Energy-momentum puzzle in a bianchi-type II universe with $f(T)$ gravity

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Abstract

The energy-momentum localization problem, which was attemted by Einstein himself for the first time, has been continued to the present day. Recently, new prescription obtained by modifying the torsion theory and these results shed light on the solution of the energy momentum localisation problem. Focusing this purpose, we consider a Locally Rotationally Symmetric Bianchi Type-II model in the teleparallel framework and calculate the modified energy and momentum density for the general case. We also obtain the energy and momentum density for some special cases of the modified theory and compare our results with previous work in the literature.

1. Introduction

The problem of energy and momentum localization has been studied for almost a century. While it has not yet been solved, there have been many efforts to understand it. The first attempt was made by Einstein himself in seeking to unify electromagnetism and gravitation [1]. After this important work, many additional prescriptions have been proposed to solve the energy-momentum localization problem [2-10]. Except for Møller's approach [5] which is defined for any coordinate system, they produce meaningful results only when we transform the line-element coordinate system into quasi-local Cartesian coordinates. Using those prescriptions, many authors have calculated the energy and momentum densities for various space-time models [11-19]. Vargas [20] used a Friedman-Robertson-Walker space-time model to demonstrate the equivalence of the teleparallel energy and momentum densities with those obtained from general relativity. Studies of this type have shown the equivalence between general relativity and teleparallel gravity theories [21-28].

On the other hand, recent observations of supernovae of Type Ia, the Cosmic Microwave Background Radiation, Baryon Acoustic Oscillations, etc., have demonstrated that our universe is undergoing accelerated expansion [29-33]. There are three candidates to explain this accelerating expansion: (i) a time-independent cosmological constant (Λ), (ii) dark energy, and (iii) modified gravity. The latter is based on a generalization of the Einstein-Hilbert action, as done-for example-in the so-called "$f(T)$ gravity," where T is the torsion scalar. Note that $f(T)$ theory reduces to teleparallel gravity if we choose $f(T) = T$. Various $f(T)$ models have been proposed to explain the late-time cosmic expansion without the need for exotic dark energy [34, 35]. Last five year there have been a lot of studies about $f(T)$ gravity in the literature. For example Myrzakulov analyzed relation between $f(T)$ models and purely kinetic k-essence [36].

The problem of localizing the energy-momentum distribution can brought to a new level by taking into account the dark-energy/matter distributions obtained in recent observations. The problem has therefore become once again a real and important puzzle. In the present study, we address the problem of localizing the energy-matter distribution taking into account the dark-energy/matter contributions. This paper is organized as follows: In Section 2 we summarize the problem of energy and momentum localization in $f(T)$ gravity for a Bianchi Type-II model. Next, in Section 3, we calculate the energy and momentum density for four well-known $f(T)$ models. The final section is devoted to a discussion of our results.

2. Preliminaries: Modified Energy And Momentum Scenario

The tetrad fields $\{h_\mu^a\}$ are defined at each point of a manifold and constitute the basis of teleparallel gravitation theory. The orthonormal tetrad fields satisfy the following relation:

$$h_\mu^a h_b^\mu = \delta_b^a, \qquad h_\mu^a h_a^\nu = \delta_\nu^\mu. \tag{1}$$

The metric tensor $g_{\mu\nu}$ is reconstructed from these tetrad fields:

$$g_{\mu\nu} = \eta_{ab} h_\mu^a h_\nu^b \tag{2}$$

where η_{ab} is the standard Minkowski metric defined by $\eta_{ab} = diag(1, -1, -1, -1)$. The Weitzenböck connection, which contains only the torsion not the curvature [37], can be written as:

$$\Gamma_{\alpha\beta}^\lambda = h_a^\lambda \, \partial_\beta h_\alpha^a. \tag{3}$$

