\phi' = \phi\partial$, so $(\phi',\phi):(D,S,\partial) \to (\phi_*(D),R,\partial_*)$ is a crossed module morphism. Then for $(f,\phi):(D,S,\partial) \to (B,R,\eta)$ over $\phi: S \to R$ there is a unique morphism $(f_*,id_R):(\phi_*(D),R,\partial_*) \to (B,R,\eta)$ given by $f_*(d \otimes r) = r \cdot f(d)$ for all $d \otimes r \in \phi_*(D)$ in XMod/R with $(f,\phi) = (f_*,id_R)(\phi',\phi)$.

Existence of (f_*,id_R):

$$
\begin{aligned}
f_*(r' \cdot (d \otimes r)) &= f_*(d \otimes r'r) \\
&= r'r \cdot f(d) \\
&= r' \cdot (r \cdot f(d)) \\
&= r' \cdot f_*(d \otimes r) \\
&= id_R(r') \cdot f_*(d \otimes r)
\end{aligned}
$$

and

$$
\begin{aligned}
(\eta f_*)((d \otimes r)) &= \eta(f_*(d \otimes r)) \\
&= \eta(r \cdot f(d)) \\
&= r\eta(f(d)) \\
&= r\phi(\partial(d)) \\
&= \partial_*(d \otimes r) \\
&= id_R\partial_*(d \otimes r)
\end{aligned}
$$

for each $d \otimes r \in \phi_*(D), r' \in R$ so (f_*,id_R) is a crossed R-module morphism also $f_*\phi' = f$.

Uniqueness of (f'_*,id_R): Suppose that $(f'_*,id_R):(\phi_*(D),R,\partial_*) \to (B,R,\eta)$ is a crossed module morphism with $p(f'_*,id_R) = id_R$ and $(f,\phi) = (f'_*,id_R)(\phi',\phi)$. Then we get

$$
\begin{aligned}
f_*(d \otimes r) &= r \cdot f(d) \\
&= r \cdot f'_*\phi'(d) \\
&= r \cdot f'_*(d \otimes 1) \\
&= f'_*(r \cdot (d \otimes 1)) \\
&= f'_*(d \otimes r),
\end{aligned}
$$

so $f'_* = f_*$. Thus, we have a cocartesian morphism (ϕ',ϕ) in XMod over $\phi: S \to R$ in k-Alg by Proposition [7]. That is, we obtain the following commutative diagram:

$$
\begin{tikzcd}
 & & (B,R,\eta) \arrow[dd, "p"] \\
(D,S,\partial) \arrow[r, "{(\phi',\phi)}"'] \arrow[urr, bend left, "{(f,\phi)}"] \arrow[dd, "p"'] & (\phi_*(D),R,\partial_*) \arrow[ur, dashed, "{(f_*,id_R)}"] \arrow[dd, "p"] & \\
 & & R \\
S \arrow[r, "\phi"'] \arrow[urr, bend left, "\phi"] & R \arrow[ur, equal, "id_R"'] &
\end{tikzcd}
$$

We have seen that the above result is described in terms of the crossed module $\partial_*:\phi_*(D) \to R$ induced from the crossed module $\partial: D \to S$ by a morphism $\theta: S \to R$. It is called “induced crossed module” and it can be given by the following diagram:

$$
\begin{tikzcd}
 & & B \arrow[ddl, bend left, "\eta"] \\
D \arrow[r, "\phi'"'] \arrow[urr, bend left, "f"] \arrow[d, "\partial"'] & \phi_*(D) \arrow[ur, dashed, "f_*"] \arrow[d, "\partial_*"] & \\
S \arrow[r, "\phi"'] & R. &
\end{tikzcd}
$$

We will give the following examples of induced crossed modules of commutative algebras.

Examples of induced crossed modules

Let $D = S$ and $id_S: S \to S$ be identity crossed S-modules. The induced crossed module diagram is

$$
\begin{tikzcd}
S \arrow[r, "\psi"] \arrow[d, "id_S"'] & \phi_*(S) \arrow[d, "id_{S*}"] \\
S \arrow[r, "\phi"'] & R
\end{tikzcd}
$$

where $\phi_*(S) = S \otimes_S R$.