The torsion tensor, which is the teleparallel version of the gravitational "force", is

$$T_{\alpha\beta}^\lambda = \Gamma_{\beta\alpha}^\lambda - \Gamma_{\alpha\beta}^\lambda \tag{4}$$

and the torsion scalar is obtained from the torsion as follows:

$$T = \frac{1}{4} T^{\sigma\mu\nu} \mathrm{T}_{\sigma\mu\nu} + \frac{1}{4} T^{\sigma\mu\nu} T_{\nu\mu\sigma} - T_{\sigma\mu}^\sigma T_\nu^{\nu\mu}. \tag{5}$$

If we consider a tensor that is antisymmetric in the last two indices, such as

$$S^{\sigma\mu\nu} = \frac{1}{4}(T^{\sigma\mu\nu} + T^{\mu\sigma\nu} - T^{\nu\sigma\mu}) - \frac{1}{2}\left(g^{\sigma\nu} T_\gamma^{\gamma\mu} - g^{\sigma\mu} T_\gamma^{\gamma\nu}\right) \tag{6}$$

it is possible to rewrite the torsion scalar in the form

$$T = \frac{1}{2} S^{\sigma\mu\nu} T_{\sigma\mu\nu}. \tag{7}$$

The teleparallel Lagrangian for $f(T)$ gravity can be rewritten in the following form inspired by $f(R)$ gravity:

$$\mathcal{L}_f = \frac{h}{16\pi G} f(T). \tag{8}$$

The action for this modified gravity therefore becomes

$$S = \int d^4x h \left[\mathcal{L}_f + \mathcal{L}_m\right] \tag{9}$$

where $\mathcal{L}_m$ is the matter Lagrangian. Variation of the action (9) with respect to the tetrad fields yields the field equation

$$f_T(T)\left[\partial_\sigma(hh_a^\nu S_\nu^{\lambda\sigma}) - hh_a^\sigma S^{\mu\nu\lambda} T_{\mu\nu\sigma}\right] + f_{TT}(T)hh_a^\nu S_\nu^{\lambda\sigma} \, \partial_\sigma T + \frac{h}{2} h_a^\lambda f(T) = h\Xi_a^\lambda \tag{10}$$

where we use the definitions $f_T(T) \equiv \frac{df(T)}{dT}, f_{TT}(T) \equiv \frac{d^2 f(T)}{dT}$ for convenience and Ξ_a^λ is the energy-momentum tensor. Of course, if we choose $f(T)$ to be equal to the torsion (T), $f(T)$ gravity reduces to teleparallel gravity.

After applying Noether's theorem and using the new Lagrangian, Abedi and Salti [38] obtained a new energy momentum prescription:

$$ht_\lambda^\sigma = f_T(T)ht_{\lambda(TG)}^\sigma - \frac{h}{16\pi G}\delta_\lambda^\sigma[f(T) - f_T(T)T], \tag{11}$$

and the field equation becomes

$$\mathcal{T}_\nu^\lambda = \frac{1}{8\pi hG}\partial_\sigma\left[hf_T(T)S_\nu^{\lambda\sigma}\right]. \tag{12}$$

Here $\mathcal{T}_\nu^\lambda$ is being the total energy-momentum of gravitation and matter. Note that $\mathcal{T}_\nu^\lambda$ satisfies the energy-momentum conservation law $\partial_\lambda(h\mathcal{T}_\nu^\lambda) = 0$. Finally the momentum four-vector definition in modified teleparallel gravity is given by

$$P_\mu = \int \mathcal{T}_\mu^0 dx dy dz. \tag{13}$$

3. Energy and Momentum Density in Modified Teleparallel Gravity

Recently, theoretical interest in anisotropic space-time models has increased because the homogenous isotropic Friedman-Robertson-Walker may (or may not) exactly represent our universe. Bianchi-type cosmological models may therefore play an important role in describing our universe. Here, we consider an Locally Rotationally Symmetric (LRS) Bianchi Type-II model, which is given by the following line element [39]

$$ds^2 = dt^2 - A^2 dx^2 - B^2 dy^2 - [A^2 + x^2B^2]dz^2 - 2xB^2 dydz. \tag{14}$$

where the function A and B are depend only time. Using line element (14) the metric tensor can be written as

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -A^2 & 0 & 0 \\ 0 & 0 & -B^2 & -xB^2 \\ 0 & 0 & -xB^2 & -(A^2 + x^2B^2) \end{pmatrix} \tag{15}$$

and its inverse is $g^{\mu\nu}$

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -A^{-2} & 0 & 0 \\ 0 & 0 & -\frac{A^2 + x^2B^2}{A^2B^2} & xA^2 \\ 0 & 0 & xA^2 & -A^{-2} \end{pmatrix} \tag{16}$$

Using the definition (2) we find the tetrad and its inverse to be, respectively:

$$h_\mu^a = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & B & xB \\ 0 & 0 & 0 & A \end{pmatrix}, \qquad h_a^\mu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & A^{-1} & 0 & 0 \\ 0 & 0 & B^{-2} & 0 \\ 0 & 0 & -\frac{x}{A} & A^{-1} \end{pmatrix}. \tag{17}$$

Equation (3) giv

es the following Weitzenböck connection:

$$\Gamma_{10}^1 = \Gamma_{30}^3 = \frac{\dot{A}}{A}, \qquad \Gamma_{20}^2 = \frac{\dot{B}}{B}, \qquad \Gamma_{30}^2 = \frac{x\dot{A}}{A} + \frac{x\dot{B}}{B}, \qquad \Gamma_{31}^2 = 1. \tag{18}$$

where the dot means the first derivative with respect to time. The components of the torsion tensor obtained from the Weitzenböck connections become

$$T_{01}^1 = -T_{10}^1 = T_{30}^3 = -T_{30}^3 = \frac{\dot{A}}{A}, \qquad T_{02}^2 = -T_{20}^2 = \frac{\dot{B}}{B},$$

$$T_{13}^2 = -T_{31}^2 = \frac{x\dot{A}}{A} + \frac{x\dot{B}}{B} \qquad T_{13}^2 = -T_{31}^2 = 1. \tag{19}$$

From the components of the torsion tensor, we find the following components of the antisymmetric tensor used in the energy-momentum prescription:

$$S^{023} = -S^{032} = -\frac{x\dot{A}}{A^3},$$

$$S^{101} = -S^{110} = S^{303} = -S^{330} = \frac{1}{A^3}\left(\dot{A} + \frac{A\dot{B}}{B}\right),$$

$$S^{123} = -S^{132} = S^{213} = -S^{231} = S^{312} = -S^{321} = \frac{1}{2A^4},$$

$$S^{202} = -S^{220} = \frac{1}{A^3 B^2}\left(2A^2\dot{A} + 3x^2 B^2 \dot{A} + x^2 AB\dot{B}\right),$$

$$S^{203} = -S^{230} = S^{302} = -S^{320} = -\frac{x}{A^3 B}\left(2B\dot{A} + A\dot{B}\right),$$

$$S^{212} = -S^{221} = -\frac{x}{A^4}. \tag{20}$$

$f(T)$ Models

Now we can calculate the energy and momentum densities for several well-known $f(T)$ models. We consider $f(T)$ models which are contain linear, logarithmic and exponential term to determine modified torsion gravity.

• Model I: $f(T) = \alpha(-T)^b$

In this model the quantities α and b are free parameters [40]. From equation (12) we obtain the following components of the energy and momentum densities:

$$_{1st}\mathcal{T}_0^0 = 0, \tag{21}$$

$$_{1st}\mathcal{T}_1^0 = \frac{4(b-1)bxB^2\dot{A}^2}{2^b G\pi A^5}\left(\dot{A} + \frac{A\dot{B}}{B}\right) \times \left[\frac{4A^2 B\dot{A}^2 + B^3(1-4x^2\dot{A}^2) + 8A^3\dot{A}\dot{B}}{A^4 B}\right]^{b-2}. \tag{22}$$