(Remark: S has not unit, otherwise $S \otimes_S R \cong R$). When we take $S = k^+[X]$ the positively graded part of the polynomial algebra over k on the set of generators

X, we have the induced crossed module $\partial_*: k^+[X] \otimes_{k^+[X]} R \to R$ which is the free R-module on $f: X \to R$. Thus, the free crossed modules is the special case of the induced crossed modules.

Considering the free crossed module construction given in the first section, we have a diagram

$$\begin{tikzcd}
k^+[X] \times R \arrow[rr, "\theta"] \arrow[d] & & R \\
(k^+[X] \times R)/P = k^+[X] \otimes_{k^+[X]} R \arrow[urr, "\partial_*"'] & &
\end{tikzcd}$$

with $\theta(p,r) = \partial_*(p \otimes r) = \phi(p)r$ for all $p \in k^+[X]$, $r \in R$, where P is an ideal generated by all the relations given in the proof of Proposition 12. From

$$\begin{aligned}
\theta\big((p_1,r) + (p_2,r) - (p_1+p_2,r)\big) &= \theta(p_1,r) + \theta(p_2,r) - \theta(p_1+p_2,r) \\
&= \phi(p_1)r + \phi(p_2)r - \big(\phi(p_1) + \phi(p_2)\big)r \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\theta\big((p \cdot q, r) - (q, \phi(p)r)\big) &= \theta(pq,r) - \theta(q,\phi(p)r) \\
&= \phi(pq)r + \phi(q)\phi(p)r \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\theta\big((p_1,r_1)(p_2,r_2) - (p_2, r_1\phi\partial(p_1)r_2)\big) &= \theta(p_1,r_1)\theta(p_2,r_2) - \theta(p_2, r_1\phi\partial(p_1)r_2) \\
&= \phi(p_1)r_1\phi(p_2)r_2 - \phi(p_2)r_1\phi\partial(p_1)r_2 \\
&= 0,
\end{aligned}$$

we get $\theta(P) = 0$, and have the pushout diagram

$$\begin{tikzcd}
 & & & A \arrow[dl, "\delta"] \\
X \arrow[r] \arrow[d, equal, "id"'] & k^+[X] \arrow[r, "\phi'"] \arrow[d, "\partial"] \arrow[urr] & \phi_*(k^+[X]) \arrow[d, "\partial_*"] \arrow[ur, dotted] & \\
X \arrow[r, "i"'] & k^+[X] \arrow[r, "\phi"'] & R &
\end{tikzcd}$$

where $\phi_*(k^+[X]) = k^+[X] \otimes_{k^+[X]} R$. Also, by considering the connection between free crossed module $\partial: k^+[X] \otimes_{k^+[X]} R \to R$ and the usual Koszul differential $d: \Lambda^2 R^n \to R^n$, we get

$$k^+[X] \underset{k^+[X]}{\otimes} R \cong R^n / d(\Lambda^2 R^n).$$

(see [1,3] and [14] for details.)

2. Let D be S-module and $0 = \partial: D \to S$ be zero morphism. The pushout diagram is

$$\begin{tikzcd} D \arrow[r, "\psi"] \arrow[d, "\partial=0"'] & \phi_*(D) \arrow[d, "\partial_*"] \\ S \arrow[r, "\phi"'] & R \end{tikzcd}$$

where

$$\begin{aligned} \partial_*(d \otimes r) &= \phi\big(\partial(d)\big)r \\ &= \phi(0)r \\ &= 0r = 0 \end{aligned}$$

so $\partial_* = 0$ and $P = 0$. Thus,

$\phi_*(D) = F(D \times R)$

Then, the induced crossed modules are free S-module on $D \times R$.

3. Given crossed module $i = \partial: I \hookrightarrow S$ where i inclusion of an ideal. Using any surjective homomorphism $\phi: S \to S/I$ the induced diagram is

$$\begin{tikzcd} I \arrow[r, "\psi"] \arrow[d, "i=\partial"'] & \phi_*(I) \arrow[d, "\partial_*"] \\ S \arrow[r, "\phi"'] & S/I \end{tikzcd}$$

Thus, we get $\phi_*(I) = I \otimes (S/I) \cong I/I^2$ which is an S/I-module. So ϕ_* does not preserve ideals.

Last two examples appear in [2] for associative algebras.

It is not difficult to see that the induced crossed module construction gives a functor

$$\phi_*: \mathrm{XMod}/S \to \mathrm{XMod}/R$$

called "extension along a morphism" by Porter [1].

Proposition 13. If $\phi_1: S \to R$, $\phi_2: T \to S$ are two morphisms of k-algebras, then $(\phi_1\phi_2)_*$ and $\phi_{2*}\phi_{1*}$ are naturally isomorphic.

Proposition 14. Suppose p:XMod →k-Alg, $\phi: S \to R$ in k-Alg, and a reindexing functor ϕ_*:XMod$/S \to$ XMod$/R$ is chosen. Then there is a bijection

$$\mathrm{XMod}_\phi((D,S,\partial),(B,R,\eta)) \simeq \mathrm{XMod}/R((\phi_*(D),R,\partial_*),(B,R,\eta))$$

natural in $(D,S,\partial) \in$ XMod$/S$, $(B,R,\eta) \in$ XMod$/R$ where $\mathrm{XMod}_\phi((D,S,\partial),(B,R,\eta))$ consists of those morphisms $\alpha \in \mathrm{XMod}((D,S,\partial),(B,R,\eta))$ with $p(\alpha) = \phi$.

Thus, we get a direct deduction from these discussions as follows.

Corollary 15. The category XMod is bifibred over k-Alg, by the forgetful functor p: XMod $\to k$-Alg.

Proof. For any k-algebra morphism $\phi: S \to R$, there is an adjoint functor pair (ϕ^*, ϕ_*) as we mentioned above. That is, there is a bijection

Θ:XMod$/R((\phi_*(D),R,\partial_*),(B,R,\eta)) \to$ XMod$/S((D,S,\partial),\phi^*(B),S,\eta^*))$

which is natural in $(D,S,\partial) \in$ XMod$/S$, $(B,R,\eta) \in$ XMod$/R$. It is clear that $\Theta(f_*) = f^*$ and $\Theta^{-1}(f^*) = f_*$.

$$\begin{tikzcd}
D \arrow[rrr, "\phi'"] \arrow[dr, "f^*"] \arrow[dd, "\partial"'] & & & \phi_*(D) \arrow[dl, "f_*"'] \arrow[dd, "\partial_*"] \\
& \phi^*(B) \arrow[r, "\phi^B"] \arrow[dd, "\eta^*"] & B \arrow[dd, "\eta"] & \\
S \arrow[dr, equal] & & & R \arrow[dl, equal] \\
& S \arrow[r, "\phi"'] & R. &
\end{tikzcd}$$

Properties of induced crossed module

$\phi_*(D)$ induced crossed R-module can be expressed more simply for the case when $\phi: S \to R$ is an epimorphism of k-algebras.

Epimorphism case:

Proposition 16. Let $\partial: D \to S$ be a crossed S-module and $\phi: S \to R$ epimorphism with $\mathrm{Ker}\phi = K$. Then

$\phi_*(D) \cong D/KD$

where KD is an ideal of D generated by $\{k \cdot d \mid d \in D,\ k \in K\}$.

Proof. Because S acts on D/KD, K acts trivially on D/KD and ϕ is an epimorphism, $R \cong S/K$ acts on D/KD. As follows

$(s + K) \cdot (d + KD) = s \cdot d + KD$

$\beta: D/KD \to R$ given by $\beta(d + KD) = \partial(d) + K$ is a crossed R-module. Indeed,

$$\begin{aligned}
\beta(d + KD) \cdot (d' + KD) &= (\partial d + K) \cdot (d' + KD) \\
&= \partial(d) \cdot d' + KD \\
&= dd' + KD \\
&= (d + KD)(d' + KD)
\end{aligned}$$

$(\rho, \phi): (D, S, \partial) \to (D/KD, R, \beta)$ is a crossed module morphism where $\rho: D \to D/KD, \rho(d) = d + KD$ since $\rho(s \cdot d) = \phi(s) \cdot \rho(d)$.