• Model II: $f(T) = Te^{\beta T}$

In this model, β is an arbitrary constant [35] and the momentum and energy densities are given by

$$_{2nd}\mathcal{T}_0^0 = 0, \tag{23}$$

$$_{2nd}\mathcal{T}_1^0 = \frac{\beta x \dot{A}^2 (B\dot{A} + A\dot{B})}{4G\pi A^9}[4\beta A^2 B\dot{A}^2 - 4A^4 B$$

$$+ \beta B^3 (1 - 4x^2\dot{A}^2) + 8\beta A^3 \dot{A}\dot{B}]e^{\frac{-\beta(4A^2 B\dot{A}^2 + B^3(1-4x^2\dot{A}^2) + 8A^3\dot{A}\dot{B})}{2A^4 B}}. \tag{24}$$

• Model III: $f(T) = T + aT^{1/2}\ln T$

Here a is a constant [41] and the energy and momentum densities become

$$_{3rd}\mathcal{T}_0^0 = 0, \tag{25}$$

$$_{3rd}\mathcal{T}_1^0 = \frac{axA\dot{A}(B\dot{A}+A\dot{B})\ln[B^3(4x^2\dot{A}^2)-4A^2 B\dot{A}^2-8A^3\dot{A}\dot{B}]}{\sqrt{2}G\pi B^{1/2}[B^3(4x^2\dot{A}^2)-4A^2 B\dot{A}^2-8A^3\dot{A}\dot{B}]^{3/2}} \tag{26}$$

• Model IV: $f(T) = T - \mu T(1 - e^{-bT_0/T})$

Here μ is a parameter obtained by solving modified the Friedmann equation [42]. For this model we obtain

$$_{4th}\mathcal{T}_0^0 = 0, \tag{27}$$

$$_{4th}\mathcal{T}_1^0 = \frac{8bT_0 x\mu A^7 \dot{A}^2 (B\dot{A} + A\dot{B})}{G\pi(4A^2 B\dot{A}^2 + B^3(1 - 4x^2\dot{A}^2) + 8A^3 \dot{A}\dot{B})^4}[4A^2 B\dot{A}^2 - bT_0 A^4 B$$

$$- B^3(4x\dot{A}^2 - 1) + 8A^3 \dot{A}\dot{B}]e^{\frac{-2bT_0 2A^4 B}{(4A^2 B\dot{A}^2 + B^3(1-4x^2\dot{A}^2) + 8A^3\dot{A}\dot{B})}}. \tag{28}$$

4. Discussions

The four-momentum localization problem contains many important issues and non-specific solutions [43]. For example, the study of energy-momentum localization may help to clarify our understanding of space-time, including phenomena such as gravitational lensing [11, 13, 15].

In the present work we have focused on the energy-momentum localization problem for a Bianchi Type-II universe in modified teleparallel gravity. We calculated the energy and momentum densities for four different well-known functions $f(T)$. The energy density vanishes for each $f(T)$ definition, but we may have a non-zero momentum density, depending upon the particular space-time and the parameters of the chosen model. One can easily see that if we choose the parameters of the model specifically to reduce to teleparallel gravity, the momentum density vanishes. In the first model, taking $b = 1$ and $\alpha = -1$ yields teleparallel gravity, and the momentum density vanishes. Likewise, if we take $\beta = 0$ in the second model, $a = 0$ in the third model, and $\mu = 0$ in the last model, they also reduce to teleparallel gravity, and we obtain:

$$E_{TP} = {}_{TP}P_i = 0. \tag{29}$$

These findings agree with previous results obtained by Aydogdu [22]. It is found that the total energy of all closed type of universes are zero because of the energy momentum contributions from the matter and field inside two arbitrary surfaces cancel each other [21,23,44]. If explicit forms of expansion coefficients $(A(t), B(t))$ are known, the contribution of free parameters to energy density can be more clearly apprehensible. As mentioned in the first section of this paper, one must take into account the modified energy-momentum localization puzzle in discussing the dark-energy/matter contributions. This demonstrates the importance of the new calculations we have performed and the new results obtaind in this work.

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