Suppose that the following diagram of crossed module is commutative.

$$\begin{tikzcd}
D \arrow[r, "\rho'"] \arrow[d, "\partial"'] & D' \arrow[d, "\beta'"] \\
S \arrow[r, "\phi"'] & R
\end{tikzcd}$$

Since $\rho'(s \cdot d) = \phi(s) \cdot \rho'(d)$ for any $d \in D,\ s \in S$, we have

$$\rho'(k \cdot d) = \phi(k) \cdot \rho'(d) = 0 \cdot \rho'(d) = 0$$

so $\rho'(KD) = 0$. Then, there is a unique morphism $\mu: (D/KD) \to D'$ given by $\mu(d + KD) = \rho'(d)$ such that $\mu\rho = \rho'$ and μ is well defined, because of $\rho'(KD) = 0$. Finally, the diagram

$$\begin{tikzcd} D \arrow[r, "\rho"] \arrow[d, "\partial"'] & D/KD \arrow[r, "\mu"] \arrow[d, "\beta"] & D' \arrow[dl, "\beta'"] \\ S \arrow[r, "\phi"'] & R & \end{tikzcd}$$

commutes, since for all $d \in D$

$$\begin{aligned} \beta(d+KD) &= \beta\rho(d) \\ &= \phi\partial(d) \\ &= \beta'\rho'(d) \\ &= \beta'\mu(d+KD) \end{aligned}$$

and

$$\mu(r \cdot (d+KD)) = \mu((s \cdot d) + KD) = \mu(\rho(s \cdot d)) = \rho'(s \cdot d)$$

$$= \phi(s) \cdot \rho'(d) = r \cdot \mu\rho(d) = r \cdot \mu(d+KD)$$

so μ preserves the actions.

The above proposition is given in [2] and [15] for noncommutative and Lie algebras, respectively.

When $\phi: S \to R$ is the inclusion of an ideal, we can give the following result:

Monomorphism case:

In this subsection, we consider the crossed modules induced by a morphism $\phi: S \to R$ of k-algebras, where k is a field, the particular case when S is an ideal of R. It is found the Lie algebra version of that in [15].

If $d \in D$, then the class of d in D/D^2 is written as $[d]$. Then the augmentation ideal of $I(R/S)$ of a quotient algebra R/S has the basis $\{\bar{e}_{i_1}\bar{e}_{i_2}...\bar{e}_{i_p}, i_1 \leq i_2 \leq \cdots i_p, i_j \in I\}_{(i)\neq\emptyset}$, where $\bar{e}_{i_j}$ is the projection of the basic element $e_{i_j} \in I(R)$ on R/S.

Theorem 17. Let $D \subseteq S$ be ideals of R so that R acts on S and D by multiplication. Let $\partial: D \to S$, $\phi: S \to R$ be the inclusions and let $\mathcal{D}$ denote the crossed module (D, S, ∂) with the multiplication action. Then the induced crossed R-module $\phi_*(\mathcal{D})$ is isomorphic as a crossed R-module to

$$\begin{aligned} \zeta: \quad D \times \left(D/D^2 \otimes I(R/S)\right) &\to R \\ (d, [t] \otimes \bar{x}) &\mapsto d. \end{aligned}$$

The action is given by
$r \cdot (d, [t] \otimes \bar{x}) = (r \cdot d, [d] \otimes \bar{r} + [t] \otimes \bar{r}x - [x \cdot t] \otimes \bar{r})$

for , $t \in D$; $\bar{x} \in I(R/S)$ where $\bar{r}, \bar{x}$ denote the image of r, x in R/S, respectively.

Proof. First, we will show that
$\mathcal{T} = (\zeta: T = (D \times (D/D^2 \otimes I(R/S))) \to R)$, $\zeta(d, [t] \otimes \bar{x}) = d$

is a crossed module with the given action:

$$\begin{aligned} \zeta(d', [t'] \otimes \bar{x}').(\mathrm{d}, [\mathrm{t}] \otimes \bar{x}) &= d'.(d, [t] \otimes \bar{x}) \\ &= (d'.d, [d] \otimes \bar{d}' + [t] \otimes \bar{d}'x - [x.t] \otimes \bar{d}') \\ &= (d'd, 0) \\ &= (d'd, [t't] \otimes \bar{x}'\bar{x}) \end{aligned}$$

Consider $i: D \to D \times \left(D/D^2 \otimes I(R/S)\right)$, $i(d) = (d, 0)$. We have the following diagram:

$$\begin{tikzcd}
 & & C \arrow[ddl, "\alpha"] \\
D \arrow[urr, "\beta"] \arrow[r, "i"] \arrow[d, "\partial"'] & T \arrow[ur, dotted, "\tilde{\phi}"] \arrow[d, "\zeta"] & \\
S \arrow[r, "\phi"'] & R &
\end{tikzcd}$$

Clearly we have a morphism of crossed modules $(i, \phi): \mathcal{D} \to \mathcal{T}$. We just verify that when a morphism of crossed module $(\beta, \phi): (D, S, \partial) \to (C, R, \alpha)$ is given, there is a unique morphism $\tilde{\phi}: T = D \times (D/D^2 \otimes I(R/S)) \to C$ such that $\tilde{\phi} i = \beta$ and $\alpha\tilde{\phi} = \zeta$. Since $\tilde{\phi}$ has to be a homomorphism and preserve the action, we have

$$\begin{aligned}
\tilde{\phi}\left(d, [t] \otimes \bar{e}_{(i)}\right) &= \tilde{\phi}\left((d, 0) + \left(0, [t] \otimes \bar{e}_{(i)}\right)\right) \\
&= \tilde{\phi}\left((d, 0) + \left(e_{(i)}.t, [t] \otimes \bar{e}_{(i)}\right) + (-e_{(i)}.t, 0)\right) \\
&= \tilde{\phi}\left((d, 0) + \left(e_{(i)}.(t, 0) + (-e_{(i)}.t, 0\right)\right) \\
&= \tilde{\phi}\left((d, 0)\right) + \tilde{\phi}\left(e_{(i)}.(t, 0) + (-e_{(i)}.t, 0\right)) \\
&= \tilde{\phi}\left((d, 0)\right) + \tilde{\phi}\left(e_{(i)}.(t, 0)\right) + \tilde{\phi}\left(-e_{(i)}.t, 0\right) \\
&= \tilde{\phi} i(d) + e_{(i)}.\tilde{\phi} i(t) - \tilde{\phi} i(e_{(i)}.t) \\
&= \beta(d) + e_{(i)}.\beta(t) - \beta(e_{(i)}.t)
\end{aligned}$$

for any $d \in D, \left([t] \otimes \bar{e}_{(i)}\right) \in D/D^2 \otimes I(R/S)$. This proves uniqueness of any such a $\tilde{\phi}$. We now prove that this formula gives a well-defined morphism.

It is immediate from the formula that $\tilde{\phi}: D \times (D/D^2 \otimes I(R/S)) \to C$ must be defined by $\tilde{\phi}(d, u) = \beta(d) + \gamma(u)$ where $\gamma: D/D^2 \otimes I(R/S) \to C, \gamma([d] \otimes \bar{e}_{(i)}) = \gamma_{e_{(i)}}(d)$ and $\gamma_r: D \to C, \gamma_r(d) = r \cdot \beta(d) - \beta(r \cdot d)$.

Since

$$\begin{aligned}
\alpha\gamma_r(d) &= \alpha\left(r \cdot \beta(d) - \beta(r \cdot d)\right) \\
&= \alpha\left(r \cdot \beta(d)\right) - \alpha\left(\beta(r \cdot d)\right) \\
&= r\alpha\left(\beta(d)\right) - \phi\partial(r \cdot d)\big) \\
&= r\phi\partial(d) - \phi\left(r\partial(d)\right) \\
&= r\partial(d) - r\partial(d) = 0
\end{aligned}$$

for $d \in D$, $\gamma_r(D)$ is contained in the annihilator Ann(C) of C. Also, we get

$$\begin{aligned}
\gamma_r(dd') &= r \cdot \beta(dd') - \beta\left(r \cdot (dd')\right) \\
&= r \cdot \left(\beta(d)\beta(d')\right) - \beta(r \cdot d)\beta(d') \\
&= r \cdot \beta(d)\beta(d') - \beta(r \cdot d)\beta(d') \\
&= \gamma_r(d)\beta(d') \\
&= 0 \\
&= \gamma_r(d)\gamma_r(d')
\end{aligned}$$

for $d, d' \in D$. Consequently γ_r is a homomorphism of commutative algebras that factors through D/D^2.

The function $\tilde{\phi}$ is clearly a well-defined morphism of commutative algebras:

$$\begin{aligned}
\tilde{\phi}\big((d,u)(d',u')\big) &= \tilde{\phi}(dd', uu') \\
&= \beta(dd') + \gamma(uu') \\
&= \beta(d)\beta(d') + \gamma(u)\gamma(u') \\
&= \beta(d)\beta(d') + \beta(d)\gamma(u') + \gamma(u)\beta(d') + \gamma(u)\gamma(u') \\
&= \big(\beta(d) + \gamma(u)\big)\big(\beta(d') + \gamma(u')\big) \\
&= \tilde{\phi}(d,u)\tilde{\phi}(d',u')
\end{aligned}$$

Further, $\tilde{\phi} i = \beta$ and $\alpha\tilde{\phi} = \zeta$, as $\alpha\gamma$ is trivial:

$$\begin{aligned}
\alpha\gamma\big([t] \otimes \bar{e}_{(i)}\big) &= \alpha\left(e_{(i)} \cdot \beta(t) - \beta\big(e_{(i)} \cdot t\big)\right) \\
&= e_{(i)} \cdot \alpha\big(\beta(t)\big) - \alpha\left(\beta\big(e_{(i)} \cdot t\big)\right) \\
&= e_{(i)} \cdot \phi\partial(t) - \phi\partial\big(e_{(i)} \cdot t\big) \\
&= e_{(i)} \cdot \partial(t) - \partial\big(e_{(i)} \cdot t\big) \\
&= 0.
\end{aligned}$$

Finally, we prove that $\tilde{\phi}$ preserves the action. Let $d, t \in D$, $r \in R$ and $\overline{e}_{(i)}$ be an element in the basis of $I(R/S)$, then we have

$$\begin{aligned}
\tilde{\phi}\left(r \cdot \big(d, [t] \otimes \overline{e}_{(i)}\big)\right) &= \tilde{\phi}\big(r \cdot d, [d] \otimes \overline{r} + [t] \otimes \overline{r}e_{(i)} - \big[e_{(i)} \cdot t\big] \otimes \overline{r}\big) \\
&= \beta(r \cdot d) + \gamma\big([d] \otimes \overline{r} + [t] \otimes \overline{r}e_{(i)} - \big[e_{(i)} \cdot t\big] \otimes \overline{r}\big) \\
&= \beta(r \cdot d) + \gamma_r(d) + \gamma_{re_{(i)}}(t) - \gamma_r\big(e_{(i)} \cdot t\big) \\
&= \beta(r \cdot d) + r \cdot \beta(d) - \beta(r \cdot d) + re_{(i)} \cdot \beta(t) - \beta\big(re_{(i)} \cdot t\big) \\
&\quad - r \cdot \beta\big(e_{(i)} \cdot t\big) + \beta\left(r \cdot \big(e_{(i)} \cdot t\big)\right) \\
&= \beta(r \cdot d) + re_{(i)} \cdot \beta(t) - r \cdot \beta\big(e_{(i)} \cdot t\big) \\
&= r \cdot \left(\beta(d) + e_{(i)} \cdot \beta(t) - \beta\big(e_{(i)} \cdot t\big)\right) \\
&= r \cdot \left(\beta(d) + \gamma_{e_{(i)}}(t)\right) \\
&= r \cdot \tilde{\phi}\big(d, [t] \otimes \overline{e}_{(i)}\big).
\end{aligned}$$

Conflicts of interest

The author states that she did not have a conflict of interest.

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